Acoustical Wave Propagator method for two-dimensional sound propagation

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Abstract

In this paper, the Acoustical Wave Propagator (AWP) method is used to predict sound propagation in two dimensional room. The Chebyshev polynomial expansion scheme is applied to approximate the exponential wave propagator and Fast Fourier transform method is used to evaluate the spatial derivatives. The free field prediction using AWP is compared with exact solutions for accuracy assessment. Various rooms are used to demonstrate the capacity of AWP in modelling transient sound propagation.

1 Introduction

The study of room acoustics in an enclosure is important. The sound quality of a room is often determined by the early decay time and reverberation. The accurate prediction of sound distribution would be extremely useful for rooms with specific acoustic requirement such as concert halls and provide valuable information for room designs.

Theoretical models for the prediction of sound are becoming increasingly practical due to rapid growth in computer performance and power. However, these studies have been dominated by frequency domain methods [1,2]. These methods assume that the sound source has a periodic time dependency, which is the case for machine tools and electric motors, and even as an approximation for some musical instruments. However, since the sound quality in a room is largely affected by the sound decay with time and often the sound source is an impulse or with random frequencies, a time domain analysis would be more valuable. The method should also be realistic and hence it must handle complex boundary conditions, which is another obstacle for the frequency domain methods.
Some work has been done with regard to sound propagation in the time-domain using finite-difference, finite-element or boundary-element methods[3-8]. The drawback of these methods is that the associated numerical error is usually proportional to the time step $\Delta t$ used in the simulation. Consequently, only very small time steps can be adopted and the number of steps required for modelling a complete propagation is large. This often leads to significant accumulated error in both the magnitude and the phase of the acoustical wave in propagation.

Recently Pan and Wang [9] developed an acoustical wave propagator scheme to investigate the propagation and scattering properties of acoustic waves. This scheme extended the quantum wave propagator based on time-domain schordinger’s equation [10-13] to the acoustical field and has been successfully implemented for a one-dimensional acoustic wave in a duct. It is found to be highly accurate and computationally effective for the prediction of time-domain evolution of acoustical waves, providing complete transitory information of the system under study.

In this paper, we extend this method to two dimensional sound field. The accuracy of the scheme is examined by a comparison of computed and exact solution for two-dimensional open space. This scheme is then applied to study the acoustics of six rooms with different geometries. It is found to be highly accurate and computationally effective for describing the time-domain evolution of acoustic waves.

2 Theoretical model

The propagation of a sound wave in a two-dimensional space is described by the wave equation:

$$\frac{\partial}{\partial t} \Phi(x,y,t) = -\hat{H} \Phi(x,y,t), \quad (1)$$

where $\Phi$ is a state vector representing the sound pressure $p(x,y,t)$ and particle velocity $v_x(x,t)$ and $v_y(y,t)$ in the x- and y- directions,

i.e.,

$$\Phi = \begin{bmatrix} p(x,y,t) \\ v_x(x,t) \\ v_y(y,t) \end{bmatrix},$$

and $\hat{H}$ is the system operator defined as:

$$\hat{H} = \begin{bmatrix} 0 & \rho c^2 \frac{\partial}{\partial x} & \rho c^2 \frac{\partial}{\partial y} \\ \frac{1}{\rho} \frac{\partial}{\partial x} & 0 & 0 \\ \frac{1}{\rho} \frac{\partial}{\partial y} & 0 & 0 \end{bmatrix}.$$
by setting the appropriate values for the speed of sound $c$ and the medium density $\rho$, which are defined as functions of position namely $c(x,y)$ and $\rho(x,y)$.

By integrating eqn (1) with respect to time, the solution for the state vector $\Phi$ can be obtained,

$$\Phi(x,y,t) = e^{-(t-t_0)H} \Phi(x,y,t_0).$$  \hspace{1cm} (2)

where $e^{-(t-t_0)H}$ is the acoustical wave propagator and $\Phi(x,y,t_0)$ represents the initial state vector. We can obtain the sound pressure $p(x,y,t)$ and particle velocity $v_x(x,t)$ and $v_y(y,t)$ at any time $t$ by the operation of the acoustical wave propagator acting upon the initial state vector.

Briefly, the Chebyshev scheme approximates the exponential time propagator by a Chebyshev polynomial expansion. Since the Chebyshev polynomials are defined in the range of $[-1,1]$, the matrix operator $\hat{H}$ needs to be normalized by

$$\hat{H}' = \frac{\hat{H}}{\sqrt{\lambda_{\text{max}}}},$$  \hspace{1cm} (3)

where $\lambda_{\text{max}}$ is the maximum eigenvalue of the matrix operator. Expanding the acoustical wave propagator as Chebyshev polynomials of the first kind and denoting $R = (t-t_0)\sqrt{\lambda_{\text{max}}}$, Eqn. (2) becomes:

$$\Phi(x,y,t) = e^{-R\hat{H}} \Phi(x,y,t_0) = \sum_n a_n(R)T_n(\hat{H}')\Phi(x,y,t_0)$$  \hspace{1cm} (4)

where $a_0(R) = I_0(R)$ except for $a_n(R) = 2(-1)^n I_n(R)$ and $I_n(R)$ is the nth-order modified Bessel function of the first kind. The exact number of terms required for convergence is determined by a number of parameters such as the time step, the resolution and the boundary conditions. In general, about 60 terms are required for convergence. The zero and first-order Chebyshev polynomials are defined as $T_0(\hat{H}') = I$ and $T_1(\hat{H}') = \hat{H}'$, and the higher terms can be calculated by the following recursive relations:

$$T_{n+2}(\hat{H}') = 2\hat{H}'T_n(\hat{H}') - T_{n+1}(\hat{H}').$$  \hspace{1cm} (5)

The evaluation of eqn (4) then comprises a series of the Hamiltonian operator acting upon the initial state vector $\Phi(x,y,t_0)$. This action involves mainly spatial derivatives, which can be determined accurately using the fast Fourier transformation method [9].

3 Accuracy of the results

In order to exam the accuracy of the AWP in 2D space, an exact solution is used for comparison. In a two-dimensional free space with polar symmetry, wave equation can be presented as:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2},$$  \hspace{1cm} (6)
with initial conditions:
\[ p(r,0) = p_0(r), \quad \frac{\partial p(r,0)}{\partial t} = v_0(r). \]  

(7)

We employ the zero-order Hankel transform given by:
\[ \bar{p}(\xi) = \int_0^\infty r p_0(r) J_0(\xi r) \, dr, \]

(8)
\[ \bar{v}(\xi) = \int_0^\infty r v_0(r) J_0(\xi r) \, dr. \]

(9)

The sound pressure at time \( t \) can be written as [14]:
\[ p(r,t) = \int_0^\infty \bar{p}(\xi) \cos \xi \xi J_0(\xi r) d\xi + \frac{1}{c} \int_0^\infty \bar{v}(\xi) \sin \xi \xi J_0(\xi r) d\xi. \]

(10)

If the initial sound field in the two-dimensional open space is generated by an impulse excitation, the distribution of sound pressure and air particle velocity can be written as:
\[ p_0(r,0) = e^{-r/a}, \]

(11)
\[ v_0(r,0) = 0. \]

(12)

Substitute eqns (11) and (12) to eqn (10), we have an analytical solution:
\[ p(r,t) = \frac{a^2}{2} \int_0^\infty \xi \exp \left( -\frac{\xi^2 a^2}{4} \right) \cos \xi \xi J_0(\xi r) d\xi. \]

(13)

The relative errors for the sound pressure can be defined as:
\[ \delta(r,t) = \left| \frac{p_{\text{emp}}(r,t) - p_e(r,t)}{p_{\text{emp}}(r,t)} \right|, \]

(14)

where \( p_{\text{emp}}(r,t) \) and \( p_e(r,t) \) are the sound pressure by AWP method and exact solution respectively. Note that \( p_{\text{emp}}(r,t) \) is the maximum value of \( p_e(r,t) \).

Figure 1 shows the relative error as a function of position \( r \) defined in eqn (14). The maximum error is less than \( 3.1 \times 10^{-3} \), demonstrating that the acoustical wave propagator scheme provides a highly accurate prediction to sound wave propagation in time domain.

4 Application

In this section, we present the results obtained using the AWP method to illustrate its usefulness for acoustical design. The various rooms examined are illustrated in Fig. 2-7. The rooms have a dimension about \( 3.5m \times 3.5m \). The boundary conditions are defined by specifying the wave speed and density for the boundary medium. A spatial resolution of \( 128 \times 128 \) is used in our calculations, with the following initial conditions:
\[ p(x,y,0) = e^{-\left( \frac{x-x_0}{2a} \right)^2 - \left( \frac{y-y_0}{2a} \right)^2} \quad \text{and} \quad \frac{\partial p(x,y,0)}{\partial t} = 0. \]

(15)
These symmetric initial conditions indicates that the sound propagates uniformly in all directions of free space. The choice of \( x_0 \) and \( y_0 \) defines the position of the initial sound disturbance.

![Graph showing relative errors for exactly solution and the AWP method](image)

Figure 1: The relative errors for exactly solution and the AWP method (t=0.002s)

Fig. 2 shows the sound pressure distribution in a square room with the initial sound field at the center (1.75m,1.75m). Note that some sound wave was absorbed into the walls. In this study we only exam the effect of reflective wave on the resultant field. This also applies to the results presented below. Since the boundary reflection pattern are fairly intuitive, we can see the accuracy of AWP method.

Fig. 3 shows the sound pressure distribution in a square room with the initial sound source at the bottom left corner (1.25m,1.25m). Since the position of the initial sound source was symmetrical about the diagonal line and closer one corner than others, the sound pressure distribution is symmetrical about the diagonal of the room and need more time to develop a pattern in the room.

The sound pressure distribution for a semi-circular room is very interesting, since the concave shape tends to focus the sound into the centre of the room, as shown in Fig. 4. This explains why concave shapes are generally avoided in acoustical design of concert or lecture halls, because one position in the room receives a greater proportion of the sound than other sections.

A room with a barrier in the middle of the room was modelled to investigate the effects of the barrier. Fig. 5 shows that the AWP method can effectively model the sound diffraction around the barrier. This figure shows the propagation of
sound from the top left-hand corner \((0.75m,2.5m)\) of the room as it makes its way around the corner via a series of reflections.

A room with a truncated edge is shown in Fig. 6. The sound source located at the center \((1.75m,1.75m)\). By comparing the sound distribution with the square room shape for the same propagation time, it is evident that the side without the truncated edge acts exactly as it would for a square room. The walls or ceilings of a room are often truncated to improve the room's acoustics, since it projects the sound wave into the room rather than focusing the sound into the corner.

Finally, A room with a column at \((2.375m,2.375m)\) is shown in Fig. 7. The sound source located at \((1.25m,1.25m)\). The sound pressure distribution is symmetrical about the diagonal of the room, due to the position of the initial sound source and the column located symmetrical about the diagonal line of the room.

5 Conclusions

In this paper, the acoustic wave propagator scheme is extended to two-dimensional space. The qualitative aspects of the sound wave evolution in 2D space with complex boundary geometry are demonstrated. It shows that the AWP method are computational efficient and accurate for determining the time domain propagation of acoustic waves in a room.

Figure 2: Sound wave propagation in a square room with the initial sound field at the center \((1.75m,1.75m)\) and \((t=0.012s)\).
Figure 3: Sound pressure distribution in a square room with the initial sound source at the corner (1.25m,1.25m) and (t=0.016s).

Figure 4: Sound pressure distribution in a semi-circular shaped room, sound source at (1.75m, 1.25m) and (t=0.013s).
Figure 5: Sound pressure distribution in a room with a barrier, sound source at (0.75m, 2.5m) and (t=0.018s).

Figure 6: Sound pressure distribution in a truncated square room, sound source at (1.75m, 1.75m) and (t=0.012s).
Figure 7: Sound pressure distribution in a room with a column (2.375m, 2.375m), sound source at (1.25m, 1.25m) and (t=0.013s).

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References

Modelling and Experimental Measurements in Acoustics III


