Time-harmonic Green’s function for the half-plane with quadratic-type inhomogeneity

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Abstract

This work presents closed form solutions for point force generated motions in a continuously inhomogeneous half-plane, which represent a complete elastic wave-train in the interior domain obeying traction-free boundary conditions at the horizontal surface. A special type of material inhomogeneity is studied, where both shear modulus and material density vary quadratically with respect to the depth coordinate. Furthermore, Poisson’s ratio remains fixed at one-quarter. Next, numerical results serve to validate the aforementioned model, and to show the differences in the wave motion patterns developing in the presence of a free surface for media that are inhomogeneous as compared to a reference homogeneous background. These singular solutions are useful within the context of boundary element formulations for the numerical solution of problems involving non-homogeneous continua, which find applications in fields as diverse as composite materials, geophysical prospecting, oil exploration and earthquake engineering.

Keywords: Inhomogeneous media, Elastic waves, Fourier transforms, Singular solutions.

1 Introduction

Detailed knowledge of wave motions produced by point forces in the elastic half-plane \cite{1} are of paramount importance in mechanics, since they form the
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backbone of any integral equation formulation whose numerical treatment yields boundary element method (BEM) solutions to a wide range of boundary-value problems in elastodynamics [2]. More specifically, let $O\mathbf{x}$ be the Cartesian coordinate system in $R^2$ shown in Figure 1 and denote the lower half-plane as $R^2 = \{(x_1, x_2) : x_2 < 0\}$. Consider the following boundary-value problem defined in the frequency domain, where all variables have an $\exp(i\omega t)$ dependence:

$$L^s(G) = \left( C_{j\beta pq} G_{\gamma\alpha ij} \right) - \rho \omega^2 G_{ik} = -\delta(x - \xi)e_{ik}$$  \hspace{1cm} (1)

$$T^s(G) = C_{j\beta pq} G_{\gamma\alpha ij} = 0, \text{ on } x_2 = 0, \hspace{1cm} (2)$$

$$G \to 0, \hspace{0.5cm} \text{for } x_2 \to -\infty. \hspace{1cm} (3)$$

We have that Green’s tensor $G$ satisfies the Sommerfeld radiation condition along lines parallel to $\{x_2 = 0\}$, i.e., $\{(x_1, x_2), x_1 \to \pm \infty\}$. In the above, $x, \xi \in R^2$ and $x = (x_1, x_2), \xi = (\xi_1, \xi_2)$ are the source (S) and receiver (R) points in the continuum; $C_{j\beta pq} = h(x_2)C_{j\beta pq}^0$ is the elasticity tensor; $\rho = h(x_2)\rho_0$ is the material density and $h(x_2) = (a x_2 + 1)^2, \ a \leq 0$ is the material profile indicating a quadratic-type variation with respect to the depth coordinate.

Figure 1: Elastic half-plane with quadratically varying material properties in the depth coordinate as described by profile function $h(x)$. 

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In terms of quantities defined for the corresponding homogeneous material background, we have that
\[ C_{j kp q}^0 = \mu_0 (\delta_{jk} \delta_{pq} + \delta_{jp} \delta_{kq} + \delta_{jq} \delta_{kp}) \], where \( \mu_0 > 0 \) is the shear modulus, \( \delta_{jk} \) is the Kronecker’s delta, \( \rho_0 > 0 \), and frequency \( \omega > 0 \). Furthermore, \( \delta \) is Dirac’s delta function, \( \varepsilon = \varepsilon_{\delta} \) is the unit tensor, commas denote partial differentiation with respect to the spatial coordinates and summation is implied over repeated indices.

In elastodynamics, the problem defined by eqns (1)–(3) models an isotropic medium in \( \mathbb{R}^2 \) with a point force at \( \xi \) and traction-free boundary conditions. Poisson’s ratio is fixed at a value of \( \nu = 0.25 \), while the shear modulus \( \mu \) and the density \( \rho \) depend in the same manner on depth coordinate \( x_2 \). A fundamental solution to eqn (1) of this problem was derived in Manolis and Shaw [3] for \( a \neq 0 \), while a solution of eqns (1)–(3) defining a Green’s function for the homogeneous half-plane, i.e., \( a = 0 \), has been obtained by Kinoshita [4] (see also Kobayashi [5]). A corresponding Green’s function in the Laplace domain for a homogeneous half-plane can be found in Guan et al. [6], while an approximate such function using image sources across the free surface was derived earlier by Kontoni et al. [7]. Finally, the transient Green’s function due to a suddenly applied load in the homogeneous half-plane, namely Lamb’s problem, can be found in the book on Compiled fundamental solutions of elastodynamics by Kausel [8].

2 Solution outline

By following the procedure as outlined in the references given above, we will now derive the unique solution to eqns (1)–(3), which corresponds to a Green’s function \( G \) for the inhomogeneous half-plane. Let matrix-valued function \( u \) be a fundamental solution to eqn (1), i.e.
\[ L^a (u) = -\delta(x - \xi) \varepsilon, \quad \text{where} \quad x, \xi \in \mathbb{R}^2, \]
and \( w \) is smooth matrix-valued function such that
\[ L^a (w) = 0, \quad x, \xi \in \mathbb{R}^2, \]
\[ T^a (w) = -T^a (u), \quad \text{on} \quad x_2 = 0, \]
where superscript \( a \) in the operators corresponds to the degree of inhomogeneity. Then, by using superposition, the complete Green’s function is \( G = u + w \).

Fundamental solution \( u \) can be expressed in the form [3]
\[ u(x, \xi, \omega) = h^{-1/2}(\xi_2) U(x, \xi, \omega) h^{-1/2}(x_2), \]
where $U$ is a fundamental solution for the corresponding homogeneous case, i.e.

$$L^0(U) = -\delta(x - \xi)\omega, \quad \text{with } x, \xi \in \mathbb{R}^2_+.$$  

Finally, the traction matrix corresponding to displacements $u$ on free surface $x_2 = 0$ is

$$T^{\omega}_{2k}(u) = \mu_0 h^{-1/2}(\xi_2)(-aU_{1k} + U_{1k,2} + U_{2k,1}),$$

$$T^{\omega}_{2k}(u) = \mu_0 h^{-1/2}(\xi_2)(-3aU_{2k} + U_{1k,1} + 3U_{2k,2}).$$  

The homogeneous matrix-valued function $U$ in $\mathbb{R}^2$ can be found in Eringen and Suhubi [9] as

$$U_{jk} = \frac{i}{4\mu_0} \left[ \delta_{jk} H^{(1)}_0(k_2r) + \frac{1}{k_2^2} \partial_{\xi_j}^2 \left( H^{(1)}_0(k_2r) - H^{(1)}_0(k_2r) \right) \right].$$  

In the above equation, the two wave numbers corresponding to pressure and shear body waves are $k_1 = \sqrt{\rho_0/3\mu_0} \omega$ and $k_2 = \sqrt{\rho_0/\mu_0} \omega$, respectively, while the radial distance between source and receiver is $r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}$ and $H^{(1)}_0(z)$ is the Bessel function of third kind (or Hankel function), zero order (see Gradshteyn and Ryzhik [10]).

In order to simplify the calculations, we fix the source point along the vertical axis as $\xi = (0, \xi_2)$, $\xi_2 < 0$. As will be shown later on, Green’s function $G$ actually depends on $x_1 - \xi_1$ and separately on $x_2, \xi_2$ due to the fact that the corresponding profile function $h$ is independent of coordinate $x_1$, which implies that assumption $\xi_1 = 0$ is not restrictive.

3 Solution methodology

The first step is to recover a general solution $w$ to eqn (5) in the form

$$w(x, \xi, \omega) = h^{-1/2}(x_2)W(x, \xi, \omega).$$  

Then, the two corresponding differential operators are related as

$$L^0(w) = h^{1/2}(x_2)L^0(W),$$

$$L^0(w) = h^{1/2}C_{j\beta kq}^{\omega} \left[ W_{q,\beta k} + h^{-1} (h^{1/2}_j W_{q,\beta} - h^{1/2}_k W_{\beta, j} - h^{1/2}_q W_{\beta, k}) + \rho \omega^2 h^{-1/2}W_{j\beta k} \right]$$

$$= h^{1/2} \left[ C_{j\beta kq}^{\omega} W_{q,\beta k} + \rho \omega^2 W_{j\beta k} \right] = h^{1/2}L^0(W).$$
Thus, if \( W \) solves eqn (5) with \( a = 0 \), then \( w \) also solves eqn (5) for \( a < 0 \) and we seek a solution \( W = \{ W_{jk} \} \) in the general Rayleigh form [1,11] as

\[
W_{jk} = \frac{1}{2\pi} \int_R S_{jk} e^{i\eta y} d\eta,
\]

where kernel function \( S_{jk} \) depends on \( \exp(\beta x_2), \eta, \omega, a \) and parameter \( \beta \) is found as solution of an algebraic system of equations to be developed in what follows.

**Remark 1:** It is not possible to proceed for the inhomogeneous case, as in Kinoshita [4], for a homogeneous material. The algebraic transformation produces a function \( \tilde{u}(x, \xi, \omega) = h^{-1/2}(-\xi_2)U(x, \xi, \omega)h^{-1/2}(x_2) \) that is not well defined for all \( \xi_2 < 0 \) and is infinite if \( h(-\xi_2) = 0 \), corresponding to a value \( \xi_2 = 1/a, \quad a < 0 \). Thus, we cannot use superposition as \( u(x, \xi, \omega) + \tilde{u}(x, \xi, \omega) \), for which \( T_1 = 0, T_2 = 0 \) on \( x_2 = 0 \), but can only use \( u(x, \xi, \omega) \) and then add a Rayleigh form to satisfy the boundary conditions.

In order to find \( S = \{ S_{jk} \} \), we use the Fourier transform \( \mathfrak{F} \) with respect to the \( x_1 \) coordinate, defined for the direct and inverse transformations as

\[
\tilde{f}(\eta, x_2) = \mathfrak{F}_{x_1 \rightarrow \eta} \{ f \} = \int_R f(x_1, x_2) e^{-i\eta x_1} dx_1,
\]

\[
f(x_1, x_2) = \mathfrak{F}^{-1}_{\eta \rightarrow x_1} \{ \tilde{f} \} = \frac{1}{2\pi} \int_R \tilde{f}(\eta, x_2) e^{i\eta x_1} d\eta,
\]

where \( \eta \) is the transform parameter. Applying the Fourier transform to \( W \), eqn (5) with \( a = 0 \) becomes

\[
L^0(\mathfrak{F}_{x_1 \rightarrow \eta}(W)) = 0,
\]

which in matrix form reads as

\[
(M(\eta, \beta) + \rho_0 \omega^2 I_2)S = 0. \tag{16}
\]

In the above equation, \( I_2 \) is the 2×2 unit matrix and

\[
M(\eta, \beta) = \begin{pmatrix}
-3\mu_0 \eta^2 + \mu_0 \beta^2 + \rho_0 \omega^2 & 2i\mu_0 \eta \beta \\
2i\mu_0 \eta \beta & -\mu_0 \eta^2 + 3\mu_0 \beta^2 + \rho_0 \omega^2
\end{pmatrix}. \tag{17}
\]
For every fixed value of $\eta$, a non-zero solution of eqn (16) exists if \( \det \{\mathcal{M}(\eta, \beta)\} = 0 \), which gives the following quartic equation for parameter $\beta$:

\[
3\mu_0^2\beta^4 - 2\mu_0(3\mu_0\eta^2 - 2\rho_\nu\omega^2)\beta^2 + \rho_\nu^2\omega^4 + 3\mu_0^2\eta^4 - 4\mu_\nu\eta^2\rho_\nu\omega^2 = 0 \tag{18}
\]

By denoting $\gamma_j^2 = \eta^2 - k_j^2$, eqn (18) simplifies to

\[
\beta^4 - (\gamma_1^2 + \gamma_2^2)\beta^2 + \gamma_1^2\gamma_2^2 = 0, \tag{19}
\]

and the solutions are $\beta_j^2 = \pm \gamma_j^2$. In order to satisfy the radiation condition of eqn (3), the positive root is retained:

\[
\beta_j = \gamma_j = \sqrt{\eta^2 - k_j^2}. \tag{20}
\]

Since $rk\mathcal{M}(\eta, \beta_j) = 1$ (i.e. the rank of the matrix $\mathcal{M}(\eta, \beta_j)$ for $j = 1, 2$ is one), there are two eigenvectors, namely,

\[
v^1 = \begin{pmatrix} \eta \\ -i\beta_1 \end{pmatrix}, v^2 = \begin{pmatrix} i\beta_2 \\ \eta \end{pmatrix}, \tag{21}\]

and every solution to eqn (16) has the standard form

\[
S = \{S_{jk}\} = \sum_{m=1}^{2} C_m^k v^m e^{ikx} \tag{22}
\]

Recapitulating, the matrix form of eqn (11) using indicial notation is

\[
w_{jk}(x, \xi, \omega) = h^{-1/2}(x_2)W_{jk}(x, \xi, \omega) \tag{23}
\]

and the remaining step is to determine functions $C_m^k(\eta, \xi, a)$ such that the boundary condition for zero tractions in eqn (6) is satisfied. The traction field corresponding to displacement field $w$ on $x_2 = 0$ is

\[
T_{ik}^a(w) = \frac{1}{2\pi} \int_{\mathbb{R}} \mu_0 \left[ \eta(-a + 2\beta_i)C_{1k}^k + i(-a\beta_2 - 2\eta^2 - k_2^2)C_{2k}^k \right] e^{\eta y_i} dx_i, \tag{24}
\]

\[
T_{2k}^a(w) = \frac{1}{2\pi} \int_{\mathbb{R}} \mu_0 \left[ i(3a\beta_1 - 2\eta^2 + k_2^2)C_{1k}^k + \eta(-3a + 2\beta_2)C_{2k}^k \right] e^{\eta y_i} dx_i.
\]

In order to determine the traction field corresponding to displacement field $u$ on $x_2 = 0$, we use the representation of $H_0^{(1)}$ based on a Fourier transform with
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respect to $x$ (see Gradshteyn and Ryzhik [10]; formulas 6.677 3&4; also Section 8.42):

$$H_0^{(1)}(r_k) = \frac{i}{2\pi} \int R_{\beta j}^\kappa e^{i\eta \xi \cdot x_k} d\eta.$$  \hspace{1cm} (25)

Employing eqns (7) and (10) for $u$ and $U$, respectively, we obtain

$$T_{\mu k}^m (u) = \frac{1}{2\pi} \int R_{\mu k}^\eta e^{i\eta \xi \cdot x_k} d\eta,$$  \hspace{1cm} (26)

where matrix components $D_{\mu k}$ are given below:

$$D_{11} = \frac{h^{-1/2}(\xi_2)}{2k^2_2} \left[ (-a\beta_2 - 2\eta^2 + k^2_2) e^{i\xi \beta_2} + \eta^2 \left( \frac{a}{\beta_1} + 2 \right) e^{i\xi \beta_1} \right],$$

$$D_{21} = \frac{i\eta h^{-1/2}(\xi_2)}{2k^2_2 \beta_1} \left[ \beta_1 (-3a - 2\beta_2) e^{i\xi \beta_1} + (3a\beta_1 + 2\eta^2 - k^2_2) e^{i\xi \beta_2} \right],$$

$$D_{12} = \frac{i\eta h^{-1/2}(\xi_2)}{2k^2_2 \beta_2} \left[ (-a\beta_2 - 2\eta^2 + k^2_2) e^{i\xi \beta_2} + \beta_2 (a + 2\beta_1) e^{i\xi \beta_1} \right],$$

$$D_{22} = \frac{h^{-1/2}(\xi_2)}{2k^2_2} \left[ \eta^2 \left( \frac{3a}{\beta_1} + 2 \right) e^{i\xi \beta_1} + (-3a\beta_1 - 2\eta^2 + k^2_2) e^{i\xi \beta_2} \right].$$  \hspace{1cm} (27)

By combining eqns (24) and (26), a system of two linear equations is recovered for $C_m, C^2_m$ that appear as kernels of integral equations when substituted in the boundary condition of eqn (6). The determinant of this system is

$$\Delta^a = \frac{h^2}{4\pi^2} \left| \begin{array}{cc}
\eta (-a + 2\beta_1) & i(\eta a + 2\eta^2 - k^2_2) \\
(3a\beta_1 - 2\eta^2 + k^2_2) & \eta (-3a + 2\beta_2)
\end{array} \right|,$$  \hspace{1cm} (28)

and is computed as

$$\Delta^a = \frac{h^2}{4\pi^2} \left[ 3(\eta^2 - \beta_1 \beta_2) a^2 - ((\beta_1 + \beta_2) k^2_2 + \eta^2 \beta_1) a - \Delta^0 \right],$$  \hspace{1cm} (29)

where $\Delta^0 = 4\eta^2 \beta_1 \beta_2 - (2\eta^2 - k^2_2)^2$ is a Rayleigh function [5].

Functions $C_m, C^2_m$ are unique solutions of eqn (6), since for every $\eta \in \mathbb{R}, a < 0, \omega > 0, \rho_0 > 0, \mu_0 > 0$, the condition $\Delta^a \neq 0$ holds true. Possible
combinations of values of parameter $|\eta|$ as compared to the two wave numbers $k_1, k_2$ yield the following cases:

- If $|\eta| < k_1$, then $\text{Im} \Delta^\alpha = -\left[ |\beta_1| + |\beta_2| |k_2^2 + \eta^2| \right] a > 0$,
- If $|\eta| = k_1$, then $\text{Im} \Delta^\alpha = -\left[ |\beta_2| k_2^2 a + (2\eta^2 - k_2^2)^2 \right] > 0$,
- If $k_1 < |\eta| \leq k_2$, then $\text{Re} \Delta^\alpha = 3\eta^2 a^2 - \beta_1(k_2^2 + \eta^2)a + (2\eta^2 - k_2^2) > 0$,
- If $k_2 < |\eta|$, then $\Delta^\alpha > \Delta^0 > 0$.

Applying Kramer’s rule yields matrix functions $C_m^k, C_m^s$ as

$$C_m^k = \Delta_{mk}^\alpha / \Delta^\alpha,$$  \hspace{1cm} (30)

where the sub-determinants $\Delta_{mk}^\alpha$ are given below:

$$\Delta_1^\alpha = \begin{vmatrix} -D_{11} & \mu_0 i(-a\beta_2 + 2\eta^2 - k_2^2) \\ -D_{21} & \mu_0 i(3a\beta_1 - 2\eta^2 + k_2^2) \end{vmatrix}, \quad \Delta_2^\alpha = \begin{vmatrix} \mu_0 \eta(-a + 2\beta_1) & -D_{11} \\ \mu_0 i(3a\beta_1 - 2\eta^2 + k_2^2) & -D_{21} \end{vmatrix},$$

$$\Delta_1^s = \begin{vmatrix} -D_{11} & \mu_0 i(-a\beta_2 + 2\eta^2 - k_2^2) \\ -D_{21} & \mu_0 i(3a\beta_1 - 2\eta^2 + k_2^2) \end{vmatrix}, \quad \Delta_2^s = \begin{vmatrix} \mu_0 \eta(-a + 2\beta_1) & -D_{11} \\ \mu_0 i(3a\beta_1 - 2\eta^2 + k_2^2) & -D_{21} \end{vmatrix}.$$  \hspace{1cm} (31)

Finally, the radiation boundary condition in eqn (3) holds because of the presence of multiplier $h^{-1/2}(x_2)$ for $u$ and $h^{-1/2}(x_2) \exp(x_2\beta)$ under the integral on $\eta$ for $w$ in eqn (13).

**Remark 2:** The above method can be applied for complex wave numbers, i.e. $k_j = k_{jr} + ik_{jt}$, $k_{jr} > 0, k_{jt} > 0$, and the structure of Green’s function remains the same. This is because the representations for the fundamental solution of eqn (10) and for the Bessel function, eqn (25) are valid for complex numbers as well. However, the proof that $\Delta^\alpha \neq 0$ in this case is more complicated.

**Remark 3:** The same method can be applied to obtain a transient Green’s function in the inhomogeneous half-plane for the equations of motion defined in the time domain as

$$L^\alpha(G) = \left(C_{j\rho q} G_{\rho q, j}\right)_{ij} - \rho G_{ik,j} = -f(t)\delta(x - \zeta)\epsilon_{ik},$$  \hspace{1cm} (32)

where $f(t) \in L^1_{loc}(R^1)$, with $L^1_{loc}$ i.e., the localised elastodynamic operator and $f = 0$ for $t < 0$. More specifically, $f(t) = H(t)F(t)$, with $H(t)$ the Heaviside
function and $|F(t)| \leq Ae^{ct}$ for $t \to \infty$. The transient Green’s function is obtained by applying Laplace transform to eqn (32) and using a Kelvin function representation of the type $K_0(z) = (i\pi/2)H_0^{(1)}(iz)$. Formally, the Green’s function in the Laplace domain is obtained by replacing frequency $\omega$ with the Laplace transform parameter $s$ written as a purely imaginary number $is$ and then applying the inverse Laplace transform. This path was followed for the homogeneous case, i.e. $a = 0$ and with $F(t) = 1$, by Guan et al. [6].

**Remark 4:** Green’s function $G(x, \xi, \omega, a)$ converges in the weak sense to $G(x, \xi, \omega, 0)$ for $a \to 0$, i.e. for every $\varphi(\xi) \in C^0_c(R^2)$, the following holds true:

$$\int_{R^2} G(x, \xi, \omega, a) \varphi(\xi) d\xi \to \int_{R^2} G(x, \xi, \omega, 0) \varphi(\xi) d\xi, \quad \text{for } a \to 0. \quad (i)$$

Also, Green’s function $G(x, \xi, \omega, a)$ converges in the weak sense to $G(x, \xi, 0, a)$ for $\omega \to 0$, i.e. for every $\varphi(\xi) \in C^0_c(R^2)$, the following holds true:

$$\int_{R^2} G(x, \xi, \omega, a) \varphi(\xi) d\xi \to \int_{R^2} G(x, \xi, 0, a) \varphi(\xi) d\xi, \quad \text{for } \omega \to 0. \quad (ii)$$

### 3.1 Recovery of the homogeneous case

In order to check that it is possible to recover the homogeneous half-plane solution by setting the inhomogeneity parameter $a = 0$ (and, correspondingly, $h(x_2) = h(\xi_2) = 1$ for the profile function) in the solution derived above, we start with the results presented in Kobayashi [5]. In that case, eqn (24) reads as

$$T_{10}^0(w) = \frac{1}{2\pi} \int_R \mu_0 \left[ 2\eta \beta_1 C_1^k + i(2\eta^2 - k_2^2)C_2^k \right] e^{\eta x_1} dx_1,$$

$$T_{20}^0(w) = \frac{1}{2\pi} \int_R \mu_0 \left[ i(-2\eta^2 + k_2^2)C_1^k + 2\eta \beta_2 C_2^k \right] e^{\eta x_1} dx_1. \quad (33)$$

Also, in place of $u(x_1, x_2 - \xi_2)$ we use $u(x_1, x_2 - \xi_2) + \tilde{u}(x_1, x_2 + \xi_2)$, where \(\tilde{u}(x_1, x_2 + \xi_2)\) is a smooth in $R^2$ matrix-valued function defined in reference to eqn (10) as

$$\tilde{u}_{jk}(x_1, x_2 + \xi_2) = \frac{i}{4\mu_0} \left[ \delta_{jk} H_0^{(1)}(k_2 \hat{r}) + \frac{1}{k_2^2} \delta_{jk} (H_0^{(1)}(k_2 \hat{r}) - H_0^{(1)}(k_1 \hat{r})) \right], \quad (34)$$
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with \( \tilde{r} = \sqrt{x_1^2 + (x_2 + \xi_2)^2} \) the radial distance between source and receiver. Furthermore, corresponding to eqn (25), the integral representation for the Hankel function is

\[
H^{(1)}_0(\tilde{r} k_j) = \frac{i}{2\pi} \int_R \frac{1}{\beta_j} e^{i(\xi_2 + x_2)\beta_j} e^{i\eta\eta} d\eta. \tag{35}
\]

Finally, the traction vector on the free surface \( x_2 = 0 \) for the complete displacement field \( u + \tilde{u} \) that replaces eqn (26) is

\[
T^0_{jk}(u + \tilde{u}) = \frac{1}{2\pi} \int_R \tilde{D}_{jk} e^{i\eta\eta} d\eta, \tag{36}
\]

with the following new definitions:

\[
\begin{align*}
\tilde{D}_1 &= 0, \quad \tilde{D}_2 = \frac{i\eta}{2k_2^2 \beta_2} \left[ -2\beta_1 \beta_2 e^{i\xi_2\beta_1} + (2\eta^2 - k_2^2) e^{i\xi_2\beta_1} \right], \\
\tilde{D}_3 &= \frac{i\eta}{2k_2^2 \beta_2} \left[ -(2\eta^2 - k_2^2) e^{i\xi_2\beta_2} + 2\beta_1 \beta_2 e^{i\xi_2\beta_2} \right], \quad \tilde{D}_{22} = 0.
\end{align*} \tag{37}
\]

The new sub-determinants \( \tilde{\Delta}_{mk}^0 \) are now

\[
\begin{align*}
\tilde{\Delta}_{11} &= \begin{vmatrix} 0 & \mu_0 i (2\eta^2 - k_2^2) \\ -\tilde{D}_{21} & \mu_0 2\eta \beta_2 \end{vmatrix}, & \tilde{\Delta}_{21} &= \begin{vmatrix} \mu_0 2\eta \beta_1 & 0 \\ -\mu_0 i (2\eta^2 - k_2^2) & -\tilde{D}_{21} \end{vmatrix}, \\
\tilde{\Delta}_{12} &= \begin{vmatrix} -\tilde{D}_{22} & \mu_0 i (2\eta^2 - k_2^2) \\ 0 & \mu_0 2\eta \beta_2 \end{vmatrix}, & \tilde{\Delta}_{22} &= \begin{vmatrix} \mu_0 2\eta \beta_1 & -\tilde{D}_{12} \\ -\mu_0 i (2\eta^2 - k_2^2) & 0 \end{vmatrix}.
\end{align*} \tag{38}
\]

and the solution for the matrix functions is

\[
\tilde{C}_{mk}^h = \tilde{\Delta}_{mk}^0 / \Delta^0. \tag{39}
\]

Finally, reconstruction of the complete Green’s function that replaces eqn (22) is

\[
\tilde{S} = \{\tilde{S}_{jk}\} = \sum_{m=1}^2 \tilde{C}_{mk}^{\text{tr}} v_n e^{i\beta_n x_2}, \tag{40}
\]
whose components are now written explicitly below as

\[
\begin{align*}
\hat{S}_{11} &= \frac{i\eta\mu_0}{\Delta_0} \left[ (2\eta^2 - k_2^2) e^{i\omega t} - 2\beta \beta_2 e^{i\omega t} \right] \hat{D}_{21}, \\
\hat{S}_{21} &= \frac{\beta_1 \mu_0}{\Delta_0} \left[ (2\eta^2 - k_2^2) e^{i\omega t} - 2\eta^2 e^{i\omega t} \right] \hat{D}_{21}, \\
\hat{S}_{12} &= \frac{\beta_2 \mu_0}{\Delta_0} \left[ -2\eta^2 e^{i\omega t} + (2\eta^2 - k_2^2) e^{i\omega t} \right] \hat{D}_{12}, \\
\hat{S}_{22} &= \frac{i\eta\mu_0}{\Delta_0} \left[ -2\beta_1 \beta_2 e^{i\omega t} + (2\eta^2 - k_2^2) e^{i\omega t} \right] \hat{D}_{21},
\end{align*}
\]

(41)

**Remark 5:** The half-plane Green’s function derived above can be used for solving general types of boundary-value problems in the half-plane enclosing singularities such as cracks, holes, cavities, etc. This can be done using BEM formulations [12] and the advantage now is that free surface (line \( x_2 = 0 \)) discretisation is unnecessary.

### 4 Numerical example

As an example, we consider the lower part of an original full-plane, keeping in mind that in the upper half-plane the material function \( h \) has a line of degeneracy (see Figure 1). Some results will be given here for the inhomogeneous half-plane Green’s functions derived herein, for the following source/receiver configuration:

\[
(\xi_1, \xi_2) = (0.0, -300.0 \text{ m}); \quad (\chi_1 = 30.0 \text{ m}, x_2 = 0.0).
\]

(42)

The background homogeneous material corresponds to relatively firm soil and has the following values for the pressure (P) and shear (S) wave speeds and for the density:

\[
c_1 = 621.0 \text{ m/sec, } c_2 = 359.0 \text{ m/sec, } \rho = 2100.0 \text{ kgm}.
\]

(43)

The inhomogeneity parameter is assigned the value of \( \alpha = -0.0005 \text{ (1/m)} \), which implies that the \( h(\xi) \) profile at the source depth is stiffer by a factor of 1.32 (i.e. about 30%) compared with the reference value \( \mu_0 = 270.0 \times 10^6 \text{ N/m}^2 \) at the free surface level. The travel time for the S-wave to reach the receiver starting from the source is \( t_2 = r / c_2 = 301 / 359 = 0.84 \text{ sec} \) in the reference homogeneous background material, and the frequency scale is set up according to a total time duration of the disturbance phenomenon of \( T = 2.0 \text{ sec} \). This gives a frequency value \( f = 1.0 / T = 0.50 \text{ Hz} \), which is
rounded off to 0.64 Hz so that it corresponds to \( \Omega = 4.0 \text{ rad/sec} \), swept in 40 increments of \( \Delta \omega = 0.1 \text{ rad/sec} \) starting from zero where the static solution \( G(x, \xi, 0; a) \) is used (see Remark 4 above).

In reference to the one-sided Fourier transform of eqn (14), this is performed numerically using the fast Fourier transform (FFT) [13]. More specifically, we use the positive side of the horizontal axis going up to \( X = 40.0 \text{ m} \). For \( N = 1024 \) data points, the wave number spectrum is set up according to the following formulas:

\[
\Delta x = 2X / N = 0.078125 \text{ (m)}, \quad \Delta \eta = 2\pi / N\Delta x = 0.07854 \text{ (1/m)} \quad (44)
\]

We note in passing that it is possible to introduce viscoelastic material behaviour using the Kelvin model with complex values for the material parameters [14], which is compatible with the static solution at zero frequency.

Figure 2(a) plots the amplitude of the Green’s function component \( G_{11}(x, \xi, \omega) = u_{11}(x, \xi, \omega) + w_{11}(x, \xi, \omega) \), where both full-plane \( u_{11}(x, \xi, \omega) \) and correction \( w_{11}(x, \xi, \omega) \) parts are separately plotted as functions of the inhomogeneity parameter \( a \), with the latter part restoring traction-free conditions at the free surface of the half-plane. Similarly, Figure 2(b) plots component \( G_{12}(x, \xi, \omega) \). For a value of \( a = 0, \ h = 1 \) and the equivalent homogeneous material Green’s functions can be obtained.

We observe that the Rayleigh integral yields a low frequency correction, which is very pronounced in the horizontal direction compared with the vertical one. Also, because the displacements are plotted at the free surface, the effect of inhomogeneity is rather small in the \( u_1 \) part and of the order of about 5–10% less (the homogeneous material is softer), since the inhomogeneity function \( h(\xi) \) becomes active with increasing depth from surface. Thus, corresponding plots are not shown here in the interest of brevity. We note in passing that approximate solutions using an image source [7] lead to a doubling of selected components of the displacement field in order to erase their corresponding traction components from the free surface, but it is never possible to completely reproduce traction-free conditions for all components simultaneously, unless additional sources (dipoles, etc.) are added.

### 5 Conclusions

In this work, a new point-force solution was derived for the continuously inhomogeneous half-plane with quadratic-type variation of all material parameters in terms of the depth coordinate. The solution represents a complete elastic wave-train propagating outwards from the loaded area and satisfies
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traction-free boundary conditions along the horizontal surface. As such, solutions of this type are useful as kernel functions in BEM formulations for problems of engineering importance in elastodynamics and other fields of mechanics.

References

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