CHAPTER 5

Mechanics of compliant structures

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Abstract

Biological organisms must be structurally efficient. Hence it is not surprising that nature has embraced the compliant membrane structure as a central element in higher biological forms. This chapter discusses the unique challenges of modeling the mechanical behavior of compliant membrane structures. In particular, we focus on several special characteristics, such as large deformation, lack of bending rigidity, material nonlinearity, and computational schemes.

1 Introduction

1.1 Motivation

Biological organisms must be structurally efficient. They have benefited from millions of years of evolution to achieve, for example, high load carrying capacity per unit weight. It is not surprising then to find that compliant membrane structures play a significant role in nature, for membranes are among the most efficient of structural elements. Long ago, engineers adopted membrane structures for solutions where structural efficiency was critical. As an example, consider the working balloon. For a given balloon volume, the total lift available is fixed, implying a zero-sum trade between structure and payload. Modern high-altitude scientific balloons (Fig. 1), made of thin polymer film fractions of a millimeter thick with areal densities of a few grams per square meter, support payloads of a few tons!

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Figure 1: Preparing to launch a high-altitude scientific balloon in Antarctica (courtesy NASA).

The subject of the mechanics of membranes is quite broad and challenging, being a special application of nonlinear continuum mechanics, and entire books are written on the topic. This chapter seeks to provide a very concise summary of the important features of membrane mechanics. We hope to give the reader some background and exposure to the subject, a "flavor" if you will, with references provided for those who desire to pursue deeper study. The first part of the chapter is a general foundation used in many applications besides membrane mechanics, while the latter part is more specific to membrane applications

1.2 Brief historical review

The first membrane structures were biological organisms, which may well represent the widest usage of this structural type even today. Examples range from cell walls (Fig. 2) and bat wings, to the bullfrog's inflatable throat. Historical use of membranes in engineering structures may be traced to the sail (Figure 3) and the tent. Kites, parachutes, balloons, and other flying structures followed. Musical instruments, notably drums, were comprised of membranes formed from stretched animal skin or parchment. In modern times, membranes have seen increasing use in civil structures such as temporary storage facilities and large-span roofs. Recently, there has been considerable interest in large membrane/inflatable structures for space applications.

The earliest formal analysis of membranes was begun by acousticians during the Renaissance period. This dynamic analysis was limited of course to simple geometries and linear problems. The first nonlinear analysis occurred during the early part of the 1900s, as a membrane solution to the so-called von Karman plate equations. Only since the advent of modern computers has the solution to problems with strongly nonlinear membranes of arbitrary geometry been accomplished. [For a complete review, see references 1–3.]

Figure 2: The paramecium wall is typical of cell walls formed from compliant membranes.

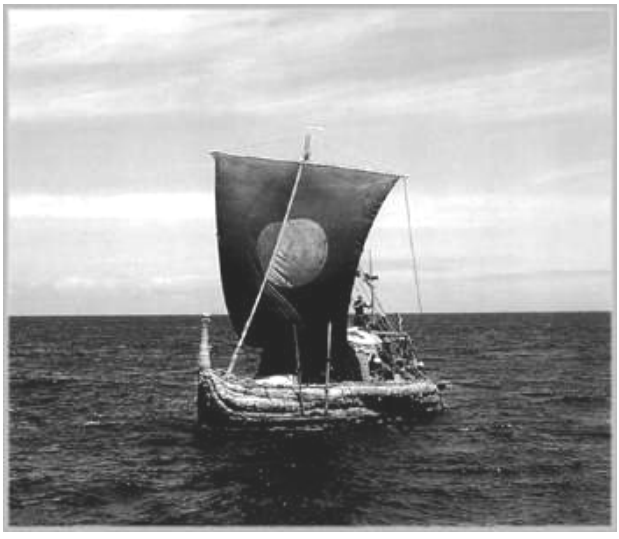

Figure 3: The sail may be one of the first human engineered compliant structures.

1.3 Definition and unique behavior of compliant membrane structures

To the structural engineer, membrane may mean an idealized model of a plate or shell structure, wherein the in-plane response dominates away from domain and load boundaries; hence, the stress couples may be neglected in this interior region. To the applied mechanician, membrane may mean a surface (thin film) with zero bending rigidity, resulting in nonexistent compressive solutions. In the present work, the term "membrane model" will be used in the sense of the structural engineer and the term "membrane" will be reserved for zero bending rigidity structures;

the term elastic "sheets" will be adopted for those structures with small but non-negligible bending rigidity.

In the case of membranes or sheets, their lack of bending rigidity, due to extreme thinness and/or low elastic modulus, leads to an essentially under-constrained structure that has equilibrium configurations only for certain loading fields. Under other loading conditions, large rigid-body deformations can take place. In addition, these same characteristics lead to an inability to sustain compressive stress. Time-dependent and nonlinear behaviors are also common features of typical membrane materials. The formalism for describing such behaviors is that of nonlinear continuum mechanics, in which the mathematical language of tensors is central.

2 Tensor analysis

The mathematical analysis of nonlinear continuum mechanics is usually presented using tensors. This is primarily due to the *invariance* of such a formulation, in that it is then general for any coordinate system. A secondary benefit is the conciseness of the resulting mathematics. For a more complete discussion of tensor analysis than can be provided here, see for example references [4, 5].

A *tensor* is a mathematical *operator*, specifically a *linear transformation*. Physical quantities that require a number of descriptors for their specification may be represented by tensors of the required *order*. For example, a quantity like temperature requires only one descriptor (magnitude) and is representable as a *zero-order* tensor (or *scalar*). Force requires more descriptors (magnitude and direction) and is representable by a *first-order* tensor (or *vector*). Stress and strain (9 descriptors) can be represented by *second-order* tensors. In "operator language," we might say that the second-order "rotation" tensor $\boldsymbol{Q}$ <u>operates</u> on the position vector $\boldsymbol{r}$ to form a new (first order) tensor $\boldsymbol{r}^*$, i.e., $\boldsymbol{r}^* = \boldsymbol{Q}\boldsymbol{r}$.

Two types of notations are used to denote tensors: (i) direct notation – similar to matrix or vector notation, e.g., $\boldsymbol{x}$ or $\{x\}$; and (ii) indicial notation – subscripts or superscripts attached to main letters called "kernel letters," e.g., x_i or x^i, $i = 1, 2, 3$. The number and range of the indices depends on the number of descriptors of the quantity. There is a physical meaning associated with the subscript or superscript notation. Subscripts and superscripts denote "covariant" and "contravariant" components, respectively; the physical meaning is given below.

Two other concepts are important for the understanding of tensor analysis. The first is the *summation convention*. We replace the summation operator Σ with a simple rule: in any given term, whenever an index is repeated <u>once</u> it is a *dummy* variable indicating summation over the range of the index. For example:

$$\sum_{i=1}^{3} a_i x^i = a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3 \tag{1}$$

Formally, summation always occurs "along the diagonal" among sub and super indices.

The second concept is that of *expansion in terms of a basis*. Recall that a vector $\boldsymbol{r}$ in the rectangular Cartesian *x-y-z* coordinate system can be represented as a *linear combination of basis vectors*, say the unit vectors $\boldsymbol{e}_i$ associated with each coordinate direction. Then we may write

$$\boldsymbol{r} = r_1\boldsymbol{e}_1 + r_2\boldsymbol{e}_2 + r_3\boldsymbol{e}_3 = r_i\boldsymbol{e}_i \tag{2}$$

We can expand higher-order tensors in the same fashion, once a suitable basis is identified. The set of so-called *unit dyads* $\{\boldsymbol{e}_i\boldsymbol{e}_j\}$, derived from the outer product of $\boldsymbol{e}_i$ with itself (see

below), forms a basis for expansion of second-order tensors, e.g., $\boldsymbol{S} = S^{ij}\boldsymbol{e}_i\boldsymbol{e}_j$. Thus quantities like r_i or S^{ij} are seen to be *components* of the tensor relative to a specific basis (or coordinate system).

A sample of important tensor operations is given below:

- Kronecker delta: $\delta_{ij} = 1,\ i \neq j;\ 0,\ i = j$
- Inner product of two vectors: $\boldsymbol{u}\cdot\boldsymbol{v} = u_i\boldsymbol{e}_i\cdot v_j\boldsymbol{e}_j = u_k v_k$
- Outer product of two vectors: $\boldsymbol{u}\otimes\boldsymbol{v} = \boldsymbol{uv} = u_i v_j \boldsymbol{e}_i\boldsymbol{e}_j$
- $\boldsymbol{uv}(\boldsymbol{w}) = \boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{w})$
- Outer product of two 2nd-order tensors: $\boldsymbol{ST} = S_{ij}T_{kl}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l$
- Single inner product of two 2nd-order tensors: $\boldsymbol{S}\cdot\boldsymbol{T} = S_{ij}T_{jk}\boldsymbol{e}_i\boldsymbol{e}_k$
- Double inner products of two 2nd-order tensors: $\boldsymbol{S}:\boldsymbol{T} = S_{ij}T_{ij}$, $\boldsymbol{S}\cdot\cdot\boldsymbol{T} = S_{ij}T_{ji}$

The following conventions are used: Latin indices take the values 1,2,3; Greek indices take the values 1,2; capital and lower case Latin letters refer to the undeformed and deformed state, respectively; and bold type indicates vector or tensor quantities.

3 Coordinate systems and configurations

Geometric nonlinearity requires formulating the equilibrium equations in the *deformed* configuration of the body, which may be substantially different from the *undeformed* configuration (and which is not known in advance). We also, however, consider an elastic body to have a *natural*, *unstressed, reference state*, and we expect the elasticity to be derivable from a thermodynamic potential written in this reference state. Hence, we have a need for coordinate systems in the reference configuration and in the current configuration. Also, for analysis of membranes (as in shells), it will prove convenient to have coordinate systems aligned with the membrane *mid-surface*.

Consider the rectangular Cartesian (RC) coordinates *X-Y-Z* of a point *X* on the reference membrane midsurface that becomes point x on the current midsurface with RC coordinates *x-y-z* (Fig. 4). (Note that *rectangular* refers to *orthogonal* coordinates, while *Cartesian* refers to *straight* coordinate lines.) We also define *curvilinear midsurface* coordinates X^I and x^i in the reference and current configurations, respectively. These coordinate lines will be in general neither straight nor orthogonal, but they remain tangential and normal to the midsurface at every point. It is obvious, then, that a single coordinate system will not suffice for midsurface coordinates. Also, the reference and current configurations are typically taken as the undeformed and deformed states, respectively.

Vectors (*basis vectors*) are chosen that characterize the coordinates (see Fig. 4). In the RC coordinates, the vectors $\boldsymbol{E}_I$ and $\boldsymbol{e}_i$ are orthogonal and of unit magnitude (*orthonormal*). In the curvilinear coordinate system, *covariant* basis vectors $\boldsymbol{G}_I$ and $\boldsymbol{g}_i$ lie tangent to the respective coordinate curves X^I and x^i. In general, they are neither orthogonal nor of unit magnitude. (*Reciprocal bases*, consisting of the *contravariant* vectors $\boldsymbol{G}^I$ and $\boldsymbol{g}^i$ can also be defined, such that $\boldsymbol{G}_I\cdot\boldsymbol{G}^J = \delta_I^J$ and $\boldsymbol{g}_i\cdot\boldsymbol{g}^j = \delta_i^j$, but these bases will not be discussed further here.)

A *metric tensor* $\boldsymbol{g}$ in the deformed state is defined by *metric coefficients* $g_{kl} = \boldsymbol{g}_k\bullet\boldsymbol{g}_l$, where $\boldsymbol{g}_i = \partial\boldsymbol{r}/\partial x^i$ is the basis vector in x^i and $\boldsymbol{r}$ is the position vector from o to x; similarly, in the undeformed state, $G_{KL} = \boldsymbol{G}_K\bullet\boldsymbol{G}_L$, where $\boldsymbol{G}_I = \partial\boldsymbol{R}/\partial X^I$ is the basis vector in X^I and $\boldsymbol{R}$ is the position vector from 0 to X. The significance of the *metric tensor* is that it contains all the information about how length

(specifically squares of length) is measured in different coordinate systems (hence the name *metric*). As a simple example, consider that the (square of the) length of the infinitesimal element dl in rectangular Cartesian coordinates (*x, y, z*) is

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{3.1}$$

But in cylindrical polar coordinates (r, θ, z), the same value is given by

$$(dl)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2 \tag{3.2}$$

since the angular coordinate θ is not of and by itself a measure of length. So to measure length in the θ-direction, in this case one uses the metric coefficient $(r)^2$.

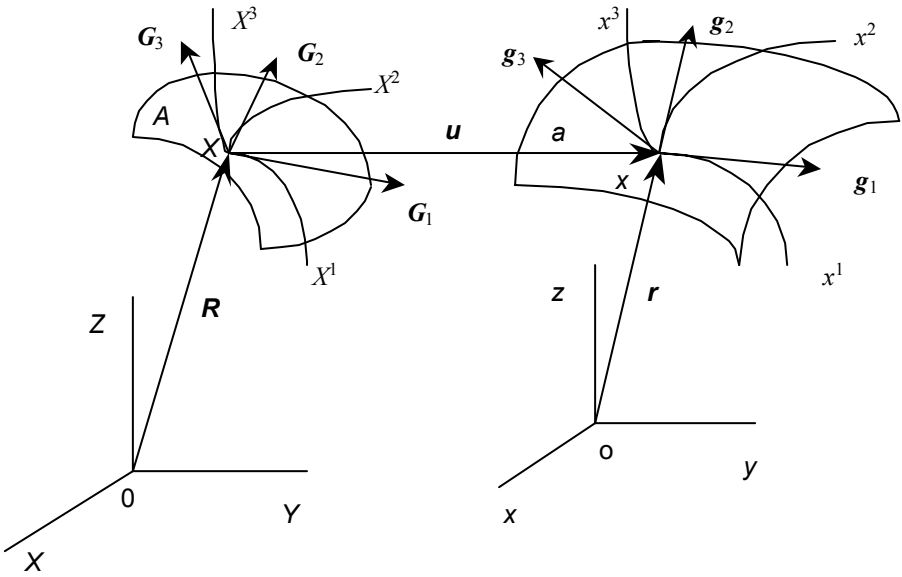

Figure 4: General configuration sketch for nonlinear continuum mechanics.

In the general curvilinear coordinates above, we would have similiarly:

$$(dr)^2 = d\boldsymbol{r}\cdot d\boldsymbol{r},\ (dR)^2 = d\boldsymbol{R}\cdot d\boldsymbol{R} \tag{4}$$

In order to evaluate these expressions, we realize that

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial x^i} dx^i\,, \quad d\boldsymbol{R} = \frac{\partial \boldsymbol{R}}{\partial X^I} dX^I \tag{5}$$

Substituting in from above we have

$$d\boldsymbol{r} = \boldsymbol{g}_i dx^i \ , \quad d\boldsymbol{R} = \boldsymbol{G}_I dX^I \tag{6}$$

Then, using the fact that $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ and $G_{IJ} = \boldsymbol{G}_I \cdot \boldsymbol{G}_J$, we combine (4) and (6) to get

$$(dr)^2 = g_{ij} dx^i \, dx^j \ , \ (dR)^2 = G_{IJ} dX^I dX^J \tag{7}$$

The use of general curvilinear coordinates (i.e., non-rectangular, non-Cartesian) provides for an elegant and fundamental formalism when developing the general theory of nonlinear membrane response. Noting, however, that the general curvilinear basis is always referred to a rectangular Cartesian basis, some simplifications can be made in the formulation, particularly for computations and in reporting of engineering quantities. To this end, we can establish a *local* rectangular Cartesian coordinate system at each point on the deformed midsurface. We will see how this is accomplished later.

4 Kinematics of Deformation

4.1 Motion and deformation

We take a material particle originally at location $\boldsymbol{X}$ in the reference configuration, and track its *motion* (viz., time-dependent response to loading) to its new location at position $\boldsymbol{x}$ in the current configuration, or

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) \tag{8}$$

We assume that matter cannot be created nor destroyed (the *axiom of continuity*), hence the motion is invertible or

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t) \tag{9}$$

Figure 4: General configuration sketch for nonlinear continuum mechanics.This invertability can equally be expressed through the so-called *Jacobian J* of the deformation as

$$J \equiv \left| \frac{\partial x^k}{\partial X^K} \right| \neq 0 \tag{10}$$

where $|\ |$ denotes the determinant operation.

We can define the *deformation* for each instant of time t of the motion, or as a *quasi-static process*:

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}) \tag{11}$$

4.2 Deformation gradient, stretch, and polar decomposition

Consider two neighboring material particles, located at $\boldsymbol{X}$ and $\boldsymbol{X} + d\boldsymbol{X}$, which deform to $\boldsymbol{x}$ and $\boldsymbol{x} + d\boldsymbol{x}$, respectively. Then we can determine $d\boldsymbol{x}$ from

$$d\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} d\boldsymbol{X} = \boldsymbol{F} \cdot d\boldsymbol{X} \tag{12}$$

where $\boldsymbol{F}$ is called the *deformation gradient tensor*. Using (12), the initial *gage length* dL ($dL = (d\boldsymbol{X}^T \cdot d\boldsymbol{X})^{1/2}$ is a natural and convenient choice for measuring deformation) is stretched to a length dl by

$$d\boldsymbol{x}^T \cdot d\boldsymbol{x} = d\boldsymbol{X}^T \cdot d\boldsymbol{F}^T \cdot d\boldsymbol{F} \cdot d\boldsymbol{X} \tag{13}$$

The *stretch ratio* of the gage lengths is

$$\Lambda = \frac{dl}{dL} = \sqrt{\frac{d\boldsymbol{x}^T \cdot d\boldsymbol{x}}{d\boldsymbol{X}^T \cdot d\boldsymbol{X}}} \tag{14}$$

An important postulate in continuum mechanics is that a deformation can be decomposed into a rigid rotation $\boldsymbol{Q}$ followed by a stretch along principal directions (the maximum or minimum stretches possible), or stretch first then rotation, such that

$$\boldsymbol{F} = \boldsymbol{U} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{V} \tag{15}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are the left and right stretch tensors, respectively. Consequently, $\boldsymbol{V}$ and $\boldsymbol{U}$ describe the stretch relative to the reference and current configurations, respectively.

The rotation tensor $\boldsymbol{Q}$ can be used to define a local RC basis for every point on the membrane surface, as discussed below. Following our convention that the X^I and x^i are RC, the current basis vectors $\boldsymbol{g}_i$ are related to the reference basis $\boldsymbol{G}_I$ by

$$\boldsymbol{g}_i = \boldsymbol{Q} \cdot \boldsymbol{G}_I \tag{16}$$

4.3 Strain definitions

Although stretch provides an adequate measure of deformation, it is convenient to formulate measures that have <u>zero</u> value when <u>undeformed</u> (stretch equals unity in the undeformed state). These measures are called *strain* and are non-unique. It is intuitive, then, to refer strain from the undeformed or reference configuration, i.e., a Lagrangian strain definition.

A very fundamental approach to developing a strain definition is to simply measure the difference in gage lengths before and after deformation (stretching and/or rotation). This accounts for the fact that rigid body rotation should not contribute to strain, i.e., $(dr)^2 = (dR)^2$. Then using (6) and taking the difference:

$$(dr)^2 - (dR)^2 = g_{ij} dx^i\, dx^j - G_{IJ} dX^I dX^J \tag{17a}$$

$$\equiv 2E_{IJ}(\boldsymbol{X},t)\, dX^I dX^J \tag{17b}$$

Now recalling (12), $dx^i = F^i{}_J\, dX^J$, where $F^i{}_J = \partial x^i / \partial X^J$, and equating (17a) and (17b) above, it must be that

$$E_{IJ}(\boldsymbol{X},t) = \tfrac{1}{2}\,(g_{ij}F^i{}_I\, F^j{}_J - G_{IJ}) \tag{18}$$

which are the components of the *Green–Lagrange strain tensor*, and $g_{ij}F^i{}_I\, F^j{}_J$ are components of the so-called *Green's deformation tensor*. It is readily seen that this strain measure is referred to the reference configuration. If, as we have assumed, the midsurface coordinates are RC coordinates, $\boldsymbol{g}$ and $\boldsymbol{G}$ become the identity tensor $\boldsymbol{1}$, and the Green–Lagrange strain tensor may be written as:

$$\boldsymbol{E} = \tfrac{1}{2}\,(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{1}) \tag{19}$$

(A similar development leads to an *Eulerian* strain tensor, but this will not be discussed further here. Moreover, many materials, including some that may be useful for compliant membranes, have a response that is dependent on the *rate* of straining; this topic will also not be discussed further here.)

5 Stress and balance laws

5.1 Concept of stress

The idea of stress in a body is a way of characterizing the internal force reaction to loads. Physically, the loads create deformations that lead to changes in interatomic distances relative to the unloaded equilibrium state. Resistance to the deformation (e.g., due to van der Waals forces), averaged over a large group of atoms (the continuum *point*), is what we call *stress*. The goal then is to relate the loads to the (internal) stress.

A remarkable hypothesis attributed to Cauchy allows us to do this. It is an extension to deformable media of Newton's law of action/reaction. The first step is to define a *stress vector* or *traction* $\boldsymbol{t}$ that is the limit of an increment of force Δp per unit increment of current area Δa:

$$\boldsymbol{t} = \lim_{\Delta a \to 0} \frac{\Delta p}{\Delta a} \tag{20}$$

Without additional development, we then simply state *Cauchy's Stress Hypothesis*:

> The surface traction $\boldsymbol{t}$ (the body load $\boldsymbol{b}$ vanishes in the limit) acting on a body, or portion of a body, is related to the *stress* in the neighborhood of the traction by:
>
> $$\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{\nu} \tag{21}$$
>
> where $\boldsymbol{\nu}$ is the outward unit normal to the surface, and $\boldsymbol{\sigma}$ is the Cauchy (or true) stress tensor.

Newton's law of action/reaction follows since $\boldsymbol{t}(-\boldsymbol{\nu}) = -\boldsymbol{t}(\boldsymbol{\nu})$. The components of $\boldsymbol{t}$ and likewise $\boldsymbol{\sigma}$ depend on the coordinate system chosen as a basis.

5.2 Stress definitions

The Cauchy stress is the most accurate measure of stress at a point. However, for constitutive relation development, other forms of the stress tensor are desirable (see the discussion below on *stress conjugacy*.) Several options are available:

- *Kirchhoff* stress: $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ (simply a "weighted" Cauchy stress)
- *1st Piola–Kirchhoff* stress: $\boldsymbol{P} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}$ (which is not a symmetric tensor)
- *2nd Piola–Kirchhoff* stress: $\boldsymbol{S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$ (for small strain, the 2nd P – K stress can be shown to be merely the Cauchy stress rotated as if acting on the originally oriented surface)

5.3 Energy, mass, and momentum balance

The principle of energy balance (energy conservation) states that the time rate of change of the kinetic plus internal energy is equal to the sum of the rate of work of external forces plus all other power (energy/time) sources or sinks (e.g., from heat energy, electrical energy, chemical energy, etc.). Mass balance (conservation of mass) provides the mathematical description for the physical observation that matter can neither be created nor destroyed. Moreover, mass must be invariant under motion.

For linear elastostatics, the equilibrium equations are special cases of the balance of momentum equations. In this case, the inertia term is neglected, either due to vanishing mass and/or acceleration of the body. The equilibrium equations can be grouped into force (translational) and moment (rotational) equilibrium.

For *global* translational equilibrium, we sum all of the surface tractions $\boldsymbol{t}$ and body forces $\boldsymbol{b}$ over any arbitrary portion of the body of volume v and enclosing surface area a in the current configuration:

$$\oint_a \boldsymbol{t} da + \int_v \boldsymbol{b} dv = \boldsymbol{0} \tag{22}$$

The internal stress within the volume can be related to the surface tractions through the Cauchy stress hypothesis (21):

$$\oint_a \boldsymbol{\sigma} \cdot \boldsymbol{n} da + \int_v \boldsymbol{b} dv = \boldsymbol{0} \tag{23}$$

The surface integral on the left can be converted to a volume integral through use of Gauss's theorem [6], giving, after some manipulation:

$$\int_v \left(\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} \right) dv = \boldsymbol{0} \tag{24}$$

Since the original volume v was arbitrary, it must follow that

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \tag{25}$$

This single *local* vector equation contains the familiar three scalar equations of translational equilibrium.

Rotational or moment equilibrium follows in a similar fashion, namely

$$\oint_a (\boldsymbol{t} \times \boldsymbol{x})da + \int_v (\boldsymbol{b} \times \boldsymbol{x})dv = \boldsymbol{0} \tag{26}$$

Again, use is made of Gauss's theorem to convert the closed surface integral to a volume integral. The conclusion drawn from the result is that the stress tensor $\boldsymbol{\sigma}$ must be symmetric. Conversely, the symmetry of the stress tensor automatically satisfies rotational equilibrium. (This result assumes, however, that there are no local "stress couples").

For nonlinear elastostatics, the moment equilibrium may be satisfied in an interesting way. Since the body can deform largely, global moment equilibrium is automatically satisfied, as shown in Figure 5.

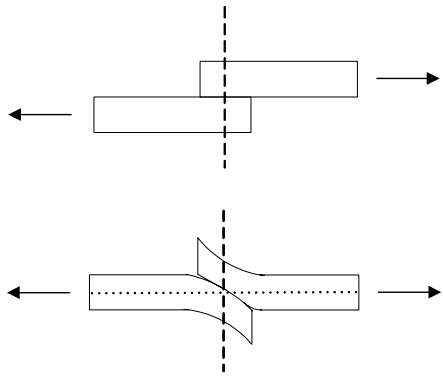

Figure 5. A single lap joint rotates to satisfy moment equilibrium

5.4 Weak form of translational equilibrium

For computational purposes, it is convenient to replace (25) with an equivalent <u>single scalar</u> *weak* or integral form. The *pointwise* equation (25) is multiplied by a test function and integrated over the *entire* body. If the test function is the virtual velocity field $\delta\boldsymbol{v}$ (which is an arbitrary function that must satisfy kinematic constraints), the weak form is called the *principle of virtual power*:

$$\int_v (\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b}) \cdot \delta\boldsymbol{v}\ dv = 0 \tag{27}$$

After some manipulation [6], we get

$$\int_v \boldsymbol{\sigma} : \delta\boldsymbol{d}dv = \int_a \boldsymbol{t} \cdot \delta\boldsymbol{v}da + \int_v \boldsymbol{b} \cdot \delta\boldsymbol{v}dv \tag{28}$$

where $\boldsymbol{d}$ is the deformation rate tensor. Stated plainly, the rate of work done by the external forces for any virtual velocity field is equal to the rate of work done by the internal stresses for the same velocity field.

5.5 Conjugate stress and strain

As stated earlier, we expect the elastic response of a material to be derivable from a thermodynamic potential function that is referred to the original, undeformed state (i.e., has zero potential when undeformed, which implies that the stress and strain are zero as well). Hence, we form the rate of work per unit volume in the reference state as

$$d\dot{\mathrm{W}} = [\text{stress}]: d[\text{strain}]$$

where [stress] and [strain] are *work conjugate* stress and strain tensors, respectively.

We now generalize the concept of conjugacy by considering the virtual power (work rate) equation previously derived:

$$\int_v \boldsymbol{\sigma} : \delta \boldsymbol{d} dv = \int_a \boldsymbol{t} \cdot \delta \boldsymbol{v} da + \int_v \boldsymbol{b} \cdot \delta \boldsymbol{v} dv \tag{29}$$

where the integration was taken over the current volume v. Knowing that $dv = J\, dV$, we can take the integration over the reference volume (recall that work conjugacy is referred to the reference state). Then the internal virtual power (left hand side) can be taken as:

$$\int_v \boldsymbol{\sigma} : \delta \boldsymbol{d} dv = \int_V J\boldsymbol{\sigma} : \delta \boldsymbol{d} dV \tag{30}$$

We conclude that $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is work conjugate to the strain whose (deformation) rate is $\boldsymbol{d}$; $\boldsymbol{\tau}$ is the *Kirchhoff* stress tensor.

Other conjugate stress–strain pairs are possible, including that the 2nd Piola–Kirchhoff stress $\boldsymbol{S}$ is work conjugate to the Green–Lagrange strain $\boldsymbol{E}$, and that the Cauchy stress $\boldsymbol{\sigma}$ is work conjugate to the *Hencky* or logarithmic strain

$$\ln(\boldsymbol{U}) = \tfrac{1}{2} \ln (\boldsymbol{F} \cdot \boldsymbol{F}^T) \tag{31}$$

where $\boldsymbol{F} \cdot \boldsymbol{F}^T$ is the *Finger* tensor. (Remark: although it is conceivable to write the constitutive relations in the current configuration, there are fundamental reasons [7] and practical reasons [6, 8] that call this approach into question.)

6 Constitutive equations

6.1 Introduction

Many structural materials exhibit at sufficiently small strains, i.e. under sufficiently small stresses, a linear-response regime. In that regime the accumulation of damage due to cyclic loading is typically relatively small. At higher stress levels the strain response typically deviates from linearity. Generally, the rate of accumulation of damage due to cyclic loading or due to sustained loading increases with the stress level. The onset of non-linear response often signals some irreversible changes and the acceleration of damage accumulation. Therefore, in technological applications, the operational design-strength threshold, at least in a nominal sense, is often limited to well below the onset of significant deviation from linear response.

Some materials deviate from this response behavior. Materials that exhibit linear strain response up to failure are brittle. Their technological application is quite limited; it requires special precautions. Successful technological applications of brittle materials typically avoid features that locally raise stress levels as predicted by the theory of elasticity solutions, such as holes and voids within the load path, or re-entrant corners. These features cause local stress peaks that exceed the stress level of the remote uniform cross section in the load path by a large factor (in strength of material terminology, this is the stress concentration factor). To reduce that effect it is generally necessary to round out re-entrant corners using as large a radius as possible. Unless the stress peaks in a brittle component are accurately predicted, it is generally necessary to severely limit the exploitation of the tested material strength. Applications may require the limitation to be one quarter to less than one tenth of the tested strength.

Technological applications of the materials discussed so far justify the use of linearly elastic constitutive laws in the analysis of structural systems. In general, theories developed for linearly elastic materials are limited to small strains.

There are materials that exhibit elastic, hence reversible response well into their non-linear range. Some of those materials remain elastic for very large strains. These are elastomers. In continuum mechanics, they are termed hyper-elastic. Theories developed for hyper-elastic materials typically extend into large strains.

The analytical methods and tools available since the last quarter of the 20th century, such as the non-linear finite element method, allow in principal the analysis of structural systems that use a far broader collection of classes of material. In principal, in an incremental analysis such as performed via the non-linear finite element method, any response mechanism can be programmed and hence executed during the numerical process. For the outcome of the solution process to make sense, however, it is necessary that the implemented constitutive laws mimic the material responses and satisfy the conservation laws of mechanics and the first and second fundamental laws of thermodynamics. In developing new constitutive relations that do not fit within established formulations from empirical observations on test specimens, it is necessary to assure that these laws are not violated. For a loading process that is adiabatic and isothermal, this requirement can be simply stated: Under monotonic up-loading from the unstrained state the external load system must perform non-negative (non-negative rather than positive is used here to include the friction of an incompressible material) work on the solid element upon uploading, that is:

$$d\sigma_{ij}\, d\varepsilon_{ij} \geq 0. \tag{32}$$

This requirement demands that the incremental (or tangent) moduli of the material for direct stress, shear stress, and hydrostatic stress must be positive. An application of this requirement yields the range of possible Poisson's ratios for isotropic solids.

6.2 Thermomechanics

Events in solid mechanics are governed by three types of equations that must be solved simultaneously. These are 1) the equilibrium equations, which may be either static or dynamic, 2) the kinematic relations, and 3) the constitutive laws. The equilibrium equations and the kinematic relations are general. The equilibrium equations are directly derived from physics. The kinematic relations connect displacements to the deformation quantities, i.e., they are just geometric relations. The constitutive laws by contrast are material specific; and, they can be quite complex. Indeed it may be difficult if not impossible to specify such a law for a specific material that suits all its applications. Often the analyst must be satisfied with a constitutive law that in some narrow way captures the material response that is pertinent for his particular

application. In an earlier time, when analytical solutions were principally of the closed-form type, i.e. the strong solution of differential equations, constitutive laws were sought that made the solution of solid mechanics problems by these methods tractable. This limitation on the choice of constitutive equations was quite severe. Modern analysis methods such as the various discretization methods and semi-discretization methods accept a much broader class of constitutive laws. Using these methods, which are usually available to the analyst as a commercially supplied numerical tool, even material failure modes can be included in the analysis. This section describes a number of mechanical material response phenomena connecting stress, strain and temperature and it provides an overview of their mathematical description in sufficient detail to be useful to the analyst. A given material may exhibit one or more of these phenomena. Where appropriate and expedient, the description is limited to the two-dimensional sheet.

Not until the second half of the 20th century did the field theory of thermomechanics come into being. By contrast the field theory of mechanics dates back to 1775 for fluids, due to Euler, and 1822 for solids, due to Cauchy. Thermodynamics, which treats the state of matter and transformation of energy within matter is due to Carnot (1824) and Clausius (1850). Until recently, it has been limited to the treatment of homogeneous states. Still, its importance in the development of constitutive relations in mechanics has long been recognized. The limitations placed on constitutive laws by the first and second fundamental laws of thermodynamics have been properly regarded. An example of this will be discussed in the section that deals with linear elasticity.

The first fundamental law of thermodynamics, when written for an element of matter, states that there exists a state function $u(a_k, \theta)$, the internal energy per unit mass, such that:

$$du = dw + dq \tag{33}$$

The second fundamental law of thermodynamics identifies a second state function $s(a_k, \theta)$, the entropy per unit mass, that satisfies

$$\theta\, ds \geq 0 \tag{34}$$

Here θ is the absolute temperature, dw is the external work supply per unit mass, dq is the heat supply per unit mass, and the a_k are the collection of external and internal parameters that combine with external and internal force quantities in products that have the dimensions of work per unit volume. The entropy increment has two parts, a reversible part

$$d^{(rev)}s = dq/\theta \tag{35}$$

which is due to heat flow, and an irreversible part, the entropy production inside the element

$$d^{(ir)}s \geq 0 \tag{36}$$

Unlike ds, $d^{(rev)}s$ and $d^{(ir)}s$ are not total differentials. In the dynamic case, i.e., when matter accelerates, the first fundamental law as stated above must be amended by the addition of kinetic energy supply.

6.3 The linearly elastic isotropic solid

The constitutive relation that governs the response of a linear-elastic solid in an isothermal and isentropic process is Hooke's law. Written for the general stress state of an anisotropic solid it is given by

$$\sigma_{ij} = D_{ijkl}\, \varepsilon_{kl} \,. \tag{37}$$

Both the stress tensor and the strain tensor are symmetric in their subscripts, and in addition the tensor $D_{(ij)(kl)}$ is symmetric with respect to the 1st and 2nd group of index pairs

The symmetries of the 4th-rank material tensor reduce the number of independent constants to 21. The additional symmetries of an isotropic solid reduce the independent elastic constants to two. These are the Lamé constants λ and μ. Hooke's law for this solid is given by

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\, \mu \varepsilon_{ij} \tag{38}$$

The response to uniaxial stress is

$$\varepsilon_{11} = (\lambda+\mu)/[\mu\,(3\lambda+2\mu)]\; \sigma_{11} = \sigma_{11}/E \,, \tag{39}$$

where E is the elastic modulus (or Young's modulus).

The shear response is

$$\varepsilon_{ij} = \sigma_{ij}/(2\,\mu) = \sigma_{ij}/(2\,G) \,, \tag{40}$$

where $G = \mu$ is the shear modulus.

The response to hydrostatic stress is

$$\varepsilon_{kk} = \sigma_{kk}/(3\lambda+2\mu) = \sigma_{kk}/(3\,K) \,, \tag{41}$$

where K is the bulk modulus.

The two independent elastic constants used in the engineering literature are Young's modulus and Poisson's ratio

$$\nu = (\lambda/2)/(\lambda+\mu) \tag{42}$$

The shear modulus in terms of these constants is

$$G = E/[2(1+\nu)] \tag{43}$$

Young's modulus characterizes the strain response in the direction of a uniaxial applied stress. Poisson's ratio is the ratio of contraction in a direction that is perpendicular to an elongation caused by a tensile stress that is in line with the elongation.

Thermodynamic considerations require that E, G, and K are non-negative. These requirements limit Poisson's ratio for an isotropic material to the range between –1.0 and + 0.5. There is no real isotropic material known that has a negative Poisson's ratio. Therefore, one may postulate for the

range of Poisson's ratio $0.0 \leq \nu \leq 0.5$. Typically, cork and other porous materials have Poisson's ratios near the lower end and elastomers such as rubbers have Poisson's ratios near the upper end of that range.

For later convenience, the stress tensor and the strain tensor may be decomposed into their isotropic part σ_{kk} and deviatoric part

$$\sigma'_{ij} = \sigma_{ij} - \delta_{ij}\, \sigma_{kk}/3 \tag{44}$$

for stress, and isotropic part ε_{kk} and deviatoric part

$$\varepsilon'_{ij} = \varepsilon_{ij} - \delta_{ij}\, \varepsilon_{kk}/3 \tag{45}$$

for strain. The shear response can then be written as

$$\varepsilon'_{ij} = \sigma'_{ij}/(2\,G)\,. \tag{46}$$

6.4 The linearly elastic isotropic membrane

The stress state of interest in compliant membrane mechanics is plane stress. For this state the constitutive equation for the isotropic membrane can be reduced to

$$\begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_6 \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 2(1+\nu)/E \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \tag{47}$$

The inverse relation is an obvious simplification of the corresponding equation for the orthotropic membrane discussed below. The notation used here is common in the engineering community that deals with thin sheets and laminates:

$$\gamma_1 = \varepsilon_{11},\ \gamma_2 = \varepsilon_{22}, \text{ and } \gamma_6 = \varepsilon_{12};\ \sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22}, \text{ and } \sigma_6 = \sigma_{12} \tag{48}$$

6.5 The linearly elastic orthotropic membrane

The relations connecting stresses and strains in orthotropic films, when written in the material principal directions, are

$$\begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_6 \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \tag{49}$$

and

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} E_1/(1-\nu_{12}\nu_{21}) & -\nu_{12}E_1/(1-\nu_{12}\nu_{21}) & 0 \\ -\nu_{21}E_2/(1-\nu_{12}\nu_{21}) & E_2/(1-\nu_{12}\nu_{21}) & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_6 \end{Bmatrix} \tag{50}$$

In any other reference coordinate system, the forms of the constitutive relations for the orthotropic membrane are indistinguishable from those for a general anisotropic membrane, i.e. the matrix $[D_{ij}]$ and its inverse $[C_{ij}]$ are fully populated.

In determining the thermodynamic restrictions for the Poisson's ratios ν_{12} and ν_{21}, it is necessary to take the through-thickness direction into account. These Poisson's ratios may indeed be larger than 0.5. There is no general relation for G in terms of the E_i and the ν_{ij}.

An orthotropic membrane may be isotropic in the plane. In that case (47) holds but without the restriction on Poisson's ratio in the plane to not more than 0.5. Also, balanced orthotropy in the plane, i.e. $E_1 = E_2$, does not imply isotropy in the plane. While $E_1 = E_2$ implies $\nu_{12} = \nu_{21}$, it does not imply $G = E/[2(1 + \nu)]$.

6.6 Generalized elasticity: the Cauchy, hyperelastic, and hypoelastic solids

Linear elastic materials obey (37), where the coefficients C_{ijkl} are independent of stress and strain; they may, however, be dependent on temperature. These coefficients are restricted on grounds of the symmetries of the stress and strain tensors and the fundamental laws of thermodynamics. In particular, specific linear combinations of them, the elastic moduli, the shear moduli, and the bulk modulus, must be positive. The concept of elasticity can be expanded to more generality. Truesdell [9–11] categorized three classes of elasticity, which are:

1) A material is said to be elastic (Cauchy material) if it possesses a homogeneous stress-free state, the natural state, and if in some neighborhood of this state there exists a one-to-one correspondence between a work-conjugate pair of stress and strain tensors

$$\sigma_{ij} = F(\varepsilon_{kl}) \tag{51}$$

2) A material is said to be hyperelastic if it possesses a homogeneous stress-free state, and if there exists a strain-energy density function $\rho_0 W$, which is an analytic function of the strain tensor, the work done by the stresses equalling the gain in strain energy

$$\partial(\rho_0 W)/\partial\varepsilon_{ij} = \sigma_{ij} \tag{52}$$

3) A material is said to be hypoelastic, if the stress rate is a homogeneous linear function of the rate of deformation

$$d\sigma_{ij}/dt - \Omega_{kj}\,\sigma_{ik} - \Omega_{ki}\,\sigma_{kj} = C_{ijkl}\,d_{kl}\,. \tag{53}$$

Here

$$\Omega_{ij} = ½\,(\,\partial v_j/\partial x_i - \partial v_i/\partial x_j\,) \tag{54}$$

is the spin tensor.

In this definition the coefficients C_{ijkl} are functions of stress or strain. The definition of the hypoelastic material can be restated by substituting the time derivatives of any work-conjugate pair of stresses and strains so that

$$d\sigma_{ij}/dt = C_{ijkl}\,d\,\varepsilon_{kl}/dt. \tag{55}$$

Furthermore, without violating the concept, one can replace the time derivatives with derivatives of a generalized evolution parameter. With this identification it is clear that the only difference in the definitions of the Cauchy material and the hypoelastic material is the requirement of the homogeneous stress-free state in the Cauchy material. Clearly, the Cauchy material includes the hyperelastic material. Thus, the hypoelastic material is the most general definition of an elastic material.

Note: Some writers have questioned the admissibility of hypoelastic materials on thermodynamic grounds. While we caution anyone who develops a mechanical constitutive model for a material to make sure that under a complete loading cycle net energy cannot be extracted from a material, it is quite conceivable that the micro-structure of a continuum may contain elastic instabilities that exhibit snap-through like response at the sub-continuum mechanics scale. Such a material can be modeled as hypoelastic.

Interest in hyperelastic models has focused primarily on large deformation for which the dilatory component is deemed insignificant, so that only distortion is of concern; in that case the body is considered incompressible. It is clear from the context that the strain under discussion here is the logarithmic strain and the stress is the Cauchy stress in a reference frame that is rotated with the principal stretch axes. Also, since there is no volume change, the hydrostatic pressure does no net work.

A particular form of the elastic potential, which is due to Mooney, is

$$W(I_1, I_2) = C_1(I_1 - 3) - C_2(I_2 - 3)\,, \tag{55}$$

where I_1, I_2 are "stretch invariants", and C_1 and C_2 are constants, seems to provide a suitable energy function for certain rubber-like materials. With $C_2 = 0$ this function becomes the strain energy function for the so-called neo-Hookean solid of Rivlin.

The coefficients in the constitutive equation of a hyperelastic material satisfy the fundamental laws of thermodynamics automatically. In developing constitutive equations for a Cauchy solid or for a hypoelastic solid, it is necessary to assure that the coefficients of the tangent stiffness are such that during monotonic uploading the external load system performs positive work on the element for each load increment.

6.7 Visco-elasticity

Visco-elastic solids are dissipative; i.e., they can undergo thermodynamically irreversible processes during deformation. They may or may not be restorable to their initial state by simply unloading and letting sufficient time pass. A convenient way to characterize the response behavior of these materials, at least in a qualitative sense, is to envision them to be performing like viscous elements (dash-pots) and elastic elements (springs) in some combination of series and parallel arrangement. Some examples of such visualizations in linear visco-elasticity are the Maxwell model – a non-restorable system, the Voigt model – a restorable system, and the standard linear model – also a restorable system.

A model that exhibits, at least qualitatively, most observed phenomena in the early response phases of many technological materials used in structural systems combines the Maxwell model and the Voigt model in series. This model, when loaded by a step load, which is maintained constant over time (this is the creep test), exhibits an instantaneous elastic response, a primary creep phase, which is transient, and a secondary creep phase with a constant flow rate.

Real materials that may, at least qualitatively, be characterized by the Maxwell and Voigt model in series, exhibit a tertiary creep phase of accelerated creep towards failure as a result of cross section reduction due to Poisson's effect.

Particularly in polymers, but in other materials as well, visco-elastic response is very sensitive to temperature. For some materials it has been observed that there is a time–temperature correspondence that allows establishing a master curve together with a time-shift rule (these materials are sometimes referred to as thermorheologically simple). This reduces the analysis of visco-elastic response for a variable-load/variable-temperature history to a formulation with the dummy time variable of integration replaced by a scaled time.

If non-linear response can be similarly reduced to the master curve by some other time-shift rule, then the analysis of visco-elastic response to a variable-load/variable-temperature history can again be reduced to a formulation with the dummy time variable of integration substituted by the scaled time.

The creep function $J(t)$ and the stress-relaxation function $G(t)$ for complex visco-elastic response behavior can typically be characterized by a series made up of exponential terms, a so-called *Prony series*. Such a series is not an orthogonal series. Fitting a Prony series to test data can be done in two ways. The more traditional way is to choose exponential terms with the time decay coefficient separated by about a decade, and then using some weighted-integral method to determine the appropriate coefficients. Another way that generally requires a smaller number of terms is to determine both the coefficients associated with each term and the time decay coefficients in each term, using an optimization method. Both processes require the availability of appropriate software.

While with today's computers and software the determination of the coefficients in a Prony series is rapid, the acquisition of suitable creep data or stress relaxation data is time consuming and expensive. Structural analysis tools, such as non-linear finite element codes, may require both the creep formulation and the stress-relaxation formulation during the incremental analysis. To avoid resource-intensive test repetition, one can invert the Prony series for creep to obtain a Prony series for stress relaxation and vice versa [12].

6.8 Fabrics

While fabrics were one of the first structural materials used by humans in applications to membrane structures, to the structural analyst they are the least understood. Contrary to claims otherwise, a fabric does not behave like a two-dimensional continuum. Fabrics have internal mobility. This characteristic can be further classified into two distinct types of mobilities; these are the angular mobility and the transport mobility. Both types of mobilities contribute in different ways to the early success of fabric-made structural membranes, and to the difficulties that they pose to the analyst.

Angular mobility refers to the lack of resistance to shear distortion in the material reference frame. If this mobility is the only one active, then during deformation adjacent material points remain adjacent to each other, and in the topological sense in the same relative position. In particular, the crossing points of warp yarns and fill yarns do not shift. The recently developed solid mechanics subfield, theory of nets [13], is capable of addressing this aspect. As of now, it is still primarily an academic enterprise.

The standard apparatus of the theory of continuum mechanics is an inadequate tool for the analysis of structures that possess this characteristic. This is easily demonstrated by the following experiment. Fabrics have significant stiffness in both the warp and the fill (weft) direction. These can be characterized to some approximation by the elastic moduli E_W and E_F, respectively. Fabrics exhibit Poisson's effect, which can be characterized to some

approximation by the Poisson's ratios ν_{WF} and ν_{FW}. The resistance to shear (angular distortion) in the material reference frame of the fabric (x_W,x_F) is typically several decades smaller than the resistance to extension in the material directions. Hence G, the shear modulus, is very small. It is observed when subjecting a simple weave fabric to uniaxial tension diagonally (in a reference direction rotated $\pi/4$ relative to the warp direction) to the material reference coordinates, then, at least for small deformations, little resistance to the deformation is experienced.

Transport mobility refers to yarns sliding relative to each other. This aspect is even more difficult to formulate in a theoretical model. It is also responsible for the incredible toughness of fabrics, particularly loosely woven ones. This mobility prevents the stress concentrations that are found in linearly elastic solid continua to occur.

Still, fabrics make excellent structural membranes. Successful designs that use fabrics as structural membranes usually are constructed such that, under design critical loadings, the material directions are nearly coincident with the principal stress directions. To small deviations from this condition the fabric will respond by undergoing some significant but most often harmless angular distortion. Fabric structures, designed to this condition, can be analytically assessed by the methods of continuum mechanics as the shear stiffness, or lack thereof, plays a negligible role.

7 Approximations

Similar to other technical fields, membrane theory is characterized by principles that can be stated in a relatively straightforward form, and problems that may involve prohibitive difficulties in using those principles. These difficulties necessitate the careful choice of a line of attack that often involves approximations. However, approximations may affect the predicted responses of membranes more profoundly than of other, more traditional structures. To appreciate and quantify these effects in the context of a given technical application, the engineer must be aware of how the final results may be influenced.

The quality degradation of response predictions due to solution approximations depends on the problem and is generally not straightforward to assess. We here address some selected aspects of the issue only. First, the relevance of the problem of approximations is placed in the context of up-to-date computational tools. The general sensitivity of membrane solutions to approximations is illuminated next. The discussion of some selected approximations follows, and a simple illustration is given. Finally, the notion of solution accuracy is addressed.

7.1 Approximations in the era of computational mechanics

Familiar with some of the gross simplifications needed for the solution of some mechanical problems prior to the computer age, one may be tempted to ignore the issue of approximations in the context of current technology. Indeed, solution approaches such as the finite element method (FEM) are almost universally applicable and are capable of close to arbitrary solution accuracy for certain problems. However, the need for the awareness of approximations is not eliminated by the availability of such tools. One should not ignore the issue of approximations because:

- Numerical solutions also involve approximations.
- Before their predictions can be deemed acceptable, numerical models must be benchmarked *both* within their own context (convergence and parameter sensitivity studies) *and* against alternative solutions. The alternative solutions often need to be

derived symbolically because physical test results for membranes with a quality comparable to numerical/theoretical predictions are virtually non-existent and are often prohibitively difficult to obtain.

- Parametric studies with a need for only limited accuracy may be based on closed-form formulas, as opposed to numerical models.
- The knowledge of underlying approximations may prevent the abuse of numerical analysis in some cases.

By the way of the last point we stress that the unintentional abuse of a numerical tool may be due to attempts to obtain results despite the unusual difficulties of a membrane problem. Such difficulties are typically related to overcoming a common limitation of many analysis programs, namely, the inability of most regular numerical solution procedures to handle singular structural states such as a flat membrane state with no lateral stiffness, before pressurization or prestressing. In order to obtain results at all, one may have to "trick" a program through such a singular configuration. In so doing, one should take care not to modify the problem in a mechanically or mathematically illegitimate manner, such as via adding bending stiffness to the film.

Regardless of whether an analysis tool has to be outsmarted to produce results, typical membrane problems generally necessitate *full* geometric nonlinear capabilities in the program. One of the problem features necessitating nonlinear analysis is the extreme sensitivity of membranes to physical details, and membrane models to modeling details. As geometric nonlinearity is a standard feature of modern computational tools, one may safely require its use in *all* membrane analyses. This requirement becomes imperative for precision applications such as inflatable RF reflectors.

So far in this section we peculiarly omitted concerns of material nonlinearity. This omission reflects the state of the art in membrane engineering in the aerospace industry where, despite the availability of more "exotic" material models, the engineer is still practically limited to linear elasticity in the overwhelming majority of cases. This limitation need not cause concern when the subject membrane is smooth and is operated with sufficiently low stresses. However, stresses potentially higher than the material proportionality limit *or* a creased membrane state (which increases global film compliance in a highly nonlinear fashion) clearly call for nonlinear material models. Unfortunately, properly quantified (measured) parameters of constitutive behavior beyond the proportional limit of films commonly used in space are rarely available and the rigorous test study of wrinkled membranes is still in its infancy [14, 15]

7.2 The nonlinear nature of membrane problems

Film bending and compressive compliance entails a malleable global geometry: membranes re-configure their shape, at the cost of wrinkling if necessary, to enable the bearing of certain loads. "While their thinness makes them incapable of sustaining bending moments, it also renders them incapable of ... preserving their shape under certain kinds of loading" [16].

That, except for some simplistic examples, the re-configuration of geometry is an integral aspect of their response renders membrane behavior similar to post-buckling response in that geometrically linear analysis is simply incapable of capturing their mechanics. Thus geometric simplifications ─ approximations ─ can have a significant impact on a membrane solution. A pronounced example of this influence is illustrated in Figure 5.16 on page 263 of reference [17]. Shown among the insets of this figure are the shapes of a membrane cylinder subject to a perimeter load as predicted with shell theory with bending effects, with small-strain nonlinear

membrane theory, and with large-strain nonlinear membrane theory. The three shapes differ significantly.

7.3 Approximations in the context of the governing equations

A subject of mechanics, the behavior of membranes is governed by the three sets of equations stating equilibrium, stress–strain relations, and geometric compatibility. To prepare the illustration of solution approximations, we first present these equations in a complete and general form uncorrupted by any simplification. (Equivalent forms of these equations other than those presented below also exist: our choice is for convenience only.) We then discuss some classic types of approximations.

7.3.1 An exact form of the governing equations

Describe membrane mechanics locally at an internal, smooth point. Use a rectangular coordinate system with its x and y axes tangent to the membrane mid-surface and aligned in the directions of the surface principal curvatures in the load-carrying state. (These mutually perpendicular directions exist at any point on any smooth surface.) Thus the third, z, axis will be the surface normal. Define all (stress, strain, etc.) quantities in this coordinate system, and interpret all derivatives with respect to these coordinates.

Also, ignore membrane thickness: use through-thickness resultant quantities (such as membrane forces obtained by stress integration through the membrane thickness). Accordingly assume that, where necessary, even relations often stated in a general continuum-based form (such as the constitutive law) are now modified to involve through-thickness resultant quantities.

State equilibrium as

$$\partial n_x/\partial x + \partial n_{xy}/\partial y + q_x = \rho a_x \tag{56}$$

$$\partial n_y/\partial y + \partial n_{yx}/\partial y + q_y = \rho a_y \tag{57}$$

$$n_{xy} = n_{yx} \tag{58}$$

$$p - n_x/R_1 - n_y/R_2\, p = \rho a_z \tag{59}$$

where n_x, n_y, and n_{xy} are the direct and shear membrane stress resultants in units of force per deformed length, q_x and q_{qy} are the in-plane surface loads in the x and y directions, p is the surface load normal to the surface (such as pressure), ρ is the membrane surface density, a_x, a_y, and a_z are the components of the acceleration vector, and R_1 and R_2 are the principal radii of curvature along the properly aligned x and y coordinate axes.

The constitutive law F can be generally stated as

$$\boldsymbol{N} = F(history(\boldsymbol{E})) \tag{60}$$

where $\boldsymbol{N}$ is the tensor of membrane stress resultants and the term "$history(\boldsymbol{E})$" refers to the full history of the strain tensor $\boldsymbol{E}$ to accommodate time- or path-dependent nonlinear material responses. As an example we recall that for linear elasticity F is

$$\boldsymbol{N} = F(history(\boldsymbol{E})) = F(\boldsymbol{E}) = \boldsymbol{C}:\boldsymbol{E} \tag{61}$$

where $\boldsymbol{C}$ is a fourth-order tensor and ":" denotes double contraction. As no time dependence is involved, history is limited to the current instant. (By rearranging the membrane force and strain tensor components into a vector and those of $\boldsymbol{C}$ into a simple matrix, Eq. (61) is often presented as a matrix equation.)

Finally, geometric compatibility implies a proper displacement-based definition of the strains

$$\boldsymbol{E} = L(\boldsymbol{u}) \tag{62}$$

where L is a differential operator and $\boldsymbol{u}$ is the displacement vector field.

The constitutive and geometric relations, Eqs. (60) and (62), have been presented in a rather general form in comparison to the equilibrium equations to permit variations. Strains can be defined in a number of ways, with the constitutive law varied accordingly and also according to the assumed material behavior. The equilibrium conditions, however, can only vary in form, not in essence.

The difficulties in directly using relations, Eqs. (56) through (62), for the solution of a particular problem can be enormous. Consider, for example, the xyz coordinate system used. This reference frame, which permits writing the *exact* equilibrium equations in the relatively simple form Eqs. (56) through (59), is defined according to the instantaneous geometric conditions on the load-bearing membrane. These conditions generally vary over the surface as well as through the loading process, as the membrane shape evolves. Moreover, it is typically this very evolution — or its final state — that is sought by the solution. To manage the field equations in a frame of reference defined point-wise over a spatial surface among ephemeral conditions varying in an initially unknown manner would be difficult, to say the least.

To illustrate another practical difficulty, consider the membrane stress resultants n_x, n_y, and n_{xy}. For the equilibrium conditions Eqs. (56) through (59) these are defined as force per *distorted length* — a membrane-thickness resultant version of what is called the Cauchy stress in continuum mechanics. However, the constitutive laws often relate strain to the material response referred to the *stress-free* state. (For example, according to the common approach to linear elasticity, the Young's modulus E is obtained by dividing the load endured by a test coupon with the *initial* coupon cross section, as opposed to the one laterally contracted under the load due to Poisson's effect.) To avoid all approximations, the governing equations should involve the translation of the latter membrane force definition to the former.

Similar difficulties could be identified for most of the other variables as well. As a result, the above equations are rarely used as is. Instead, they are cast in equivalent forms in coordinate systems defined conveniently for particular problems. To manage the complexities of these alternative forms, approximations must be used. Resulting equations, derived for particular (classes of) problems, are established, analyzed, and solved in a number of reference works for shells and membranes — see, for example, [17] or [18]. (Shell-governing equations stated in terms of stiffness reduce to membrane equations as the bending and lateral shear stiffnesses diminish.)

7.3.2 Some geometric approximations

We here review three classes of geometric approximations. For this illustration, we borrow the nomenclature of [17], and denote by α the angle of slope with respect to a reference plane (at a generic point) of the unloaded membrane surface. Furthermore, the angular change of this slope during the loading process is denoted β.

Moderate rotation theory

Moderate rotation theory assumes that the angles of rotation β experienced by membrane surface points are small: $o(\beta) << 1$ Consequently, $o(\beta^2) <<< 1$ is also implied and the following substitutions can be made:

$$\cos(\alpha+\beta) = \cos(\alpha) - \beta \sin(\alpha)$$
$$\sin(\alpha+\beta) = \sin(\alpha) - \beta \cos(\alpha)$$
$$\cos(\alpha) - \cos(\alpha+\beta) = \beta \sin(\alpha) + 1/2\, \beta^2 \cos(\alpha)$$
$$\sin(\alpha+\beta) - \sin(\alpha) = \beta \cos(\alpha) - 1/2\, \beta^2 \sin(\alpha)$$

where α is the initial slope of the surface with respect to a reference plane, measured in the direction where β is taken.

Shallow shell theory

If, in addition to moderate rotations, the initial slope α of the surface is also low ($o(\alpha)<<1$, $o(\alpha^2)<<<1$), the following approximations can be used:

$$\cos(\alpha+\beta) = 1$$
$$\sin(\alpha+\beta) = \alpha + \beta$$
$$\cos(\alpha) - \cos(\alpha+\beta) = \alpha\beta + 1/2\, \beta^2$$
$$\sin(\alpha+\beta) - \sin(\alpha) = \beta$$

The same approximations can be stated in other forms as well, depending on the frame of reference adopted. For example, an alternative but equivalent condition is $\sqrt{(z_{,x}^2 + z_{,y}^2)} << 1$, where x and y are coordinates within the particular reference plane, and z is the surface position normal to this plane [17, p.442].

Föppl–Kármán equations

The Föppl–Kármán equations, a classic formulation of the axishell (axisymmetric shells under axisymmetric loads) equations, involve small-strain approximations in the context of a particular version of shallow shell theory [17].

7.4 On accuracy and modeling

The need for faithful response prediction pervades by nature the entire gamut and history of engineering. The following discussion of this need is particularly relevant for precision inflatable space structures where application tolerances, manufacturing reliability, and modeling capabilities often appear at conflict.

Accuracy ─ in the sense of how well intent or assessment turns out to coincide with reality ─ is directly relevant to almost all steps of the engineering process. A few of these steps are presented in Table 1 with the errors qualitatively referred to and the statistical issues ignored.

One can state as a general condition for successful engineering endeavor that

$$e_a > e_f \tag{63}$$

In plain English: fabrication must adhere to tighter tolerances than necessary for the operation of the product.

Table 1: Modeling errors.

Error	Associated accuracy	Example
e_a ("application")	The accuracy required of a particular application: how close to ideal the hardware should be in terms of shape, material, etc.	The maximum *rms* surface error acceptable for a reflector.
e_f ("fabrication")	Workshop accuracy: how close to the specifications can the product be fabricated	Fabrication tolerances, quality scatter.
e_m ("modelling")	The accuracy of modeling assumptions: how well the principles underlying a model correspond to reality.	Is the material really linear elastic? Can dynamic effects really be ignored?
e_s ("solution")	Accuracy of solving the model: how well the mathematical and physical principles in the focus of the model are actually reflected by the solution.	Prediction errors due to the math. approximations that rendered the governing equations solvable. Numerical errors.
e_{pr} ("prediction")	Response prediction accuracy: how well physical reality can be predicted. e_{pr} is a compound of e_m and e_s.	Calculated vs. measured response, if all significant aspects of the test are accounted for in the calculation.

For the responses that drive a design (those onto which much of the engineering effort focuses), prediction must be better than the allowable tolerances for the operability of the final product for the governing responses:

$$e_a > e_{pr} = e_m + e_s \tag{64}$$

Furthermore, one can also observe that, in terms of the governing responses, an analysis tool must also reliably predict the impact of likely (fabrication and other) errors on the operational conditions for the governing responses:

$$e_a > e_f > e_{pr} = e_m + e_s \tag{65}$$

The state of the art of space inflatable structural engineering in general, and of precision inflatables in particular, does not yet consistently reflect the relative order of error magnitudes just outlined.

Historically, classical shell and membrane theory has been primarily concerned with e_s, with improving model solution accuracy by alleviating as many of the solution approximations as possible for membranes of various characteristics and geometries. In some way, the development of solution methods for additional classes of problems (such as wrinkled membranes) also falls in this category because it aims at enabling the solution at all of certain models.

The concern with the solution error e_s is secondary today because properly designed and used numerical tools can reduce e_s to limits of computer arithmetic and discretization.

However, these methods do not reduce any of the other errors reviewed above. Actually, it is the uncertainties and the magnitudes of the fabrication error e_f and of the modeling error e_m that primarily hamper precision membrane engineering. While generally no established estimates, prediction methods, or rules of thumb exist for the assessment of these errors, they clearly violate the rules spelled out above in some of cases. For example, the modeling with a linear elastic material model of a reflector canopy subject to low to moderate pressurization after deployment from a creased stowed state is clearly inadequate for a precision application because the creases entail a highly nonlinear material behavior. Such a wrong modeling approach entails

$$e_m > e_a \tag{66}$$

which is in obvious violation of Eq. (64) and thus renders the results useless. Another faulty approach that entails the same contradiction is to benchmark precision membrane shape predictions to approximate symbolic or empirical membrane shape formulas (such of those collected in [19]).

Modeling errors similar to those just highlighted continue to haunt recent membrane engineering efforts. (Some of these common mistakes are examined quantitatively in [20].) The trend to overlook such mistakes in the context of newly pursued precision membrane applications (which include RF and even optical reflectors) is unfortunate. The operational error limit e_a of such devices can be orders of magnitude lower than common structural engineering tolerances. An engineer not keenly aware of this fact may consider his results acceptable because e_f, e_m, and e_s are within the limits he is used to. However, a low value of e_a can still render the predictions unacceptable by violating conditions (64) through (66).

8 Analysis of wrinkled membranes

8.1 Introduction

Due to their lack of bending stiffness, membranes cannot sustain compressive stresses. The membrane responds to an in-plane contraction, due to external agencies other than that of Poisson's effect from tensile stresses that act perpendicular to the contraction, by out-of-plane displacements that oscillate about the mean plane; i.e., the membrane wrinkles (Fig. 6). Wrinkles are seen in biological organisms, such as the wrinkles in the skin of humans (Fig. 7).

The strength of material model for the membrane cannot model this response behavior. A suitable model for this response behavior is the tension field (TF). Different from other strength of material models, the TF model is non-linear. This is the case even when the membrane material is linearly elastic.

The TF responds to a planar strain field where one of the principal strains is extensional the other contractive by a stress field with a single non-zero stress component of tension in the direction of the extensional principal strain. All other components of the stress tensor are zero.

The first mention of a tension field model was due to Wagner (1929) [21]. Wagner was concerned with the load-carrying capability of web-stiffened steel plate girders that were capable of carrying loads with a buckled web far in excess of their load carrying capability in the unbuckled state. The mode of performance of these plate girders is akin to the mode of performance of parallel cord trusses, where the bending moment is carried by the cords in tension and compression, respectively, and the shear stress resultant is carried by vertical cross members in compression and by diagonal cross members in tension. In the case of the plate girder, the stiffeners take the role of the vertical members of the truss, and the wrinkled web

takes the role of the diagonal tension members. Wagner's theory is an equilibrium theory only. The full set of solid mechanics equations as specialized to the structural elements is only used to predict the onset of buckling, not the tension field response. By contrast, the TF model for membrane mechanics uses the full set of solid mechanics equation as specialized to the two-dimensional sheet.

Figure 6. Wrinkles in a thin metallized polymeric foil are easily observed.

Figure 7. Wrinkles in the skin are seen under the eye.

TF theory ignores the minutia of out-of-plane wrinkling, neither the amplitude, nor the frequency of wrinkles are of concern. The direction of the wrinkles is determined and the total amount of in-plane contraction of the mean mid-plane of the field in excess of that due to Poisson's effect are determined by displacements at the boundary of the tension field.

Since Wagner's early work, numerous researchers have contributed to the development of TF theories. No attempt is made here to give a full account if these efforts. Reissner (1938) [22], Kondo et al. (1955) [23], and Mansfield (1970) [24] developed solutions for geometrically linear problems. Wu (1981) [25] developed a model for finite plane stress theory. Pipkin (1986) [26], and Steigmann and Pipkin (1988) [27] used the concept of relaxed strain energy density. Roddeman et al. (1987) [28] modified the deformation tensor. Jenkins and Leonard (1993) [29]

used modified strain energy and a modified dissipation function in the analysis of the dynamic wrinkling of visco-elastic membranes.

A particular simple approach for solving membrane mechanics problems that include wrinkling response is due to Schur (1994) [30, 31]. This approach is particularly suitable for use with a non-linear finite element (FE) code (be it a special purpose type or a commercially supplied general purpose code) [32, 33]. It treats the wrinkled region as a degenerate membrane for which it modifies the constitutive equation via a penalty parameter. This process diminishes the stiffness of the membrane in the wrinkling direction without rendering the stiffness matrix non-singular. By nature, this method is approximate. This method is outlined below.

8.2 Tension-Field modeling via a penalty parameter modified constitutive law

The modification of the analysis process so to enable TF response remains entirely within the material module (i.e. the application of the constitutive law) of the FE code. In a non-linear finite element analysis an interim solution step starts with the strained state ${}^{(i-1)}\varepsilon$ of the previous instance and advances under a load increment to the new strained state ${}^{(i)}\varepsilon = {}^{(i-1)}\varepsilon + \Delta\varepsilon$. The inverse of the tangent stiffness matrix D_{TAN} is used to advance the stress to its new state ${}^{(i)}\sigma$. If the principal stresses of this state are both (plane stress) positive, then the membrane is non-degenerate, i.e., the membrane model is appropriate and the constitutive model for the membrane sheet can be applied unmodified for the determination of the element tangent stiffness matrix. However, if one of the principal stresses is negative, then the membrane is degenerate.

There are two degrees of degeneracy. When both principal stresses are negative then the membrane is locally fully degenerate. The TF state exists when one of the principal stresses is positive and the other is negative.

For analysis to proceed in the fully degenerated state, it is necessary to return a zero stress state and a tangent stiffness that is severely diminished but not identically zero. If a zero tangent stiffness matrix were to be returned, then the analysis would terminate due to a singular stiffness matrix. The magnitude of that diminution must be supplied by the analyst, and methods are available to assist in choosing the optimum value [32]. It should be such that the solution can proceed yet the results of the analysis remain meaningful.

In the TF case the stiffness matrix is transformed to the principal stress axes. The off-diagonal coefficients in the transformed matrix are set to zero. The stiffness coefficient on the diagonal of the transformed matrix that is associated with the compressive principal stress is diminished by a penalty parameter of the analysts choosing and the compressive stress is set to zero.

Inversion of the diagonal tangent stiffness matrix is trivial. The tangent stiffness (or compliance) and the stresses are returned after transformation back to the material reference coordinate system. The transformation matrices that are required for this process are presented in the section on multi-layered membrane sheets.

This process is well suited for single-integration-point finite elements. In the case of elements with more than one integration point there exists the possibility that the iterative analysis process toggles indefinitely; thus preventing the progress of the solution process.

Although the classical TF model ignores details of the wrinkles themselves, progress has been made on predicting the wrinkle parameters of number of waves, wavelength, and wave height [32, 34].

9 Experimental analysis

9.1 Unique challenges for experimental analysis of membrane structures

Their extreme thinness is central to the unique challenges when making experimental measurements on membrane structures. Thinness contributes in large part to the lightweight and high compliance of these structures. In addition, membrane materials are often comprised of polymer films, with lower modulus, higher elongations at failure, and time, temperature, and frequency dependence. Also, membrane space structures may be considerably larger than more conventional space structures.

All of this points to the fact that, in most measurement situations involving membrane structures, noncontact methods are usually called for. The following table summarizes some of these issues.

Table 2. Contact problems in membrane measurements, and noncontact solutions.

Measurement	Contact Issues	Noncontact Methods	Limitations
Static Deformation	Artificial stiffening Mass loading	Eddy-current probes Capacitance probes Moiré Electronic speckle	Measurement range Single-point measure Measurement range Single-point measure Optically quiet environ. Set up, grid placement Expensive
Dynamic Deformation	Artificial stiffening Mass loading	Eddy-current probes Capacitance probes Laser vibrometer Holography Moiré Electronic Speckle	Measurement range Single-point measure Measurement range Single-point measure Expensive Not true full-field Sensitivity to noise, vibration Optically quiet environ. Set up, grid placement Expensive
Thermal Deformation	Artificial stiffening Mass loading	Infrared thermography	Surface measure only Calibration

Details about each of these issues and solutions will be briefly discussed below.

9.2 Static deformation measurement

Measurement of displacement and strain are often required and essential in membrane mechanics. However, conventional contact instruments like dial gages and strain gages cannot be used; the former may artificially add displacement, while the latter artificially stiffens material immediately surrounding the gage. Clip-on extensometers can be used for testing of some material coupons.

The simplest noncontact techniques are electric field techniques, such as eddy current and capacitance probes. These inexpensive techniques are usually fairly precise over a small measurement range, which is limited typically to a few millimeters. They also are restricted to single-point measurements. Jenkins and co-workers reported on using capacitance techniques for measuring membrane wrinkling [35].

Optical techniques offer high precision, full-field measurement, but only a few of the techniques are applicable to large deformations. The moiré family of interferometric techniques has considerable history in structural measurements. Since a typical object grating applied to the membrane would provide significant artificial stiffness, shadow moiré techniques are often used [36]. Speckle methods also hold much promise for displacement and strain measurements. A serious drawback is their expense and the need for complex data-analysis routines. Optical extensometers are available.

9.3 Dynamic displacement measurement

Conventional vibration measurement using accelerometers will not work on membrane structures, due to mass loading and artificial stiffening. Electric field techniques mentioned above are also applicable for dynamic displacement measurements. However, high frequency response may be a problem, and they are still single-point techniques.

Moiré interferometry described above may also be used for dynamic measurements, but data analysis becomes more challenging as the frequency increases, and lack of time-series may also be a problem. Holography has been used for the vibration analysis of plates for many years, but it requires a very quiet optical environment, which makes the technique less robust for many applications.

The laser vibrometer is a powerful tool for noncontact membrane vibration analysis. Vibrometers measure velocity changes due to the doppler (frequency) shift of laser light reflected from the moving surface. Lock-in amplifiers give good noise rejection. Scanning systems allow for fast raster scanning of the object, but this is only quasi full-field (in steady-state vibration). Full systems are expensive. Jenkins and co-workers have reported on laser vibrometer measurements of membrane structures [3, 37].

9.4 Thermal measurements

Conventional structural temperature measurements are performed using the ubiquitous thermocouple. Thermocouples are inexpensive and relatively precise. As in the other cases described above, attaching thermocouples to membranes artificially stiffens and loads them.

Infrared (IR) techniques provide good noncontact alternatives to thermocouples [38, 39]. High-precision IR sensors are available, some of them supercooled by liquid nitrogen for good noise rejection. The primary disadvantage of IR techniques is that they are surface temperature measurements only (thermocouples can be embedded). For thin membranes at thermal equilibrium this is not too much of a problem. The other disadvantage is that the IR/temperature conversion is dependent on the emissivity of the membrane surface, which may be a function of temperature. Hence careful and frequent calibration may be required.

10 Conclusion

The analysis of compliant structures in general and membrane structures in particular, is complicated by the nonlinear nature of the deformations and/or the materials involved. The natural language to describe such behavior is nonlinear continuum mechanics. This chapter has

attempted to provide some insight into the mathematical formalism, physical quantities, material constitution, and analysis issues associated with the nonlinear continuum mechanics.

References

[1] Jenkins, C.H. & Leonard, J.W., Nonlinear Dynamic Response of Membranes: State of the Art. *Appl. Mech. Rev.*, **44**, pp. 319–328, 1991.

[2] Jenkins, C.H., Nonlinear Dynamic Response of Membranes: State of the Art – Update. *Appl. Mech. Rev.*, **49(10)**, pp. S41–S48, 1996.

[3] Jenkins, C.H., Membrane Vibrations: a Review and New Experimental Results. *1999 ASME Joint Applied Mechanics and Materials Summer Conference*, Blacksburg, VA, 1999.

[4] Eringen, A.C., *Nonlinear Theory of Continuous Media*, McGraw-Hill, 1962.

[5] Fung, Y.C., *Foundations of Solid Mechanics*, Prentice-Hall, 1965.

[6] Bonet, J. & Wood, R.D., *Nonlinear Continuum Mechanics for Finite Element Analysis*, Cambridge University Press, 1997.

[7] Ogden, R.W., *Nonlinear Elastic Deformations*, Dover, 1997.

[8] Leonov, A.I., On the Conditions of Potentiality in Finite Elasticity and Hypo-Elasticity. *Int. J. Solids Structures*, **37**, pp. 2565–2576, 2000.

[9] Truesdell, C., Hypo-elasticity. *J. Rational Mech. Anal.*, **4**, pp. 83–133, 1019–1020, 1955.

[10] Truesdell, C., Foundation of Elasticity Theory. *Int. Sci. Rev. Ser.*, Gordon & Breach: New York, 1965.

[11] Truesdell, C. & Toupin, R., Handbuch der Physik. **3(1)**, *The Classical Field Theories*, 1960.

[12] Taylor, R.L., Inversion of Prony Series Characterization for Visco-elastic Stress Analysis. *Int. J. Num. Meth. Eng.*, **5**, pp. 499–502, 1973.

[13] Steigmann, D.J. & Pipkin, A.C., Equilibrium of Elastic Nets. *Phil. Trans. R. Soc. Lond. A.*, **335**, pp. 419–454, 1991.

[14] Ruggiero, T.J. & Mikulas, M., A One-Dimensional Constitutive Model for Wrinkled Thin Polymer Films. *Contractor Report CR-1999-209129*, NASA Langley Research Center, Hampton, VA 23681–2199, April 1999.

[15] Murphey, T.W., A Nonlinear Elastic Constitutive Model for Wrinkled Thin Films. *Ph.D. Dissertation*, University of Colorado, Boulder, 2000

[16] Calladine, C.R., *Theory of Shell Structures*, First edn, Cambridge University Press, 1983.

[17] Libai, A. & Simmonds, J.G., *The Nonlinear Theory of Elastic Shells*, Second edn, Cambridge University Press, 1998.

[18] Timoshenko, S. & Woinowsky-Krieger, S., *Theory of Plates and Shells*, Second edn, McGraw Hill, 1959.

[19] Young, W.C., *Roark's Formulas for Stress and Strain*, Sixth edn, McGraw-Hill, 1989.

[20] Greschik, G., Palisoc, A. & Mikulas, M., Approximations and errors in pressurized axisymmetric membrane shape predictions. *Proceedings of the 39th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference and AIAA/ASME Adaptive Structures Forum*, April 20–23, Long Beach, CA, **4**, pp. 2761–2771, AIAA-98-2101-CP, 1998.

[21] Wagner, H., Ebene Blechwandtraeger mit Sehr Duennem Stegblech. *Z. Flugtechnik u. Motorluftschiffahrt*, 1929.

[22] Reissner, E., Tension-field Theory. *Proc. 5th Int. Congr. Appl. Mech.*, pp. 88–92, 1938.

[23] Kondo, K., Iai, T., Moriguti, S. & Murasaki, T., Tension-Field Theory. *Memoirs of the Unifying Study of Basic Problems in Engineering Science by Means of Geometry*, Vol. 1, C.-V., 61–85,; Gakujutsu, Bunken Fukyo-Kai, Tokyo 1955.
[24] Mansfield, E.H., Load Transfer via a Wrinkled Membrane. *Proc. R. Soc. Lond. A.*, 316, 269, 1970.
[25] Wu, C.-H., Wrinkling in Finite Plane Stress Theory. *Q. Appl. Math.*, **39**, pp. 179–199, 1981.
[26] Pipkin, A., The Relaxed Energy Density for Isotropic Elastic Membranes. *J. Appl. Math.*, **36**, pp. 85–99, 1986.
[27] Steigmann, D.J. & Pipkin, A., Wrinkling of Pressurized Membranes. *ASME, Trans. J. Appl. Mech.*, **56**, pp. 624–628, 1989.
[28] Roddeman, D.G., Drukker, J., Oomens, W.J. & Jannsen, J.D., The Wrinkling of Thin Membranes. *J. Appl. Mech.*, pp. 884–892, 1987.
[29] Jenkins, C.H. & Leonard, J.W., Dynamic Wrinkling of Viscoelastic Membranes. *J. Appl. Mech.*, **60**, pp. 575–582, 1993.
[30] Schur, W.W., Tension Field Material Model for Thin Films. *32nd AIAA Aerospace Science Meeting*, Reno, NV, 1994.
[31] Schur, W.W., Tension Field Modeling by Penalty Parameter Modified Constitutive Law. *24th Midwestern Mechanics Conference*, Iowa State University, Ames, Iowa, 1995.
[32] Liu, X., Jenkins, C.H. & Schur, W.W., Fine Scale Analysis of Wrinkled Membranes. *Int. J. Computational Engr. Sci.*, 2000.
[33] Liu, X., Jenkins, C.H. & Schur, W.W., Large Deflection Analysis of Pneumatic Envelopes using a Penalty Parameter Modified Material Model. *Finite Elements in Analysis Design*, **37**, pp. 233–251, 2001.
[34] Liu, X., Jenkins, C.H. & Schur, W.W., Fine Analysis of Degenerate Membrane States using a Penalty Parameter Modified Constitutive Relation. *Fifth U.S. National Congress on Computational Mechanics*, Boulder, CO, 1999.
[35] Jenkins, C.H., Haugen, F. & Spicher, W.H., Experimental Measurement of Wrinkling in Membranes Undergoing Planar Deformation. *Exp. Mech.*, **38**, pp. 147–152, 1998.
[36] Jenkins, C.H. & Khanna, S.K., Determination of Membrane Wrinkling Parameters using Shadow Moire. *1999 Spring Conference, Society for Experimental Mechanics*, Cincinnati, OH, 1999.
[37] Jenkins, C.H. & Tampi, M., Local Membrane Vibrations and Inflatable Space Structures. *Space 2000*, Albuquerque, NM, 2000.
[38] Jenkins, C.H. & Faisal, S.M., Thermal Load Effects on Precision Membranes. *J. Spacecraft Rockets*, 2001.
[39] Jenkins, C.H. & Faisal, S.M., Thermal Load Effects on Precision Membranes. *1999 AIAA Adaptive Structures Forum*, St. Louis, MO, 1999.