Chapter Eight
Random multiaxial fatigue loading

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Abstract

A new spectral method of fatigue-life calculation under random multiaxial loading has been shown. The method consists in extension of the known formulae of Miles, Kowalewski, Rajcher, Bolotin and Wirsching and Light, based on the power-spectral density function (PSDF) of stresses under uniaxial random loading. PSDF of the equivalent stress, determined according to linear failure criteria of multiaxial random fatigue, was introduced to the method. It has been shown that while reducing the multiaxial state of stress to the uniaxial one with linear failure criteria, the frequency bands of stress-state components are transformed to the frequency band of equivalent stress without increase of its width. Such a favorable result cannot be obtained if the equivalent stress is calculated according to the nonlinear multiaxial-fatigue-failure criteria.

The chapter contains an approach for estimation of fatigue life in the HCF regime. Loading of Gaussian distribution and narrow- and broadband frequency spectra were assumed. Various characteristic states of multiaxial loading were considered. The reduced stress history was determined with use of failure criteria of multiaxial fatigue based on the critical plane. For determination of the critical-plane position, the methods of variance and damage accumulation were applied. During simulation, the authors compared the results obtained by the spectral method in the frequency domain with those from the rain-flow algorithm in the time domain. The chapter also contains the results of fatigue tests for 18G2A structural steel subjected to combined bending with torsion. The tests were performed in order to verify the proposed algorithms for determination of fatigue life. It has been shown that under multiaxial random loading results of fatigue
life calculated according to the considered algorithms in the frequency and time
domains are well correlated with the results of experiments.

1 Introduction

The known methods of evaluation of fatigue life of machine elements and
structures under random loading can be divided into two groups. The cycle-
counting approach, based on various numerical algorithms of cycle and/or half-
cycle counting (for example, the rain-flow algorithm), is typical of the first
group [1–3]. The second one is characterized by the spectral analysis of
stochastic processes [4–13]. In these two groups, fatigue damages are usually
accumulated with the linear Palmgren–Miner hypothesis where a standard
fatigue characteristic of the material, obtained under cyclic loading, is applied.
Under multiaxial loading also a suitable fatigue-failure criterion should be used.
The criterion allows reducing a triaxial random stress state to the equivalent
uniaxial one [14–18]. Then, the history of the equivalent stress (or strain) or its
suitable probabilistic characteristics may be applied for calculations of fatigue
life, as under uniaxial random loading. In the analytical methods, the power
spectral density function (PSDF) is usually used [4–13].

In this chapter, the authors present an analytical method of fatigue-life
calculation of machine elements and structures. The method requires application
of the multiaxial fatigue-failure criteria for random loading and the power
spectral density function of the equivalent stress (or strain). It is assumed that the
random stress (or strain) tensor is the six-dimensional stationary and ergodic
process with the mean values equal to zero and the low-band spectrum of
frequency. Many simulation calculations were done in order to compare the
proposed method with the cycle-counting method.

The chapter also contains the fatigue-test results for 18G2A steel subjected to
random bending with torsion. The obtained lives were compared with the
calculated ones with use of the modified criterion of maximum normal stress in
the critical plane and the Serensen–Kogayev hypothesis of damage
accumulation [3].

2 Random stress state

Random stress (or strain) state can be expressed by the six-dimensional vectorial
process

\[ X(t) = [X_1(t), \ldots, X_6(t)], \]

where \( X_k(t) = \sigma_{ij}(t), (k = 1, \ldots, 6; i, j = x, y, z). \)

According to the correlation theory, the stationary and ergodic vectorial
process \( X(t) \) is usually described with
its mean value, \( \hat{x} = [\hat{x}_1, ..., \hat{x}_6] \) and
its correlation matrix \( R_\tau(\tau) \) or covariance matrix \( \xi(\tau) \) where \( \tau = t_2 - t_1 \).

For the stationary and ergodic Gaussian process the joint probability density function (PDF) is expressed by

\[
f_{x_1, ..., x_6}(x_1, ..., x_6, \tau) = \frac{1}{\sqrt{(2\pi)^{A_1} \xi(\tau)}} \exp \left[ -\frac{1}{2} \xi^{-1}(\tau) \sigma_x^T \right],
\]  

where

\[
\begin{align*}
\xi(\tau) &= \begin{bmatrix}
\mu_{x_1}(\tau) & \cdots & \mu_{x_6}(\tau) \\
\vdots & \ddots & \vdots \\
\mu_{x_1}(\tau) & \cdots & \mu_{x_6}(\tau)
\end{bmatrix} \\
\sigma_x &= \begin{bmatrix}
x_1 - \hat{x}_1, ..., x_6 - \hat{x}_6
\end{bmatrix} ; \\
\mu_{x_1}(\tau) &= \text{positive definite covariance matrix of random variables } x_1, ..., x_6; \\
\xi(\tau) &= \text{determinant of the matrix } \xi(\tau); \\
\mu_{x_1}^{-1}(\tau) &= \text{inverse matrix to } \mu_{x_1}(\tau); \\
\sigma_x &= \text{row vector of variables } x_1, ..., x_6 \text{ and their mean values } \hat{x}_1, ..., \hat{x}_6 ; \\
\sigma_x^T &= \text{column vector } (\sigma_x \text{ transposed}).
\end{align*}
\]

In practice, random tensors expressed by (2) can be applied if the acting loading is similar to or if it even has distribution different from normal but its number is large and the effects of the law of large numbers are visible.

Generally speaking, if the joint probability density function of the random tensor, (2), must be determined, six mean values \( \hat{x} \) and twenty one elements of the covariance matrix \( \xi(\tau) \) should be known. The remaining fifteen elements are determined from \( \xi_{ij}(\tau) = \xi_{ji}(\tau) \), \((i, j = 1, ..., 6)\). While performing the spectral analysis \([13]\) we mainly concentrate on the power distribution, i.e. distribution of mean square values of amplitudes of particular harmonic components appearing in the random process and in the frequency bandwidth. As the random processes are components of the vectorial process, probabilistic relations between them are also important. These properties are expressed with PSDF \( G(f) \) that for the random tensor gives a rectangular 6×6-dimensional matrix

\[
G(f) = \begin{bmatrix}
G_{11}(f) & \cdots & G_{16}(f) \\
\vdots & \ddots & \vdots \\
G_{61}(f) & \cdots & G_{66}(f)
\end{bmatrix}.
\]

One-sided PSDF \( G_{ij}(f) \) \((i, j = 1, ..., 6)\) of the stress-state components are determined for frequency \( f \geq 0 \) and they are equal to double values of two-sided PSDF \( S_{ij}(f) \).
where

\[ G_{ij}(f) = \begin{cases} 2S_{ij}(f), & \text{for } 0 \leq f < \infty, \\ 0, & \text{for } f < 0 \end{cases} \]

and

\[ G_{ij}(f), S_{ij}(f) \] autospectral density function of stresses \(X_i(t)\),

\[ G_{ij}(f), S_{ij}(f) \] cross-spectral density function between stresses \(X_i(t)\) and \(X_j(t)\) for \(i \neq j\).

The cross PSDFs are complex functions

\[ G_{ij}(f) = \text{Re}[G_{ij}(f)] + j \text{Im}[G_{ij}(f)], \]

where \(\text{Re}[G_{ij}(f)]\) - coincident spectral density function, a real part of the complex function \(G_{ij}(f)\), \(\text{Im}[G_{ij}(f)]\) - quadratic spectral density function, an imaginary part of the complex function \(G_{ij}(f)\).

Since the relationship \(S_{ij}(f) = S_{ij}(-f)\) for \(i \neq j\) appears, for description of the frequency structure of the random stress (or strain) tensor twenty one PSDFs should be known.

### 3 The fatigue-failure criteria for multiaxial random loading

The fatigue-failure criteria presented in [14–18] are based on the assumption that fracture is influenced by stress- or strain-state components acting on the critical plane. The fatigue strength theory for multiaxial random loading can be written as

\[ S(t) = \left\{ D_{ij}(t), P_n, C_k \right\}, \]

where \((i, j = x, y, z; k, n = 1, 2, ...), D_{ij}(t)\) - components of the stress or strain tensor, stochastic processes, \(P_n\) - parameters determining the critical fracture plane position (the mean direction cosines of principal axes), \(C_k\) - parameters characterizing a material.

The surface of limit states determining fatigue life under random complex state of stress is described by the limit value of the fatigue-strength function

\[ \max_i S(t), \]

corresponding to the fatigue strength of the material under the alternating cycle of uniaxial tension-compression \(\sigma_{af}\) or alternating torsion \(\tau_{af}\). Expression (7) should also be read as a 100% quantile of the random variable \(S\).

On the basis of [14, 15] we can distinguish the following generalized criteria for fatigue failure under multiaxial random loading:
I. Generalized criterion of maximum shear and normal stresses in the critical fracture plane for long lifetime.

II. Generalized criterion of maximum shear and normal strains in the critical fracture plane for long and short lifetimes.

Assumptions underlying the two criteria can be written as:

a. The fatigue failure is determined by normal stress $\sigma(t)$ (normal strain $\varepsilon(t)$) and shear stress $\tau(t)$ (shear strain $\gamma(t)$) in a given direction $\bar{F}$ in the critical fracture plane with the normal $\bar{F}$.

b. Direction $\bar{F}$ in the critical plane coincides with the mean direction of maximum shear stress $\bar{F}$ and shear strain $\gamma(t)$.

c. In the limit state corresponding to fatigue strength the maximum value of linear combination of $\tau(t)$ and $\sigma(t)$ under multiaxial random loading satisfies the following equations:

- according to stress criterion I

$$\max_{\tau} \left\{ B\tau(t) + K\sigma(t) \right\} = F,$$  \hspace{1cm} (8)

- according to strain criterion II

$$\max_{\gamma} \left\{ b\varepsilon(t) + k\varepsilon(t) \right\} = q,$$  \hspace{1cm} (9)

where constants $B$ and $b$ are applied for selection of a particular form of (8) and (9), and $K$, $F$, $k$, $q$ are material constants determined in cyclic fatigue tests. The positions of unit vectors $\bar{F}$ and $\bar{F}$ are determined by the mean direction cosines of principal axes of strains or stresses.

Results of fatigue tests under multiaxial cyclic and random loading often testify that for brittle materials the fatigue-fracture plane is perpendicular to the mean direction of normal stress with the maximum amplitude. For plastic materials the fatigue-fracture plane is one of two planes in which shear stresses have the maximum amplitude. By selecting constants $B$, $K$, $F$ or $b$, $k$, $q$ and specifying the critical fracture-plane position one obtains particular forms of (8) and (9). The following particular forms of the generalized stress criterion and strain criterion can be distinguished:

1. Criterion of the maximum normal stress in the critical fracture plane determined by the mean position of the maximum principal stress $\sigma(t)$ [14] ($B = 0$, $K = 1$). The normal stress in the expected fracture plane is equal to the equivalent stress and expressed by

$$\sigma_{eq}(t) = \sigma_n(t) = \hat{\sigma}_n(t) + \hat{\sigma}_n(t) + \hat{\sigma}_n(t) + \hat{\sigma}_n(t) + \hat{\sigma}_n(t),$$  \hspace{1cm} (10)
where \( \hat{l}_i, \hat{m}_i, \hat{n}_i \) \( (n = 1, 2, 3) \) – mean direction cosines of principal axes of stresses or strains, \( \sigma_0(t) \) – components of the stress tensor, random processes.

2. Criterion of the maximum shear stress in the critical fracture plane determined by the mean position of one of two planes in which the maximum shear stress \( \tau(t) \) occurs [14], \( (B = 1, K = 0) \). Shear stress acting in the expected fracture plane is equal to

\[
\tau_{\text{eq}}(t) = \frac{\hat{l}_1^2 + \hat{l}_2^2}{2} \sigma_{xx}(t) + \frac{\hat{m}_1^2 + \hat{m}_2^2}{2} \sigma_{yy}(t) + \frac{\hat{n}_1^2 + \hat{n}_2^2}{2} \sigma_{zz}(t) \\
+ (\hat{l}_3 \hat{m}_1 - \hat{l}_1 \hat{m}_3) \sigma_{xy}(t) + (\hat{l}_3 \hat{n}_1 - \hat{l}_1 \hat{n}_3) \sigma_{xz}(t) + (\hat{m}_3 \hat{n}_1 - \hat{m}_1 \hat{n}_3) \sigma_{yz}(t)
\]

(11)

and according to this criterion

\[
\sigma_{\text{eq}} = 2 \tau_{\text{eq}}(t).
\]

(12)

3. Criterion of the maximum shear and normal stresses in the critical fracture plane [14], \( (B = 1) \). If the critical fracture plane coincides with the mean position of one of two planes in which the maximum shear stress \( \tau(t) \) acts, the equivalent stress can be written as

\[
\sigma_{\text{eq}}(t) = \frac{1}{1 + K} \left\{ \left[ \hat{l}_1^2 - \hat{l}_3^2 + K (\hat{l}_1^2 + \hat{l}_3^2) \right] \sigma_{xx}(t) \\
+ \left[ \hat{m}_1^2 - \hat{m}_3^2 + K (\hat{m}_1^2 + \hat{m}_3^2) \right] \sigma_{yy}(t) \\
+ \left[ \hat{n}_1^2 - \hat{n}_3^2 + K (\hat{n}_1^2 + \hat{n}_3^2) \right] \sigma_{zz}(t) \\
+ 2 (\hat{l}_3 \hat{m}_1 - \hat{l}_1 \hat{m}_3 + K (\hat{l}_1 + \hat{l}_3) (\hat{m}_1 + \hat{m}_3)) \sigma_{xy}(t) \\
+ 2 (\hat{l}_3 \hat{n}_1 - \hat{l}_1 \hat{n}_3 + K (\hat{l}_1 + \hat{l}_3) (\hat{n}_1 + \hat{n}_3)) \sigma_{xz}(t) \\
+ 2 (\hat{m}_3 \hat{n}_1 - \hat{m}_1 \hat{n}_3 + K (\hat{m}_1 + \hat{m}_3) (\hat{n}_1 + \hat{n}_3)) \sigma_{yz}(t) \right\}
\]

(13)

4. Criterion of the maximum normal strain in the critical fracture plane [15] determined by the mean position of the maximum principal strain \( \varepsilon_0(t) \), \( (b = 0, h = 1) \). The strain \( \varepsilon_0(t) \) in the direction \( \mathcal{F} \), perpendicular to the expected fracture plane is equal to the equivalent strain \( \varepsilon_{\text{eq}}(t) \) and can be expressed as

\[
\varepsilon_{\text{eq}}(t) = \varepsilon_{\text{eq}}(t) = \hat{l}_1^2 \varepsilon_{xx}(t) + \hat{m}_1^2 \varepsilon_{yy}(t) + \hat{n}_1^2 \varepsilon_{zz}(t) \\
+ 2 \hat{l}_1 \hat{m}_1 \varepsilon_{xy}(t) + 2 \hat{l}_1 \hat{n}_1 \varepsilon_{xz}(t) + 2 \hat{m}_1 \hat{n}_1 \varepsilon_{yz}(t)
\]

(14)

where \( \varepsilon_0(t) = \gamma_0(t)/2 \) – components of the strain tensor, random processes.
5. Criterion of the maximum shear strain in the critical fracture plane [15] determined by the mean position of one of two planes in which the maximum shear strain occurs \((b = 1, k = 0)\). The shear strain \(\varepsilon_{\gamma}(t)\) in the expected fracture plane is equal to

\[
\varepsilon_{\gamma}(t) = \frac{1}{2} \left[ (l^2 + \hat{l}_1^2) \varepsilon_{ss}(t) + (\hat{m}_1^2 + \hat{m}_3^2) \varepsilon_{ss}(t) + (\hat{n}_1^2 + \hat{n}_3^2) \varepsilon_{ss}(t) + (\hat{m}_1 \hat{n}_1 - \hat{m}_3 \hat{n}_3) \gamma_{\gamma}(t) + (\hat{m}_1 \hat{n}_1 - \hat{m}_3 \hat{n}_3) \gamma_{\gamma}(t) \right]
\]

and

\[
\varepsilon_{\gamma}(t) = \frac{2}{1 + \nu} \varepsilon_{\gamma}(t),
\]

where \(\nu\) is Poisson’s ratio.

6. Criterion of the maximum shear and normal strains in the critical fracture plane [15], \((b = 1, k = 1)\). If the critical plane is determined by the mean position of one of two planes in which the maximum shear strain \(\gamma_{\gamma}(t)\) occurs, the equivalent strain is expressed by

\[
\varepsilon_{eq}(t) = (l^2 + \hat{l}_1^2) \varepsilon_{ss}(t) + (\hat{m}_1^2 + \hat{m}_3^2) \varepsilon_{ss}(t) + (\hat{n}_1^2 + \hat{n}_3^2) \varepsilon_{ss}(t) + (\hat{m}_1 \hat{n}_1 - \hat{m}_3 \hat{n}_3) \gamma_{\gamma}(t) + (\hat{m}_1 \hat{n}_1 - \hat{m}_3 \hat{n}_3) \gamma_{\gamma}(t)
\]

In some particular cases, the above criteria agree with the criteria for cyclic loading. Replacing nine direction cosines, mutually related, by three Euler angles we obtain a reduced number of parameters used for description of the expected fatigue-fracture-plane position. New mathematical forms of the fatigue-failure criteria with a reduced number of independent parameters are presented in [16].

4 Statistical properties of the fatigue-failure criteria

Linear forms of the first three criteria expressed by (10), (12) and (13) allow us to present the equivalent stress in a general form as a sum of stress components multiplied by constant coefficients \(a_k\), dependent on criterion cosines of the axis of principal stresses

\[
\sigma_{eq}(t) = \sum_{k=1}^{6} a_k x_k(t),
\]

where \(a_k\) are coefficients given in Table 1.
Table 1: Coefficients $a_k$ for stress criterions.

<table>
<thead>
<tr>
<th></th>
<th>(10)</th>
<th>(12)</th>
<th>(13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$\hat{l}_1^2$</td>
<td>$\hat{l}_1^2 + \hat{l}_3^2$</td>
<td>$\frac{\hat{l}_1^2 - \hat{l}_3^2 + K(\hat{l}_1^2 + \hat{l}_3^2)}{1 + K}$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$\hat{m}_1^2$</td>
<td>$\hat{m}_1^2 + \hat{m}_2^2$</td>
<td>$\frac{\hat{m}_1^2 - \hat{m}_2^2 + K(\hat{m}_1^2 + \hat{m}_2^2)}{1 + K}$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$\hat{n}_1^2$</td>
<td>$\hat{n}_1^2 + \hat{n}_2^2$</td>
<td>$\frac{\hat{n}_1^2 - \hat{n}_2^2 + K(\hat{n}_1^2 + \hat{n}_2^2)}{1 + K}$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$2(\hat{l}_1\hat{m}_1)$</td>
<td>$2(\hat{l}_1\hat{m}_1 + \hat{l}_3\hat{n}_3)$</td>
<td>$\frac{2[\hat{l}_1\hat{m}_1 - \hat{l}_3\hat{n}_3 + K(\hat{l}_1 + \hat{l}_3)(\hat{m}_1 + \hat{n}_3)]}{1 + K}$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$2(\hat{l}_1\hat{n}_1)$</td>
<td>$2(\hat{l}_1\hat{n}_1 + \hat{l}_3\hat{n}_3)$</td>
<td>$\frac{2[\hat{l}_1\hat{n}_1 - \hat{l}_3\hat{n}_3 + K(\hat{l}_1 + \hat{l}_3)(\hat{n}_1 + \hat{n}_3)]}{1 + K}$</td>
</tr>
<tr>
<td>$a_6$</td>
<td>$2(\hat{m}_2\hat{n}_3)$</td>
<td>$2(\hat{m}_2\hat{n}_3 + \hat{m}_3\hat{n}_3)$</td>
<td>$\frac{2[\hat{m}_2\hat{n}_3 - \hat{m}_3\hat{n}_3 + K(\hat{m}_2 + \hat{m}_3)(\hat{n}_2 + \hat{n}_3)]}{1 + K}$</td>
</tr>
</tbody>
</table>

From (18) it appears that the equivalent stress $\sigma_{eq}(t)$ has the same probability distribution as the random tensor of stress $\lambda(t)$. In particular, if the vectorial process $\lambda(t)$ has the normal probability distribution $N(\tilde{\sigma},\mu_{eq})$ of (2) type, then the equivalent stress has the normal probability distribution $N(\tilde{\sigma}_{eq},\mu_{eq})$ with the following PDF

$$f_{\sigma_{eq}}(\sigma_{eq}) = \frac{1}{\sqrt{2\pi\sigma_{eq}}} \exp\left[ -\frac{(\sigma_{eq} - \tilde{\sigma}_{eq})^2}{2\mu_{eq}} \right],$$

(19)

where the mean value

$$\tilde{\sigma}_{eq} = \sum_{k=1}^{k} a_k \tilde{\sigma}_k,$$

(20)

and variance

$$\mu_{eq} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \mu_{eq} = \sum_{i=1}^{n} a_i^2 \mu_{eq} + 2 \sum_{j=i+1}^{n} a_i a_j \mu_{eq}.$$

(21)
From (20) it results that if \( \hat{x}_k = 0 \) \( (k = 1, ..., 6) \), then \( \hat{\sigma}_{eq} = 0 \). It should be emphasized that the zero mean value of the equivalent stress, according to three linear failure criteria, at the zero mean values of the stress state components is a result that is desirable and important from the physical point of view. Unfortunately, such a result cannot be obtained if the equivalent stress is calculated according to non-linear criteria with respect to \( \sigma_0(t) \), for example the criterion of maximum principal stress \( \sigma_1(t) \) [19].

While calculating the fatigue life, a frequency structure of the equivalent stress plays an important role. We are mainly interested in the frequency band width and distribution of mean square values of amplitudes of harmonic components in the random equivalent stress process. This information is included in the PSDF of the equivalent stress \( G_{\sigma_{eq}}(f) \), which is different for each criterion. PSDF of the equivalent stress \( G_{\sigma_{eq}}(f) \), assuming that the random components of stress state are correlated, can be determined from [10, 14]

\[
G_{\sigma_{eq}}(f) = \sum_{i=1}^{6} \sum_{j=1}^{6} a_i a_j G_{ij}(f) = \sum_{i=1}^{6} \left[ a_i^2 G_{ii}(f) + 2 \sum_{j=1}^{6} a_i a_j G_{ij}(f) \right].
\]  

(22)

If random components of stress are uncorrelated, the second term of (22) will be equal to zero, i.e.

\[
G_{\sigma_{eq}}(f) = \sum_{i=1}^{6} a_i^2 G_{ii}(f),
\]  

(23)

and the frequency band of the equivalent stress will depend only on the frequency bands of individual PSDF \( G_{ii}(f) \).

For example, according to the criterion of maximum normal stress on the critical fracture plane – (10) – from (22) and suitable transformations it results that the PSDF of the equivalent stress is equal to
Equation (23) allows us determine PSDF of the equivalent stress $G_{eq}(f)$ when

$$
G_{eq}(f) = \frac{1}{G_{ii}(f)} + \frac{1}{G_{jj}(f)} + \frac{1}{G_{ij}(f)}
$$

(24)

the direction cosines $\hat{l}_n, \hat{m}_n, \hat{n}_n$ ($n = 1, 2, 3$) determining the expected position of the critical fracture plane, the autospectral density function of components of the stress state tensor $G_{ij}(f)$, real parts of the cross-spectral density function of stresses Re$\{G_{ij}(f)\}$ are known.

For each frequency $f$ the following cross-spectrum inequality is correct

$$
|G_{ij}(f)|^2 \leq G_{ii}(f)G_{jj}(f).
$$

(25)

This results from the fact that the value of $|G_{ij}(f)|$ is limited for each value of frequency $f$ by suitable values of $G_{ii}(f)$ and $G_{jj}(f)$.

From (22)–(25) an important inequality concerning the frequency structure of the equivalent stress results

$$
f_{max,eq} \leq \max_{ij} \left\{ f_{max,ij} \right\}, \quad (i, j = 1, 2, \ldots, 6).
$$

(26)

This means that the maximum frequency of the equivalent stress $f_{max,eq}$ according to criteria (10), (12), (13) – is, at most, equal to the maximum frequency from all those occurring in the stress-state components $f_{max,ij}$. This results from linear criteria (10), (12) and (13) and means that while reducing the multiaxial stress state to the equivalent uniaxial one frequency bands of the stress-state components transform to the frequency bands of the equivalent stress without enlarging its width.

The equivalent stress calculated according to nonlinear multiaxial failure criteria with respect to $\sigma_{ij}(t)$ has the maximum frequency $f_{max,eq}$ larger than the maximum frequency $f_{max,ij}$ from among all the frequencies appearing in the stress-state components. Thus, the following inequality is valid.
This unfavorable increase of the frequency bandwidth of the equivalent stress caused serious difficulties while calculating fatigue life, especially while cycle counting and fatigue-damage accumulating. The increase of frequency causes a greater number of cycles to be counted in a conventional time unit and, in consequence, calculation of a larger fatigue damage. Thus, (27) a new, important limitation results for generalization of multiaxial cyclic fatigue criteria to random loading.

It should be stated here that the strain criteria equations (14), (16), (17) and the stress criteria equations (10), (12) and (13) have the same statistical properties and they can be determined in the same way.

5 The critical fracture-plane position

If the fatigue-failure criteria presented above are applied in practice, the expected direction of critical fracture plane (macro) should be known. It is assumed that the expected critical-plane position is determined by a matrix of the mean direction cosines of axes of principal stresses or strains. Such a matrix should satisfy the following requirements:

– the calculated values of the mean direction cosines should form the orthogonal matrix,
– the calculated values of the direction cosines should correspond to the fracture-plane direction obtained from fatigue tests.

Several methods of searching the matrix of mean direction cosines have been proposed so far:

– method of weight functions [21, 22],
– method of variance [23–25],
– method of damage accumulation [17].

The method of weight functions consists in averaging instantaneous values of angles $\alpha_n(t)$, $\beta_n(t)$, $\gamma_n(t)$, ($n = 1, 2, 3$) determining positions of axes of principal stresses or strain with some weight functions. The mean direction cosines are expressed by the following formulae:

\[
\begin{align*}
\hat{I}_n &= \cos \hat{\alpha}_n = \cos \left[ \frac{1}{W} \int_0^{\tau_n} \alpha_n(t)w(t)dt \right] \\
\hat{m}_n &= \cos \hat{\beta}_n = \cos \left[ \frac{1}{W} \int_0^{\tau_n} \beta_n(t)w(t)dt \right], \quad W = \int_0^{\tau_n} w(t)dt, \quad (n = 1, 2, 3) \\
\hat{n}_n &= \cos \hat{\gamma}_n = \cos \left[ \frac{1}{W} \int_0^{\tau_n} \gamma_n(t)w(t)dt \right]
\end{align*}
\]
where $T_0$ is the averaging time.

A number of weight functions were proposed and analysed for several materials. The best coincidence with the fatigue-test result was obtained with the following weight function

$$w(t) = \begin{cases} 
0 & \text{for } \sigma(t) < 0.5\sigma_{af} \\
\left(\frac{\sigma(t)}{0.5\sigma_{af}}\right)^n & \text{for } \sigma(t) \geq 0.5\sigma_{af}.
\end{cases}$$

(29)

In the averaging process this weight includes only some directions of maximum principal stress (or strain) and the influence of instantaneous values, $\sigma(t)$, fatigue limit $\sigma_{af}$ and slope of the $S-N$ curve, $m$, on the fatigue-fracture-plane position.

However, no averaging procedure gives, in general, an orthogonal matrix of the mean direction cosines, because only three of nine direction cosines are independent. It is difficult to say which three of the nine parameters should be averaged. Irrespective of the choice, the other six direction cosines must be calculated from the six nonlinear equations of orthogonality. In the papers [20–22] the authors have avoided the controversial problem of selecting three independent parameters by averaging the three Euler angles through the weight-function method.

In the method of variance it is assumed that the plane on which the maximum variance of the equivalent stress (or strain) appears is critical for the material and the fatigue fracture should be expected in this plane.

For a given covariance matrix of the random stress tensor $\mu$, the variance of the equivalent stress equation (21) is a nonlinear function of direction cosines $\hat{l}, \hat{m}, \hat{n}$. The cosines must satisfy conditions of orthogonality. Searching the maximum of the variance equation (21) depending on $\hat{l}, \hat{m}, \hat{n}$ consists in determination of the maximum of a nonlinear function with nonlinear limitations. It has been shown that – except for a particular situation – a general case of the problem cannot be efficiently solved with an analytical method and computer calculation must be applied. Several particular cases were analysed with use of the fatigue-failure criteria [23–25]. It has been shown that for each stationary random state of stress there is one or more planes that are critical for the material and in which the variance of the equivalent stress reaches its maximum.

The method of damage accumulation consists in calculation of lifetime $T$ on the basis of the equivalent stress (or strain) in many planes. The planes in which the calculated fatigue life reaches its minimum value are the expected fatigue-fracture planes. In this method the lifetime $T$ is also determined in this way. Unfortunately, this method is inconvenient because calculations are very complicated, for example an optimization algorithm for effective searching $T_{\text{min}}$ is necessary.
6 The spectral method of fatigue-life calculation in frequency domain

Under multiaxial random loading, fatigue-life calculations include reduction of a three- or two-dimensional stress state to the equivalent uniaxial one with suitable fatigue failure criteria using a concept of the critical plane. The critical-plane position is determined by the mean direction cosines of the principal stress axes \( \hat{l}_n, \hat{m}_n, \hat{n}_n \) – the life calculated for this position is the least (Fig. 1).

\[
G_{ij}(f) = \begin{bmatrix}
G_{xx}(f) & G_{xy}(f) & G_{xz}(f) \\
G_{xy}(f) & G_{yy}(f) & G_{yz}(f) \\
G_{xz}(f) & G_{yz}(f) & G_{zz}(f)
\end{bmatrix}
\]

\[
\text{PSDF matrix of stress tensor}
\]

\[
\text{Determination of the critical plane position} (\hat{l}_n, \hat{m}_n, \hat{n}_n)
\]

\[
G_{\sigma_{eq}}(f) = \sum_{i=1}^{6} \sum_{j=1}^{6} a_i a_j G_{ij}(f)
\]

\[
\text{PSDF of equivalent stress}
\]

\[
\text{Fatigue failure criteria}
\]

\[
m_k = \int_{0}^{\infty} G_{\sigma_{eq}}(f) f^k df
\]

\[
\text{Determination of moments } m_k
\]

\[
N^*_p, M^*, I, G_{\sigma_{eq}}(f)
\]

\[
\text{and other quantities}
\]

\[
\text{Damage calculating}
\]

1. S-N curve
2. Analytical formula – lifetime

\[
T_{np} = \frac{A}{(2\mu_{\sigma_{eq}})^{\frac{m}{2}} \Gamma \left( \frac{m + 2}{2} \right) f_0}
\]

Figure 1: Flowchart for calculation of fatigue life in frequency domain.
Several analytical formulae based on PSDF of stresses have been proposed for calculations of long life time under uniaxial random loadings. We can mention, for example, formulae given by Miles [9], Kowalewski [7], Raicher [11], Bolotin [5, 6] and others [4, 8]. However, they should be applied very carefully because some mistakes in evaluation of fatigue life are possible. Thus, if possible, preliminary fatigue tests should be done for a given material and a type of loading to choose a correct formula in the analysed case. For example, in [26] specimens made of 15G2ANb and 18G2A steels were tested under uniaxial random tension-compression in order to verify efficiency of formulae proposed by Miles, Kowalewski and Raicher. For these materials and the wide-band spectrum of loading, Raicher’s formula seemed to be the most efficient.

Extension of the analytical methods of fatigue-life evaluation for multiaxial random loading consists in application of PSDF of the equivalent stress (or strain) in formulae applied so far for uniaxial random fatigue. Using only some known formulae for the spectral methods of fatigue-life evaluation we obtain the following equations for lifetime (in seconds):

- according to Miles (for narrow-band Gaussian processes)
  \[ T = \frac{A}{\left(2\mu_{\sigma_m}\right)^2} \Gamma\left(\frac{m+2}{2}\right)f_0, \]  \hspace{1cm} (30)

- according to Kowalewski (for wide-band Gaussian processes)
  \[ T = \frac{A}{\left(2\mu_{\sigma_m}\right)^2} \Gamma\left(\frac{m+2}{2}\right)M^T I^m, \]  \hspace{1cm} (31)

- according to Raicher (for wide-band Gaussian processes)
  \[ T = \frac{A}{\left(2\mu_{\sigma_m}\right)^2} \Gamma\left(\frac{m+2}{2}\right)\left(\int_0^{\infty} G_0(f)f^2 df\right) I^m, \]  \hspace{1cm} (32)

- according to Miles–Bolotin (for wide-band Gaussian processes)
  \[ T = \frac{A}{\left(2\mu_{\sigma_m}\right)^2} \Gamma\left(\frac{m+2}{2}\right)\left(\int_0^{\infty} G_0(f)f^2 df\right)^{1/2}, \]  \hspace{1cm} (33)
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− according to Chaudhury–Dover

\[
T = \frac{A}{M^+ \left( 2 \mu_{\sigma_{eq}} \right)^\frac{m}{2} \sqrt{1 - I^2} \Gamma \left( \frac{m+1}{2} \right) + \frac{3I}{4} \Gamma \left( \frac{m+2}{2} \right) },
\]

(34)

\[
m_i = \int_0^\infty G_{\sigma_{eq}}(f) f^i df, \quad \mu_{\sigma_{eq}} = m_0, \quad N_o^+ = \frac{m_2}{m_0},
\]

\[
M^+ = \sqrt{\frac{m_1}{m_2}}, \quad I = \frac{N_o^+}{M^+}, \quad G_0(f) = \frac{G_{\sigma_{eq}}(f)}{\mu_{\sigma_{eq}}},
\]

where:

- \( A, m \) – parameters of the standard fatigue curve \( N\sigma_a^n = A \),
- \( \mu_{\sigma_{eq}} \) – variance of the equivalent stress expressed by (21),
- \( \Gamma() \) – gamma function,
- \( f_0 \) – dominating frequency of narrow band stress history \( \sigma_{eq}(t) \),
- \( M^+ \) – the expected rate of peaks of \( \sigma_{eq}(t) \),
- \( I \) – irregularity factor of \( \sigma_{eq}(t) \),
- \( N_o^+ \) – the expected rate of zero crossing with (+) slope of \( \sigma_{eq}(t) \),
- \( G_0(f) \) – normalized PSDF of \( \sigma_{eq}(t) \),
- \( G_{\sigma_{eq}}(f) \) – PSDF of \( \sigma_{eq}(t) \) determined by (22),
- \( f_{\text{max}} \) – maximum frequency of wide-band history of \( \sigma_{eq}(t) \),
- \( m_k \) – the \( k \)-th moment of PSDF of \( \sigma_{eq}(t) \).

The proposed spectral method of fatigue-life calculation under random multiaxial loading requires more tests because the range of its applicability must be determined.

7 Fatigue-life calculation in time domain

The algorithm of fatigue-life assessment under random loading in the time domain is shown in Fig. 2 [17]. Some versions of this algorithm have been already verified for some chosen steels [12, 25]. In the case of fatigue-life calculation under multiaxial random loading we can meet many problems, for example determination of the critical-plane position. We can use one of the proposed three methods (method of variance, method of damage accumulation or method of weight functions) in such a case.
At the first stage, histories of stress-tensor components \( \sigma_{ij}(t) \) are generated. At the next stage, the critical-plane direction is calculated with one of three proposed methods (method of variance, damage accumulation or weight functions). We can apply one of the strain or stress criteria. Next, we schematize a random history of the equivalent stress with the rain-flow method.

During damage accumulation (the fifth stage) we apply one of the known hypotheses, in this chapter they are

---

**Figure 2**: Flowchart for calculation of fatigue life in time domain.
− linear Palmgren–Miner hypothesis

\[
S_{PM}(T_0) = \left\{ \begin{array}{ll}
\sum_{i=1}^{n} \frac{n_i}{N_0 \left( \frac{\sigma_{af}}{\sigma_{ai}} \right)^m} & \text{for } \sigma_{ai} \geq a_{PM} \sigma_{af} \\
0 & \text{for } \sigma_{ai} < a_{PM} \sigma_{af}
\end{array} \right.
\]

for \(0 < \alpha < 1\), (35)

where

- \(n_i\) – number of cycles with amplitudes \(\sigma_{ai}\) at \(T_0\),
- \(m\) – coefficient of the Wöhler’s curve slope,
- \(N_0\) – number of cycles corresponding to the fatigue limit \(\sigma_{af}\), and
- \(a_{PM} = 0.5\) – coefficient including influence of amplitudes less than \(\sigma_{af}\), and

− linear Serensen–Kogayev hypothesis

\[
S_{SK}(T_0) = \left\{ \begin{array}{ll}
\sum_{i=1}^{n} \frac{n_i}{b N_0 \left( \frac{\sigma_{af}}{\sigma_{ai}} \right)^m} & \text{for } \sigma_{ai} \geq a_{SK} \sigma_{af} \\
0 & \text{for } \sigma_{ai} < a_{SK} \sigma_{af}
\end{array} \right.
\]

for \(b \geq \alpha\), (36)

In (36), the Serensen–Kogayev coefficient

\[b = \frac{\sum_{i=1}^{n} \sigma_{ai} t_i - a_{SK} \sigma_{af}}{\sigma_{a_{max}} - a_{SK} \sigma_{af}}\]

for \(b > 0.1\), where \(\sigma_{a_{max}}\) means the maximum amplitude determined from a history of loading, and \(t_i = \frac{n_i}{\sum n_i}\) is the frequency of occurrence of particular levels \(\sigma_{ai}\) at realization.

After determination of the damage degree \(S(T_0)\) at observation time \(T_0\) according to (35) or (36), the fatigue life is calculated

\[T_{RF} = \frac{T_o}{S(T_0)}.\]  

(37)

**8 Simulation tests**

The new algorithm, Fig.1, using the spectral method for fatigue-life determination was analysed for the random multiaxial stress state. It was analysed together with the known cycle-counting method, Fig. 2, according to
the flowchart presented in Fig. 3. Some blocks of these two algorithms are similar and they have been compared.

Figure 3: Algorithm for fatigue-life determination applied for simulation calculations with the marked blocks (CMP1, CMP2 and CMP3) for comparison of the cycle-counting method and the spectral method.

The following conditions of simulation were assumed:
1. The generated stress tensors have normal distributions with narrow- and wide-band frequency spectra.
2. Suitable components of the stress tensors have different correlation coefficients \( r = 1, r = -1 \) and \( r = 0 \).
3. The following stress states were considered
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- biaxial tension-compression (S1-S8),
- uniaxial tension-compression with torsion (S9-S12),
- biaxial tension-compression with torsion (S13-S16),
- full three-dimensional stress state (S17-S24).

8.1 Block 1 – loading

Components of the stress tensor, (1) with normal probability distribution and matrices of PSDF, (3) were the input data in simulation. The generated stress histories \( \sigma_{ij}(t) \) were filtered with the Chebyshev band filter of the first order. Depending on the filter settings, narrow- or wide-band courses were obtained. The matrix (3) of PSDF of stress-state components \( G_{kl}(f) \), \((k, l = 1, ..., 6)\) was estimated by the Welch method [27] from the stress histories \( \sigma_{ij}(t) \).

8.2 Block 2 – determination of the critical-plane position

During simulation, the maximum variance method was applied [23–25]. In this method, the maximum of the (21) is searched, where \( a_k \) and \( a_l \) are coefficients from Table 1. Elements of the covariance matrix \( \mu_{kl} \) are obtained directly from stress histories \( \sigma_{ij}(t) \) or from the PSDF matrix (3).

8.3 Block CMP1 – comparison of calculated critical-plane positions

If the equivalent stress histories should be determined, we must know the critical-plane position, namely direction cosines of the normal vector \( \vec{n} = \hat{n}_x, \hat{n}_y, \hat{n}_z \), when (10) is used. In order to determine the critical-plane position with the variance method we must know a covariance matrix of the stress-tensor components. In the cycle-counting method, the covariance matrix is calculated from time histories \( \sigma_{ij}(t) \) according to the following equation

\[
\mu_{kl} = E[(x_k - \hat{x}_k)(x_l - \hat{x}_l)], \quad (k, l = 1, ..., 6),
\]

and in the case of the spectral method, it is calculated directly from the matrix of power-spectral densities

\[
\mu_{kl} = \text{Re}\left[ \int_0^\infty G_{kl}(f) df \right], \quad (k, l = 1, ..., 6).
\]

For the chosen criterion, (10), it is enough to know the vector normal to the plane \( \vec{n} \) for determination of its position. Thus, the direction cosines were generated in the way allowing to fill a half of the sphere described by the end of the vector \( \vec{n} \) normal to the critical plane, Fig. 4.
During simulation, variance of the equivalent stress $\mu_{\sigma_{eq}}$ was determined for all possible positions of the critical plane. In all the considered cases (S1-S24), a good agreement between both calculation methods was observed.

8.4 Block 3 – determination of the equivalent values

In the third block of the algorithm, the equivalent values of stress histories $\sigma_{eq}(t)$ and PSDF $G_{eq}(f)$ are determined from (10) and (24), respectively.

8.5 Block CMP2 - analysis of the equivalent quantities

In the third block of the algorithm for fatigue-life determination with the cycle-counting method, Fig. 3, we obtain histories of the equivalent stresses $\sigma_{eq}(t)$, and in the third block of the spectral method – PSDF $G_{eq}(f)$. We obtain them using the criterion of multiaxial fatigue in the time and frequency domains, respectively. However, direct comparison of these two quantities is not possible because of the differences between them. Thus, the time history $\sigma_{eq}(t)$ was determined from PSDF $G_{eq}(f)$ and PSDF $G_{eq}^{FFT}(f)$ – from the equivalent stress $\sigma_{eq}(t)$. The obtained pairs of values were compared. For exact presentation of a sequence of calculations necessary for determination of these quantities, the block CMP2 from Fig. 3 has been developed and presented in Fig. 5.
The history of $\sigma_{eq}^{IFFT}(t)$ was obtained with the inverse Fourier transform (IFFT) method. It uses a fast algorithm of the inverse Fourier transform. The algorithm generates time histories based on harmonic-component amplitudes and their phase shifts. The harmonic-component amplitudes $A_i$ were obtained directly from the estimator of PSDF

$$A_i = \sqrt{2G_{eqi}\Delta f},$$  \hspace{1cm} (40)

where:

- $A_i$ – amplitude of the $i$-th harmonic component,
- $G_{eqi}$ – $i$-th value of the estimator of PSDF,
- $\Delta f$ – distance between successive discrete values of the PSDF – frequency interval.

Phase shifts were obtained by generation of random numbers with uniform distribution from the range $\langle 0, \ldots, 2\pi \rangle$.

Comparison of histories $\sigma_{eq}(t)$ and $\sigma_{eq}^{IFFT}(t)$ was done in some stages. At first, it was checked if both histories had the same probability distributions. In this order, the Kolmogorov–Smirnov significance test of probability distribution of two random variables was done where the significance level $\alpha = 0.05$ was assumed. The maximum of the distribution-function difference of the considered variables was tested. The null hypothesis for this test is that $\sigma_{eq}(t)$ and $\sigma_{eq}^{IFFT}(t)$ have the same continuous distributions. From the tests it results that we cannot reject the hypothesis that the distributions are the same. Figure 6a shows graphs of probability distribution of instantaneous values for the case S19.

Next, the amplitude distributions from histories $\sigma_{eq}(t)$ and $\sigma_{eq}^{IFFT}(t)$ obtained with use of the rain-flow algorithm were compared, Fig. 6b. In this order, amplitudes were determined from both compared histories and the Kolmogorov–Smirnov test was done. Also in this case the mentioned null hypothesis cannot be rejected.
During comparison of the obtained power-spectral densities also their moments \( m_k \), were compared since they were used for determination of statistical parameters applied in the spectral equations for fatigue-life calculations. In all the cases, the same values of estimators of PSDF for the equivalent quantities were obtained. Such a result could be expected. From the comparison it also results that numerical calculations were accurate and in further considerations any mistakes can be precluded. For example, the calculated parameters are given in Table 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( G_{eq}(f) )</th>
<th>( G_{eq}^{FFT}(f) )</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_0 ) (( \text{MPa}^2 ))</td>
<td>5 102</td>
<td>5 091</td>
<td>11</td>
</tr>
<tr>
<td>( m_1 ) (( \text{MPa}^2 \text{s}^{-1} ))</td>
<td>102 040</td>
<td>101 801</td>
<td>239</td>
</tr>
<tr>
<td>( m_2 ) (( \text{MPa}^2 \text{s}^{-2} ))</td>
<td>2 041 218</td>
<td>2 036 244</td>
<td>4 974</td>
</tr>
<tr>
<td>( m_3 ) (( \text{MPa}^2 \text{s}^{-3} ))</td>
<td>40 844 453</td>
<td>40 741 013</td>
<td>103 440</td>
</tr>
<tr>
<td>( m_4 ) (( \text{MPa}^2 \text{s}^{-4} ))</td>
<td>817 526 167</td>
<td>815 379 325</td>
<td>2 146 842</td>
</tr>
<tr>
<td>( N_0^\uparrow ) (s(^{-1}))</td>
<td>20.001</td>
<td>19.999</td>
<td>0.002</td>
</tr>
<tr>
<td>( M^\uparrow ) (s(^{-1}))</td>
<td>20.013</td>
<td>20.011</td>
<td>0.002</td>
</tr>
<tr>
<td>( I )</td>
<td>0.999</td>
<td>0.999</td>
<td>0.0</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>0.034</td>
<td>0.034</td>
<td>0.0</td>
</tr>
</tbody>
</table>

8.6 Block 4 – cycle counting and statistical parameters of amplitudes

The rain-flow algorithm was used for schematization of the equivalent stress history \( \sigma_{eq}(t) \). Statistical parameters connected with the equivalent stress history are determined directly from PSDF, \( G_{eq}(f) \). For this purpose, we use the moments of PSDF of the equivalent stress \( m_k \). Next, we determine the equivalent history variance \( \mu_{\sigma_{eq}} \), the expected number of passages through the zero level with a positive slope in time unit \( N_0^\uparrow \), the expected number of peaks (local maxima) in a time unit \( M^\uparrow \), the coefficient of irregularity \( I \) and the spectrum-width parameter \( \zeta = \sqrt{1 - I^2} \).
8.7 Block 5 and CMP3 – fatigue-life calculations

The calculated fatigue life is the time (in seconds) in which the considered material will be damaged with probability 50%. In the cycle-counting method, the Palmgren–Miner linear hypothesis of damage accumulation was used. In the spectral method, the Miles equation [9] modified with the parameter $\lambda$ proposed by Wirsching and Light [2] was applied

$$T_{sp} = \frac{A}{\lambda M^{\frac{m}{2}} \left( \frac{m + 2}{2} \right)},$$

(41)

where $\lambda(m, \zeta) = c(m) + [1 - c(m)](1 - \zeta)^{d(m)}$, $c(m)$ and $d(m)$ are empirical functions determined with computer simulations, $c(m) = 0.926 - 0.033m$ and $d(m) = 1.587m - 2.323$. A form of the spectral equation is dependent on a loading type, especially on the loading spectrum width. The results of simulations are compared in block CMP3, Fig. 7. As can be seen, both methods give similar fatigue lives.

![Figure 7: Comparison of fatigue lives obtained by the cycle-counting method $T_{RF}$ and the spectral method $T_{SP}$ for different simulated stress states.](image)
9 Experimental verification

18G2A steel was tested. The main material constants for the steel are given in Table 3.

Table 3: Selected static and fatigue properties of 18G2A steel.

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ (GPa)</td>
<td>$\nu$</td>
<td>$\sigma_{af}$ (MPa)</td>
<td>$N_0$ (cycle)</td>
<td>$m$</td>
<td>$\tau_{af}$ (MPa)</td>
<td>$A$ ((MPa)$^m$)</td>
</tr>
<tr>
<td>210</td>
<td>0.30</td>
<td>271</td>
<td>2 735 000</td>
<td>7.19</td>
<td>175</td>
<td>8.511e23</td>
</tr>
</tbody>
</table>

Fatigue tests under random loading were performed on the test stands MZGS100L and MZGS200L for combined bending with torsion. In these tests the histories of Gaussian distributions and narrow frequency bands were used. The dominating frequency was 20 Hz. The tests were performed for different coefficients of correlation $r_{\sigma\tau}$ between histories of normal stresses $\sigma(t)$ and shear stresses $\tau(t)$ and different variances $\mu_{\sigma}$ and $\mu_{\tau}$. The test results are shown in Table 4.

The calculated life was determined with the cycle-counting method and the spectral method. Calculations were done in the same way as during simulation (Fig. 3). The calculations were based on the histories of nominal stresses coming from bending $\sigma(t)$ and torsion $\tau(t)$. The modified criterion of maximum normal stress in the critical plane was used [17]. In the considered case, the criterion leads to the following expression of the equivalent stress

$$\sigma_{eq}(t) = l_2^2 \sigma(t) + 2l_2 m_2 \frac{\sigma_{af}}{\tau_{af}} \tau(t).$$

The modification includes a difference between fatigue limits for bending $\sigma_{af}$ and torsion $\tau_{af}$. The obtained history of the equivalent stress was schematized by the rain flow algorithm. The determined cycles were applied for damage accumulation according to the linear Serensen–Kogayev hypothesis [3] and the following equation for fatigue life was obtained

$$T_{RF} = \frac{T_{bA}}{\sum_{i=1}^{n} \sigma_{eqi}^m} \text{ for } \sigma_{eq} \geq a_{SK} \sigma_{af}.$$ (43)

In order to determine the life with the spectral method, we derived a suitable spectral formula, assuming that the loading had a normal probability distribution with the narrow-band frequency spectrum and damage was accumulated according to the Serensen–Kogayev hypothesis. The following final formula was obtained:

$$T_{sp} = \frac{bA}{M \left(\frac{2 \mu_{\sigma_{eq}}^2}{2} \Gamma \left(\frac{m + 2}{2}, a_{SK} \sigma_{eq}^2 \right) \right)}.$$ (44)
Table 4: Fatigue test results.

<table>
<thead>
<tr>
<th>Test (test symbol)</th>
<th>$\sqrt{\mu_r}$ (MPa)</th>
<th>$\sqrt{\mu_s}$ (MPa)</th>
<th>$T_{exp}$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bending (E01)</td>
<td>150.6</td>
<td>-</td>
<td>13 560</td>
</tr>
<tr>
<td></td>
<td>127.3</td>
<td>-</td>
<td>55 500</td>
</tr>
<tr>
<td></td>
<td>149.4</td>
<td>-</td>
<td>9 660</td>
</tr>
<tr>
<td></td>
<td>113.4</td>
<td>-</td>
<td>195 720</td>
</tr>
<tr>
<td></td>
<td>133.2</td>
<td>-</td>
<td>43 260</td>
</tr>
<tr>
<td></td>
<td>134.4</td>
<td>-</td>
<td>43 080</td>
</tr>
<tr>
<td></td>
<td>134.4</td>
<td>-</td>
<td>35 340</td>
</tr>
<tr>
<td></td>
<td>139.8</td>
<td>-</td>
<td>21 240</td>
</tr>
<tr>
<td></td>
<td>111.1</td>
<td>-</td>
<td>316 200</td>
</tr>
<tr>
<td></td>
<td>133.0</td>
<td>-</td>
<td>33 660</td>
</tr>
<tr>
<td></td>
<td>128.1</td>
<td>-</td>
<td>62 100</td>
</tr>
<tr>
<td>Torsion (E02)</td>
<td>-</td>
<td>66.0</td>
<td>448 020</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>63.0</td>
<td>653 760</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>63.0</td>
<td>226 800</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>64.0</td>
<td>352 200</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>65.0</td>
<td>219 360</td>
</tr>
<tr>
<td>Bending with torsion $r_{sr} = -0.01$ (E03)</td>
<td>96.8</td>
<td>54.2</td>
<td>191 880</td>
</tr>
<tr>
<td></td>
<td>96.8</td>
<td>54.2</td>
<td>144 240</td>
</tr>
<tr>
<td></td>
<td>118.3</td>
<td>66.2</td>
<td>30 600</td>
</tr>
<tr>
<td></td>
<td>118.3</td>
<td>66.3</td>
<td>35 100</td>
</tr>
<tr>
<td></td>
<td>106.5</td>
<td>59.6</td>
<td>118 980</td>
</tr>
<tr>
<td></td>
<td>104.0</td>
<td>58.2</td>
<td>141 300</td>
</tr>
<tr>
<td>Bending with torsion $r_{sr} = -0.01$ (E04)</td>
<td>57.9</td>
<td>56.2</td>
<td>223 320</td>
</tr>
<tr>
<td></td>
<td>57.9</td>
<td>56.2</td>
<td>416 400</td>
</tr>
<tr>
<td></td>
<td>57.9</td>
<td>56.2</td>
<td>295 740</td>
</tr>
<tr>
<td></td>
<td>63.7</td>
<td>61.8</td>
<td>168 360</td>
</tr>
<tr>
<td></td>
<td>57.9</td>
<td>56.2</td>
<td>303 660</td>
</tr>
<tr>
<td>Bending with torsion $r_{sr} = 1.0$ (E05)</td>
<td>108.8</td>
<td>54.4</td>
<td>20 700</td>
</tr>
<tr>
<td></td>
<td>108.0</td>
<td>54.0</td>
<td>22 800</td>
</tr>
<tr>
<td></td>
<td>107.4</td>
<td>53.7</td>
<td>20 580</td>
</tr>
<tr>
<td></td>
<td>96.3</td>
<td>48.1</td>
<td>41 820</td>
</tr>
<tr>
<td></td>
<td>100.0</td>
<td>50.0</td>
<td>25 020</td>
</tr>
<tr>
<td></td>
<td>89.2</td>
<td>44.6</td>
<td>81 720</td>
</tr>
<tr>
<td></td>
<td>92.0</td>
<td>46.0</td>
<td>92 280</td>
</tr>
<tr>
<td></td>
<td>79.8</td>
<td>39.9</td>
<td>380 220</td>
</tr>
</tbody>
</table>
where \( b = \frac{\sqrt{2\mu_{\sigma_{eq}}} \Gamma \left( \frac{3}{2}, \frac{\sigma_{eq}^2 \sigma_{\mu}^2}{2\mu_{\sigma_{eq}}} \right)}{\sigma_{eq_{\max}} - \sigma_{\mu_{\sigma_{eq}}}} \) is the Serensen–Kogayev coefficient in the spectral approach and \( \Gamma(w_1, w_2) = \int_{w_2}^{w_1} e^{-t} t^{w_1-1} dt \) is the incomplete gamma function and \( \sigma_{eq_{\max}} = 3.73 \sqrt{\mu_{\sigma_{eq}}} \).

The calculation results were compared with the test results in Figs. 8 and 9. Both methods give the same results close to the experimental ones, included in the scatter band of the coefficient 3.

Figure 8: Comparison of the fatigue life calculated according to the cycle-counting method \( T_{RF} \) with the experimental life \( T_{exp} \).
Figure 9: Comparison of the fatigue life calculated according to the spectral method $T_{sp}$ with the experimental life $T_{exp}$.

10 Conclusions

The proposed spectral method of fatigue-life calculation under random multiaxial loading is an extension of the known method based on the power-spectral density function (PSDF) of stresses under uniaxial random loading.

The extension mentioned above consists in application of PSDF of the equivalent stress (or strain), determined according to a fatigue-failure criterion for multiaxial random loading, in the known formulae.

The discussed fatigue-failure criteria, owing to their linear combination of stress- or strain-state components, have suitable statistical properties that make it possible to calculate fatigue life under multiaxial random loading with use of the equivalent stress or strain history.

While reducing the multiaxial state of stress to the uniaxial one with the discussed linear fatigue-failure criteria the frequency bands of stress- and strain-
state components are transformed to the frequency band of equivalent stress and strain without increase of its width. Such a favorable result cannot be obtained if the equivalent stress or strain is calculated according to the nonlinear multiaxial fatigue-failure criteria.

Comparison of fatigue lives obtained by the spectral method and by the cycle-counting method from computer simulation tests under proportional and non-proportional random loading with three different correlation coefficients between stress-state components \( r = -1, 0 \) and 1), with narrow- and wide-band frequency spectra, containing: biaxial tension-compression (8 variants), uniaxial tension-compression with torsion (4 variants), biaxial tension-compression with torsion (4 variants) and full three-dimensional stress state (8 variants) give similar results.

The conducted tests on 18G2A structural steel under random bending, torsion and combined bending with torsion for two different coefficients of correlation \( r_{\sigma_{\tau}} = -0.01 \) and 1) between normal and shear stresses and different rates of its variances have demonstrated that both spectral and cycle-counting method give the same fatigue lives close to the experimental ones, included in the scatter band of the coefficient 3.

## 11 Acknowledgements

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## References


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