Instability of flows

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Abstract

A large body of published work on the instability of laminar flows is available in the open literature. The purpose of this chapter is to present the basic equations and the mathematical techniques in a systematic manner to study the instability of both the viscous and inviscid flows. The chapter begins with the consideration of the Lorenz equations to show the dependency of instabilities on some parameters for dissipative dynamical systems. Failure of the eigenvalue analysis in some important cases, e.g., flow in a circular pipe, has been highlighted in view of the research carried out in the last ten years.

1 Introduction

In recent times the theoretical analysis of the instabilities of laminar flows from the viewpoint of small perturbation theory has reached a sort of maturity and therefore the most pertinent results only can be summarized. The basic analytical technique of the linear stability started almost a century ago through the seminal works of Rayleigh [1], Reynolds [2], Orr [3] and Sommerfeld [4]. Their works were further extended by Taylor [5], Tollmien [6], Schlichting [7], Squire [8], Lin [9], Stuart [10], and Landau [11]. This chapter starts from the basic equations in vector-invariant form so as to have equations written out in any coordinate system, e.g., refer to Warsi [12]. The problem of instability is then introduced by two illustrative problems. The first illustrative problem gives a global sense of instability and it depends on the Reynolds–Orr equation. The subsequent analysis essentially follows the technique of Serrin [13]. The second illustrative problem depends on the Lorenz [14] equations that have been used to show the initiation of instability by increasing the Rayleigh number. The Lorenz equations are first derived in
Instability of Flows

a comprehensive manner to show their link to Rayleigh–Benard flow [15]. It must be emphasized that the purpose here is not to repeat the works on chaos and strange attractors as done by Ruelle and Takens [16] among others, but to establish the initiation of instability in nonlinear dissipative dynamical systems. The rest of this chapter is devoted to linear stability and the solution of the Orr–Sommerfeld equation. At the end of the chapter we have pointed out a few anomalies of the linear theory. From these anomalies the main conclusion is that the classical linear stability theory depends in a very sensitive way on the nature of the basic laminar flow.

2 The Navier–Stokes system of equations

The system of equations that uses the constitutive equation due to Stokes has been described in practically all the available texts and monographs on fluid dynamics. Nevertheless, for the purpose of remaining consistent in notation we shall use the description as given in Warsi [12]. Since the purpose of this chapter is to study the problem of instability of incompressible flows, the pertinent equations for all considerations are

\[
\text{div } \mathbf{u} = 0, \quad (1)
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + \text{div}(\mathbf{u}\mathbf{u}) = -\frac{1}{\rho} \text{grad } p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (2)
\]

\[
\rho \frac{de}{dt} = \Phi + k \nabla^2 T, \quad (3)
\]

where \( \mathbf{u} \) is the velocity vector, \( p \) is the static pressure, and \( \mathbf{f} \) is the body force per unit mass. Furthermore, \( \rho, \nu = \mu/\rho, k, \) and \( T \) are density, kinematic viscosity, thermal conductivity, and absolute temperature, respectively. The quantity \( e \) is the specific internal energy that in terms of the fluid specific heat \( c \) is

\[
de = c \, dT, \quad (4a)
\]

and \( \Phi \) is the rate of dissipation of energy due to viscosity

\[
\Phi = 2 \mu \mathbf{D} : \mathbf{D}, \quad (4b)
\]
\[D = \frac{1}{2} \left[ \text{grad } u + (\text{grad } u)^T \right], \quad (4c)\]

where \(D\) is the rate-of-strain tensor. Following Gibbs [17],

\[D : D = D_{mn} D_{pq} \delta_{mp} \delta_{nq} = D_{mn} D_{mn}.\]

It is important to emphasize that the gradient of a vector, say \(u\), is defined in terms of a general coordinate system \(x^k\) with the basis vectors \(a_k\) as

\[\text{grad } u = \frac{\partial u}{\partial x^m} a_m, \quad (\text{grad } u)^T = a_m \frac{\partial u}{\partial x^m}, \quad (4d)\]

and in rectangular Cartesian system \(x_k\) with constant basis vectors \(i_k\) as

\[\text{grad } u = \frac{\partial u}{\partial x_m} i_m, \quad (\text{grad } u)^T = i_m \frac{\partial u}{\partial x_m}. \quad (4e)\]

Also

\[\nabla^2 () = \text{div} \left( \text{grad} () \right) = \frac{\partial^2}{\partial x^m \partial x^m}, \quad (4f)\]

where two repeated indices imply a sum. In particular

\[\text{div } u = \frac{\partial u}{\partial x_m} \cdot i_m, \quad \text{div } T = \frac{\partial T}{\partial x_m} \cdot i_m.\]

Equations (1)–(3) are to be solved under some prescribed initial and boundary conditions. For the velocity \(u\) at position \(r = (x_k)\),

\[u = u_0 (r, 0), \quad u = u_B (r, t) \text{ at the boundary.} \quad (5)\]

These conditions are very important and form a separate topic of study in their own right.

### 2.1 Equations of the perturbations

Suppose \(U(r, t), P(r, t)\) is a laminar solution of eqns (1) and (2) under the prescribed conditions (5). Then

\[\text{div } U = 0, \quad (6a)\]
\[ \frac{\partial U}{\partial t} + \text{div}(UU) = -\text{grad}\left(\frac{P}{\rho}\right) + \nu \nabla^2 U + f. \]  

(6b)

Let \( u' \) be a superposed disturbance on the laminar solution \( U \). Then

\[ u = U + u', \quad p = P + p', \]

and its use in eqns (1) and (2) along with eqns (6) yields

\[ \text{div } u' = 0, \]  

(6c)

\[ \frac{\partial u'}{\partial t} + \text{div}(UU') + \text{div}(u'U) + \text{div}(u'u') = -\text{grad}\left(\frac{p'}{\rho}\right) + \nu \nabla^2 u'. \]  

(6d)

On using the dyadic identity

\[ \text{div}(uv) = (\text{grad } u) \cdot v + (\text{div } v)u, \]

and eqns (6a, c), eqn. (6d) is written as

\[ \frac{d}{dt}u' + (\text{grad } U) \cdot u' + (\text{grad } u') \cdot u' = -\text{grad}\left(\frac{p'}{\rho}\right) + \nu \nabla^2 u', \]

(7)

where

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + U \cdot \text{grad}. \]

Equation (7) is the fundamental equation for the velocity perturbations. Along with these equations we must also add the equations of vorticity and pressure perturbations. From Warsi [12] the vorticity and pressure perturbation equations can be obtained, which are

\[ \frac{\partial \omega'}{\partial t} + (\text{grad } \omega') \cdot U + (\text{grad } \Omega) \cdot u' - (\text{grad } u') \cdot \Omega 
\quad - (\text{grad } U) \cdot \omega' = \nu \nabla^2 \omega', \]  

(8a)

where \( \Omega \) is the vorticity in the laminar flow and

\[ \omega = \Omega + \omega'. \]
In Cartesian coordinates

\[
\frac{\partial \omega_i^j}{\partial t} + U_j \frac{\partial \omega_i^j}{\partial x_j} + u_j^j \frac{\partial \Omega_j^i}{\partial x_j} - \Omega_j^j \frac{\partial u_i^j}{\partial x_j} - \omega_j^i \frac{\partial U_j^i}{\partial x_j} = \nu \nabla^2 \omega_i^j,
\]

(8b)

The equation for the pressure perturbation is

\[- \frac{1}{\rho} \nabla^2 p' = 2(\nabla U)^T : (\nabla u') + (\nabla u')^T : (\nabla u'),\]

(9a)

which in Cartesian coordinates is

\[- \frac{1}{\rho} \nabla^2 p' = 2 \frac{\partial U_i}{\partial x_j} \frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j} \frac{\partial u_j'}{\partial x_i}.\]

(9b)

3 Illustrative analysis of instability

As mentioned in Section 1, we consider two illustrative formulations that clarify succinctly the seminal ideas regarding flow instability.

3.1 Reynolds–Orr equation

Let \[E = \frac{1}{2} u' \cdot u'\] be the energy of perturbations per unit mass. To form the differential equation for \[E\], we take the dot product of eqn (7) with \[u'\]. Noting that

\[u' \cdot \nabla^2 u' = \text{div}(\nabla E') - (\nabla u') : (\nabla u'),\]

we have, on using eqn (6c)

\[
\frac{dE}{dt} = -u' \cdot D \cdot u' - \text{div}(Eu') - \text{div} \left( \frac{p'u'}{\rho} \right) + \nu \text{div}(\nabla E) - \nu(\nabla u') : (\nabla u'),
\]

(10)

where we have used the obvious identity

\[u' \cdot (\nabla U) \cdot u' = u' \cdot D \cdot u'.\]
Consider a volume $V$ of surface $S$ having normal $n$. Let

$$E_v = \int_V E \, dV.$$  

Integrating each term of eqn (10) over the volume and using Gauss’ theorem with $u' \cdot n|_S = 0$, and $\frac{\partial E}{\partial n}|_S = 0$, we get

$$\frac{dE}{dt} = -\int_V u' \cdot D \cdot u' \, dV - \Phi_v,$$  

(11)

where

$$\Phi_v = \int_V (\text{grad } u') : (\text{grad } u') \, dV.$$  

Note that in Cartesian coordinates

$$u' \cdot D \cdot u' = u'_i u'_j \frac{\partial U_i}{\partial x_j},$$

$$(\text{grad } u') : (\text{grad } u') = \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_i}.$$  

The first term on the right-hand side of eqn (11) represents the rate of capture of energy from the laminar field $U$, while the second term represents the rate of dissipation of energy due to viscosity. Following the analysis given in Serrin [13], let $-m$ be the lower bound for the eigenvalues of the symmetric tensor (matrix) $D$ in a fixed time interval $0 < t < t_1$. Since the trace of $D$ is zero, i.e.,

$$I : D = \text{div } U = 0,$$

we have $m \geq 0$. Thus,

$$u' \cdot D \cdot u' \geq -m(u' \cdot u') = -2mE,$$
and eqn (11) yields the inequality

$$\frac{dE_v}{dt} \leq 2mE_v - \Phi_v. \quad (12)$$

It is obvious that eqn (11) gives the growth or decay of the superposed disturbances of finite or infinitesimal amplitudes. In this regard the initial value $E_v(0)$ also plays a very important role. A lucid discussion on the stability classifications as developed by Joseph [18] is available in the monograph by Schmid and Henningson [19]. For our present purposes the inequality (12) is sufficient, for it clearly demonstrates that if

$$2mE_v > \Phi_v,$$

then the solution will be unstable. On the other hand if

$$2mE_v < \Phi_v,$$

which implies

$$\frac{dE_v}{dt} < 0,$$

then the solution will be monotonically stable [19].

### 3.2 Chaos and Lorenz model

The second idea regarding the instability of flows is based on the theoretical aspects of chaos and bifurcation in dissipative systems. The prediction of instability in dissipative systems tending towards chaos and bifurcation leads one to understand the basis of instability in fluid flows. In this regard we refer to the works of Joseph [18], Ruelle and Takens [16], Langford [20], and Miles [21]. These references also contain references of other authors that must be consulted.

A survey of literature on chaos theory reveals that most of the problems considered in this area are either without or with minimal fluid dynamic background, except in Joseph [18]. One that comes close to fluid dynamics is the famous model due to Lorenz [14]. Though Lorenz’s model, formed of three ordinary differential equations, has been widely studied, the detailed background of these equations has seldom been discussed. On searching it was found that beside Yorke and Yorke [22] and Busse [23] a sufficiently detailed account is available in Hilborn [24]. Below, we give a concise derivation exposing the salient points of the model for one to understand the underpinning in the context of the theory of flow instability.
The Lorenz model [14] describes the motion of a fluid under the condition of Rayleigh–Benard flow. That is, an incompressible fluid is confined between two parallel plates such that the bottom plate is at a high temperature $T_w$ and the vertically placed top plate is at a lower temperature $T_c$. The vertical distance between the plates is $h$. Let $y$ be the vertical coordinate measured from the lower plate. In the nonconvective state the temperature is a linear function of $y$ so that

$$T = T_w - (T_w - T_c) \frac{y}{h}. \tag{13}$$

In this state the temperature equation, eqn (3), is

$$\frac{\partial T}{\partial t} = D_T \nabla^2 T, \tag{14}$$

where the thermal diffusivity is

$$D_T = \frac{k}{\rho c}. \tag{15a}$$

Based on $D_T$, the Prandtl number is

$$\sigma = \frac{\nu}{D_T}. \tag{15b}$$

On dimensional considerations, eqn (14) yields a characteristic time

$$t_r = \frac{h^2}{D_T}, \tag{15c}$$

and is called the “temperature relaxation time”. The buoyant force $F_b$ is

$$F_b = \alpha \rho_0 g (T_w - T_c) \frac{\Delta y}{h},$$

where $\alpha = -\frac{1}{\rho_0} \frac{\partial}{\partial T}$ with $\rho_0 = \rho(y = 0)$ is the coefficient of thermal expansion.

The viscous force $F_v$ is

$$F_v = \mu \nabla^2 v \approx \frac{\mu v}{h^2},$$
where \( v \) is the velocity along \( y \). For nonconvective condition \( F_b = F_v \), and this gives rise to another characteristic time

\[
t_D = \frac{v}{\alpha g h (T_w - T_c)},
\]

(15d)
called the “temperature diffusion time”. With \( v \approx \frac{\Delta y}{t_D} \), the Rayleigh number \( R \) is defined as

\[
R = \frac{t_T}{t_D} \approx \frac{\alpha g h^3 (T_w - T_c)}{v D_f}.
\]

(16)
The variation of density with temperature can be considered through a series expansion as

\[
\rho(T) = \rho_0 + (T - T_w) \frac{\partial \rho}{\partial T} + \cdots
\]

(17)

\[
= \rho_0 \left[ 1 - \alpha (T - T_w) \right],
\]

where

\[
\rho_0 = \rho(T_w), \quad \alpha = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} > 0.
\]

We now consider a two-dimensional convective model. The standard equations are readily obtained from eqns (1) and (2) with \( u = (u,v,0) \) and \( f = (0,-g,0) \). The temperature equation, eqn (3), is considered without the \( \Phi \) term and \( e = cT \). Thus the case considered is that of free convection in which the dissipation due to viscosity is neglected. Furthermore, we use the Boussinesq approximation that states that the density variation can be disregarded in all terms except the term that involves the force due to gravity. Thus \( \rho \) is replaced by \( \rho_0 \) except in the \( v \)-equation where the density is written as given in eqn (17) in the body-force term only. Furthermore, we introduce the deviation of temperature in the convective state from the linear state as

\[
\tau(r,t) = T - \left\{ T_w - (T_w - T_c) \frac{y}{h} \right\},
\]

(18)
and a modified pressure

\[
p_m = p + \rho_0 g y + \alpha \rho_0 g (T_w - T_c) \frac{y^2}{2 h}.
\]
The resulting equations are nondimensionalized by introducing

\[
\begin{align*}
t^* &= \frac{t}{t_r}, \quad x^* = \frac{x}{h}, \quad y^* = \frac{y}{h}, \quad \tau^* &= \frac{\tau}{T_w - T_c}, \quad u^* = \frac{u}{u_r}, \\
II^* &= \frac{p_m}{p_{mr}}, \quad \text{grad}^* = h \text{grad}, \quad \nabla^2 = h^2 \nabla^2,
\end{align*}
\]

where \( t_r, u_r, \) and \( p_{mr} \) are some reference quantities to be described subsequently. Introducing the stream function \( \psi^* \) through (note the sign change)

\[
\begin{align*}
u^* &= -\frac{\partial \psi^*}{\partial y^*}, \quad \tau^* &= +\frac{\partial \psi^*}{\partial x^*},
\end{align*}
\]

in the equations and for brevity dropping the superscript \( \ast \), we have the non-dimensional equations

\[
\begin{align*}
-h^2 \frac{\partial^2 \psi^*}{\partial y^* \partial y^*} + hu_r \frac{\partial}{\partial t} \left( \frac{\partial \psi^*}{\partial y^*} \frac{\partial \psi^*}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial \psi^*}{\partial y^*} \right) &= -\frac{h p_{mr}}{\mu u_r} \frac{\partial II^*}{\partial x^*} - \frac{\partial}{\partial y^*} (\nabla^2 \psi^*), \\
\frac{h^2}{vt_r} \frac{\partial^2 \psi^*}{\partial x^* \partial y^*} + hu_r \frac{\partial}{\partial t} \left( \frac{\partial \psi^*}{\partial y^*} \frac{\partial^2 \psi^*}{\partial x^* \partial x^*} + \frac{\partial \psi^*}{\partial x^*} \frac{\partial^2 \psi^*}{\partial x^* \partial y^*} \right) &= -\frac{h p_{mr}}{\mu u_r} \frac{\partial II^*}{\partial y^*} + \frac{ah^2 (T_w - T_c)}{\nu u_r} \frac{\partial}{\partial x^*} (\nabla^2 \psi^*),
\end{align*}
\]

\[
\begin{align*}
\frac{h^2}{vt_r} \frac{\partial \tau^*}{\partial t} + hu_r \frac{\partial}{\partial t} \left( \frac{\partial \psi^*}{\partial y^*} \frac{\partial \tau^*}{\partial x^*} + \frac{\partial \psi^*}{\partial x^*} \frac{\partial \tau^*}{\partial y^*} \right) &= -\frac{h u_r}{D_r} \frac{\partial \psi^*}{\partial x^*} = \nabla^2 \tau^*,
\end{align*}
\]

where \( \nu = \frac{\mu}{\rho_0}, D_r = \frac{k}{\rho_0 c} \). The boundary conditions are

\[
\begin{align*}
v &= \frac{\partial \psi}{\partial x} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1, \\
\tau &= 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1.
\end{align*}
\]

Lorenz now used a finite Galerkin method and takes

\[
\begin{align*}
\psi &= \lambda(t) \sin(ax) \sin(\pi y), \\
\tau &= \tau_1(t) \cos(ax) \sin(\pi y) - \tau_2(t) \sin(2\pi y),
\end{align*}
\]

\[
\text{(22)}
\]
where \(a\) is a constant. Note that using the expression of \(\psi\) in eqn (22), we get

\[
\nabla^2 \psi = -(\pi^2 + a^2) \psi,
\]

\[
\nabla^4 \psi = (\pi^2 + a^2)^2 \psi.
\]

Substituting eqn (22) in eqns (19) and (20) and by differentiating the first equation with respect to \(y\) and the second with respect to \(x\) and subtracting to eliminate \(\Pi\), the coefficient of \(\sin(ax) \sin(\pi y)\) yields the equation

\[
\frac{h^2}{v t_r} \frac{d \lambda}{dt} = -\left(\pi^2 + a^2\right) \lambda + \frac{a gh^2 (T_w - T_r)}{v u_r} a \tau_1.
\]

Similarly substituting eqns (22) in eqn (21) we get an equation that has coefficients of \(\cos(ax) \sin(\pi y)\), \(\sin(2\pi y)\) and \(\cos(ax) \sin(\pi y) \cos(2\pi y)\). Noting that

\[
\sin(\pi y) \cos(2\pi y) = -\frac{1}{2} \sin(\pi y) + \frac{1}{2} \sin(3\pi y),
\]

and since \(\sin(3\pi y)\) lies outside our domain of interest (being of high frequency), we ignore this term. Thus the coefficients of \(\cos(ax) \sin(\pi y)\) and \(\sin(2\pi y)\) yield the equations

\[
\frac{h^2}{D_r} \frac{d \tau_1}{dt} = \frac{h u_r a \lambda}{D_r} - \frac{h u_r a \lambda}{D_r} \pi a \lambda \tau_2 - (\pi^2 + a^2) \tau_1,
\]

\[
\frac{h^2}{D_r} \frac{d \tau_2}{dt} = \frac{h u_r \pi}{2D_r} a \lambda \tau_1 - 4 \pi^2 \tau_2.
\]

We now take

\[
t_r = \frac{h^2}{(\pi^2 + a^2) D_r},
\]

\[
u_r = \frac{D_r}{h},
\]
\[ \lambda(t) = \frac{\sqrt{2}}{\pi a} (\pi^2 + a^2) X(t), \]

\[ \tau_1(t) = \frac{\sqrt{2}}{\pi a} (\pi^2 + a^2)^3 \frac{Y(t)}{R}, \]

\[ \tau_2(t) = \frac{\sqrt{2}}{\pi a^2} (\pi^2 + a^2)^3 \frac{Z(t)}{R}, \]

where \( R \) is the Rayleigh number

\[ R = \frac{ah^3(T_w - T_r)}{vD_r}, \]

then the Lorenz equations are

\[ \frac{dX}{dt} = \sigma(Y - X), \quad (26) \]

\[ \frac{dY}{dt} = rX - XZ - Y, \quad (27) \]

\[ \frac{dZ}{dt} = XY - bZ, \quad (28) \]

where

\[ \sigma = \frac{v}{D_r}, \quad \text{the Prandtl number,} \]

\[ r = \frac{Ra^2}{(\pi^2 + a^2)^3}, \]
\[ b = \frac{4\pi^2}{\pi^2 + a^2}. \]

Note that for \( r > 1 \), the convection begins. At \( r = 1 \):

\[ R = \frac{(\pi^2 + a^2)^3}{a^2}, \]

and the minimum value of \( R \) is obtained by \( \frac{dR}{da} = 0 \). This equation gives

\[ a = \frac{\pi}{\sqrt{2}}, \]

and consequently the critical Rayleigh number \( R_c \) is

\[ R_c = \frac{27\pi^4}{4} \approx 657.5, \]

and

\[ b = \frac{8}{3}. \]

The set of eqns (26)–(28) are the Lorenz’s equations with parameters \( \sigma, r, \) and \( b \). The preceding analysis clearly shows that these three equations describe a nonlinear dissipative dynamical system based on the fluid-dynamic phenomena of free convection. For establishing the concept of instability we have to solve these equations numerically since it is impossible to solve them in a closed analytic form under given initial conditions. In the sample calculations shown in Figs. 1 and 2 the value of \( \sigma \) is taken as 10 (the Prandtl number for cold water) and \( b = 8/3 \). The initial values taken are

\[ X(0) = 0.0, \quad Y(0) = -5.0, \quad Z(0) = 15.0, \]

and the purpose here is to show the occurrence of instability by increasing the reduced Rayleigh number \( r \). Though the purpose of these calculations is only to show the occurrence of instability with increasing \( r \) it will not be out of place to mention that the mixed periodic and the chaotic behavior starts occurring from \( r \geq 167 \). Refer to Hilborn [24] for further discussion on chaos.
Figure 1a. Time versus X for $r = 1$ shows a stable solution.

Figure 1b. Time versus X for $r = 5$ shows the initiation of instability.
Figure 2a. Time versus X for \( r = 28 \) shows a periodic behavior.

Figure 2b. Plot of the contours in the XZ–phase plane for \( r = 28 \).

4 Linear stability theory

In the development of the theory of instability of flows, the initial effort was directed to study the effects of infinitesimal disturbances that result in
having equations that are linear. The linear theory is due to the researches carried out by Tollmien [6], Lin [9], Schlichting [7], Squire [8], and others. Since then, much research has been done in the physical and theoretical aspects of the linear theory, e.g., Shen [25], Drazin and Reid [26], Schmid and Henningson [19]. A comprehensive bibliography of the work done in the period 1980–2000 is available in [19].

4.1 Formulation of the problem

The perturbation equations have already been derived in Section 2 as eqns (6c) and (6d). Assuming \( u' \) to be infinitesimal and, further, considering the laminar solution \( U \), and \( P \) to be steady in time, the pertinent equations in Cartesian coordinates are, (refer to [12]),

\[
\frac{\partial u'_j}{\partial x_j} = 0, \quad (29)
\]

\[
\frac{\partial u'_j}{\partial t} + U_k \frac{\partial u'_j}{\partial x_k} + u_k' \frac{\partial U_j}{\partial x_k} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_j} + v \nabla^2 u'_j, \quad (30)
\]

where \( j = 1, 2, 3 \). Since all the imposed boundary conditions have been satisfied by the laminar solution, the boundary conditions for eqns (29) and (30) are homogeneous. We now consider the disturbances as harmonic in time and as an oblique traveling wave in the \( x_1, x_3 \) plane. Thus, we take

\[
u_j = \psi_j(x_2) e^{i(-\sigma t + k_1 x_1 + k_3 x_3)}, \quad (31a)
\]

\[
p' = h(x_2) e^{i(-\sigma t + k_1 x_1 + k_3 x_3)}, \quad (31b)
\]

where \( i = \sqrt{-1} \), \( \sigma = \sigma_r + i \sigma_i \), and \( k_1, k_3 \) as the complex wave numbers of dimension \( 1/\text{length} \). Substituting eqns (31) in eqns (29) and (30) and performing the non-dimensionalization according to the following scheme:
\[ x = \frac{x_1}{L}, \quad y = \frac{x_2}{L}, \quad z = \frac{x_3}{L}, \quad \alpha = Lk_1, \quad \beta = Lk_3 \]

\[ \tau = \frac{U_\infty t}{L}, \quad U = \frac{U_1}{U_\infty}, \quad V = \frac{U_2}{U_\infty}, \quad W = \frac{U_3}{U_\infty}, \]

\[ \varphi_1 = \frac{\psi_1}{U_\infty}, \quad \varphi_2 = \frac{\psi_2}{U_\infty}, \quad \varphi_3 = \frac{\psi_3}{U_\infty}, \]

\[ \chi = \frac{h}{\rho U_\infty^2}, \quad \omega = \frac{\sigma L}{U_\infty}, \quad R_e = \frac{U_\infty L}{v}, \]

where \( L \) and \( U_\infty \) are characteristic length and velocity, respectively, the four simultaneous equations from eqns (29) and (30) are

\[
\frac{d\varphi_2}{dy} + i(\alpha\varphi_1 + \beta\varphi_3) = 0, \quad (32a)
\]

\[
i\varphi_1 (-\omega + \alpha U + \beta W) + V \frac{d\varphi_1}{dy} + \varphi_1 \frac{\partial U}{\partial x} + \varphi_2 \frac{\partial U}{\partial y} + \varphi_3 \frac{\partial U}{\partial z} = -i\alpha\chi + \frac{1}{R_e} \left( \frac{d^2}{dy^2} - \gamma^2 \right) \varphi_1, \quad (32b)
\]

\[
i\varphi_2 (-\omega + \alpha U + \beta W) + V \frac{d\varphi_2}{dy} + \varphi_1 \frac{\partial V}{\partial x} + \varphi_2 \frac{\partial V}{\partial y} + \varphi_3 \frac{\partial V}{\partial z} = -d\chi + \frac{1}{R_e} \left( \frac{d^2}{dy^2} - \gamma^2 \right) \varphi_2, \quad (32c)
\]

\[
i\varphi_3 (-\omega + \alpha U + \beta W) + V \frac{d\varphi_3}{dy} + \varphi_1 \frac{\partial W}{\partial x} + \varphi_2 \frac{\partial W}{\partial y} + \varphi_3 \frac{\partial W}{\partial z} = -i\beta\chi + \frac{1}{R_e} \left( \frac{d^2}{dy^2} - \gamma^2 \right) \varphi_3. \quad (32d)
\]
In eqns (32),

\[ \gamma^2 = \alpha^2 + \beta^2. \]

Equations (32) are linear but are complicated to solve under homogeneous boundary conditions.

The most fruitful outcome of the present effort has been in the area of plane parallel laminar flows. In this class of flows we interpret \( x, y, z \) as a local general orthogonal coordinate system with \( y \) as the coordinate normal to the main flow. Furthermore, the local variation of the laminar velocity components along \( x \) and \( z \) is assumed to be much smaller in comparison to the \( y \)-direction. Thus, locally

\[ U = U(y), \quad V = V(y), \quad W = W(y). \]

Using the continuity equation we get \( V \equiv 0 \). Using these approximations in eqns (32), we get the equations for three-dimensional perturbations superposed on three-dimensional laminar flows. If in these equations we set \( W = 0 \), then we get the equations for three-dimensional perturbations superposed on two-dimensional laminar flows, which are

\[ \frac{d\varphi_2}{dy} + i(\alpha\varphi_1 + \beta\varphi_3) = 0, \quad (33a) \]

\[ i\varphi_1 (-\omega + \alpha U) + \varphi_2 \frac{dU}{dy} = -i\alpha\chi + \frac{1}{Re} \left( \frac{d^2}{dy^2} - \gamma^2 \right) \varphi_1, \quad (33b) \]

\[ i\varphi_2 (-\omega + \alpha U) = -\frac{d\chi}{dy} + \frac{1}{Re} \left( \frac{d^2}{dy^2} - \gamma^2 \right) \varphi_2, \quad (33c) \]

\[ i\varphi_3 (-\omega + \alpha U) = -i\beta\chi + \frac{1}{Re} \left( \frac{d^2}{dy^2} - \gamma^2 \right) \varphi_3. \quad (33d) \]

Eliminating \( \chi \) between eqns (33b,c) and using eqn (33a) for \( \varphi_1 \), we obtain a fourth-order equation in \( \varphi_2 \) and \( \varphi_3 \). Next, eliminating \( \chi \) between eqns (33c,d) we
get another equation containing \( \phi_2 \) and \( \phi_3 \). Using the second equation in the first we get a single differential equation in \( \phi_2 \) as

\[
(U - c) \left( \frac{d^2}{dy^2} - \gamma^2 \right) \phi_2 - \phi_2 \frac{d^2 U}{dy^2} = -\frac{i}{\alpha R_c} \left( \frac{d^4}{dy^4} - 2\gamma^2 \frac{d^2}{dy^2} + \gamma^4 \right) \phi_2,
\]

where

\[
c = \frac{\omega}{\alpha} = c_r + ic_i, \quad \gamma^2 = \alpha^2 + \beta^2,
\]

and \( c \) is called the complex phase speed of perturbation.

It must be noted that eqn (34) contains both \( \alpha \) and \( \gamma \) that are complex. The three-dimensional perturbation of the form

\[
\phi_2 (y) e^{i(\alpha x + \beta z - \omega t)},
\]

represents an oblique wave whose wave front lies obliquely in the \( xz \)-plane. Dunn and Lin [27] pointed out that it is possible to devise a coordinate transformation in the \( xz \)-plane such that one coordinate line, say \( \xi \), is chosen along the direction of wave propagation and the other normal to it as shown in Figure 3. This transformation is equivalent to setting \( \beta = 0 \) and then interpreting \( \xi \) as \( x \). Thus, eqn (33a) becomes

\[
i \alpha \phi_1 + \frac{d\phi_1}{dy} = 0,
\]

and it is identically satisfied by introducing a function \( \varphi \) as

\[
\phi_2 = -i\alpha \varphi, \quad \phi_1 = \frac{d\varphi}{dy},
\]

which changes the form of eqn (34) to

\[
(U - c)(\varphi'' - \alpha^2 \varphi) - \varphi \frac{d^2 U}{dy^2} = -\frac{i}{\alpha R_c} (\varphi''' - 2\alpha^2 \varphi'' + \alpha^4 \varphi).
\]

Equation (35) is called the Orr–Sommerfeld equation that is a homogeneous equation to be solved under the homogeneous boundary conditions. Here a prime denotes \( \frac{d}{dy} \).
Historically, [3, 4], eqn (35) was obtained for two-dimensional perturbations superposed on two-dimensional laminar flow. This can easily be demonstrated by considering the two-dimensional Navier–Stokes equation in the stream function $\psi$ and then using

$$\psi = \Psi(y) + \psi'(x, y, z, t).$$

Neglecting the second-order terms in perturbations and noting that locally

$$\nu \nabla^4 \psi = \nu \frac{d^3 U}{dy^3} = 0,$$

while for the perturbation $\psi'$

$$\psi' = \varphi(y) e^{i\alpha(x-ct)},$$

we recover eqn (35). In eqn (34) if we write

$$\alpha R_c = \gamma R_c,$$
then both eqns (34) and (35) are exactly of the same form. Before further discussion we note that \( R_e \) and \( \overline{R}_e \) are real while both \( \alpha \) and \( \gamma = \sqrt{\alpha^2 + \beta^2} \) are generally complex. Writing

\[
\lambda = \frac{\overline{R}_e}{R_e}, \quad \alpha = |\alpha|e^{i\theta}, \quad \beta = |\beta|e^{i\beta}, \quad \theta_1 = \tan^{-1}\left( \frac{\alpha}{\overline{\alpha}} \right), \quad \theta_2 = \tan^{-1}\left( \frac{\beta}{\overline{\beta}} \right),
\]

we easily obtain

\[
|\alpha|^2 = \lambda^2 \left| \alpha \right|^2 + |\beta|^2 e^{2i(\theta_2 - \theta_1)}.
\]

Since the left-hand side term is real, \( \theta_2 = \theta_1 \), and thus

\[
\lambda = \frac{|\alpha|}{\sqrt{|\alpha|^2 + |\beta|^2}} < 1,
\]

or,

\[
\overline{R}_e < R_e.
\]

We now argue that if the solution of eqn (34) at a given value of the Reynolds number \( R_e \) is unstable, then the solution of eqn (35) at a lower value of the Reynolds number \( R_e \) is unstable. Consequently, the problem of three-dimensional perturbations at a given Reynolds number \( R_e \) is equivalent to the problem of two-dimensional perturbations at a lower Reynolds number \( R_e \), [12]. This statement is known as Squire’s theorem. Quoting Drazin and Reid [26]: “To obtain the minimum critical Reynolds number it is sufficient to consider only two-dimensional disturbances.” Another way of describing the above statements is to state that if the direction of the wave propagation is along the direction of the main laminar flow, then this flow is likely to be unstable at a lower Reynolds number in comparison to the case when the wave front is oblique to the main flow direction, (see [12]).

Before proceeding further it is instructive also to consider the vorticity perturbation equation. We consider eqn (8b) for \( \omega'_2 \) and with

\[
U_1 = U_1(x_2), \quad U_2 = U_3 = 0, \quad U_1 = U_2 = 0, \quad U_3 = -\frac{dU_1}{dx_2}.
\]

Furthermore,

\[
\omega'_2 = \frac{\partial u'_1}{\partial x_3} - \frac{\partial u'_2}{\partial x_1}.
\]
Using the non-dimensionalization used earlier with
\[ \omega' = \frac{U_\infty}{L} \eta, \quad v = \frac{u_2'}{U_\infty}, \]
we get
\[ \frac{\partial \eta}{\partial \tau} + U \frac{\partial \eta}{\partial x} + \frac{dU}{dy} \frac{\partial v}{\partial z} - \eta \frac{dU}{dy} = \frac{1}{R_e} \nabla^2 \eta. \] (37)

In eqn (37), writing
\[ \eta = \theta(y) e^{i(\alpha x + \beta z - \omega t)}, \]
\[ v = \varphi_2(y) e^{i(\alpha x + \beta z - \omega t)}, \]
we get
\[ \left[ i(-\omega + \alpha U) + \frac{1}{R_e} \left( \gamma^2 - \frac{d^2}{dy^2} \right) \right] \theta(y) = -i\beta \varphi \frac{dU}{dy}. \] (38)

Equation (38) is due to Squire [8] and should be solved with eqn (34). If \( \beta = 0 \), then eqn (38) is independent of eqn (35) but the parametric values of \( \omega \) and \( \alpha \) have to come from eqn (35).

4.2 Boundary conditions of the Orr–Sommerfeld equation

As has been mentioned earlier the boundary conditions for the perturbations are homogeneous. Here we mention only three cases that explain clearly the manner as to how the boundary conditions should be chosen. For other cases refer to Chebeci and Bradshaw [28], Warsi [12], and Drazin and Reid [26].

(i) Flow between parallel plates:

Let \( y = y_1 \) and \( y = y_2 \) be two parallel plates. Since
\[ u' = \frac{u_1'}{U_\infty} = \frac{d\varphi}{dy} e^{i(\alpha x - \omega t)}, \]
\[ v' = \frac{u_2'}{U_\infty} = -i\alpha \varphi e^{i(\alpha x - \omega t)}, \] (39)
the boundary conditions for eqns (35) and (38) are

\[ y = y_1 : \varphi = \frac{d\varphi}{dy} = 0, \quad \theta = 0 , \]  
\[ y = y_2 : \varphi = \frac{d\varphi}{dy} = 0, \quad \theta = 0 . \]  

(40a)

(ii) Flow on a plate with disturbances symmetric about a line:

Let \( y = y_1 \) be the plate and \( y = y_2 \) be the line about which the disturbances are symmetric. Using Taylor’s expansions for \( u'(y_2 + h) \) and \( u'(y_2 - h) \) and because of symmetry

\[ u'(y_2 + h) = u'(y_2 - h) , \]

we get

\[ y = y_2 : \frac{d u'}{dy} = 0, \quad v' = 0 . \]

Thus the boundary conditions are

\[ y = y_1 : \varphi = 0, \quad \frac{d\varphi}{dy} = 0 , \]  
\[ y = y_2 : \varphi = 0, \quad \frac{d^2\varphi}{dy^2} = 0 . \]  

(40b)

(iii) Boundary layers:

Let the edge of the boundary layer be denoted by \( y = \delta \) where locally \( U = U_e = \) constant. Thus eqn (35) can be written as an ordinary differential equation with constant coefficients of the form

\[ \varphi''' - (\alpha^2 + \zeta^2)\varphi'' + \alpha^2 \zeta^2 \varphi = 0 , \]  

where

\[ \zeta^2 = \alpha^2 + i\alpha R_c (U_e - c) , \]  
\[ c = \frac{\omega}{\alpha} . \]

Equation (4) being a fourth-order equation has four independent solutions that can be combined as

\[ \varphi = A_1 e^{\alpha y} + A_2 e^{-\alpha y} + A_3 e^{\zeta y} + A_4 e^{-\zeta y} . \]
At \( y = \delta \), which implies \( y \to \infty \) on the boundary scale, \( A_1 \) and \( A_3 \) should be zero. This in turn leads one to the vanishing of the two determinants (for discussion refer to Warsi [12]) giving rise to the two equations

\[
Z = 0, \quad \zeta = 0, \quad (42)
\]

where

\[
Z = (D^2 - \zeta^2)(D + \alpha)\varphi \quad \text{at} \quad y = \delta, \\
\zeta = (D^2 - \alpha^2)(D + \zeta)\varphi \quad \text{at} \quad y = \delta,
\]

\[
D = \frac{d}{dy}.
\]

Equations (42) give the relations at \( y = \delta \). At the wall, from eqn (39),

\[
y = 0 : \varphi = \varphi' = 0. \quad (43)
\]

### 4.3 Numerical solution

Numerical solution of the Orr–Sommerfeld equation has remained the central problem of linear stability theory starting from the pioneering works of Taylor [5] and Lin [9]. Here we first state a format for numerical solution. Cases of specific laminar flows employing other techniques will be discussed subsequently.

First, it must be realized that the solution of the Orr–Sommerfeld equation, eqn (35), under homogeneous boundary conditions as shown in eqns (40), (42), and (43), forms an eigenvalue problem. The basic parameters are

\[
U(y) : \text{the local laminar flow velocity} \\
\alpha : \text{the non-dimensional wave number, which is generally complex} \\
c = \frac{\omega}{\alpha} : \text{phase speed, also complex} \\
R_e : \text{the Reynolds number, which is real}
\]

Note that

\[
\alpha = \alpha_r + i\alpha_i, \\
c = c_r + ic_i, \\
c_r = \frac{\omega_r\alpha_r + \omega_i\alpha_i}{\alpha_r^2 + \alpha_i^2}, \\
c_i = \frac{\omega_r\alpha_i + \omega_i\alpha_r}{\alpha_r^2 + \alpha_i^2}.
\]
For temporal stability problems, \( \alpha_i = 0 \), and \( \alpha_r \) is a prescribed constant, while for spatial stability problems \( \omega_i = 0 \), and \( \omega_r \) is a prescribed constant. Thus, for temporal stability

\[
c_r = c_r (\alpha, R_c), \\
c_i = c_i (\alpha, R_c),
\]

where \( \alpha \) is real. If \( c_i < 0 \), then the disturbances decay in time, while for \( c_i > 0 \) the disturbances grow in time. The pair \((\alpha, R_e)\) for which \( c_i = 0 \) gives a point on a curve called the neutral stability curve. This curve separates the regions of stability and instability. The point on this curve at which \( R_e \) is lowest is called the critical point and \( R_e \) is denoted as \( R_{cr} \). Refer to Figure 4 for a neutral stability curve.

The basic format of the numerical solution of eqn (35) referred to earlier in this section is based on the fundamental solution technique for ordinary differential equations and is discussed fully in Warsi [12]. The solution of eqn (35) is written as

\[
\varphi(y) = C_i \psi_i(y), \quad \text{(sum on } i: i = 1, \ldots, 4)\]

Figure 4: Typical curve of neutral stability in the \( aR_e \) - plane, (from Ref. [12] used with permission from CRC Press).
where \( \psi_i(y) \) are the fundamental solutions satisfying the initial conditions:

\[
\begin{align*}
\psi_1(0) &= 1, & \psi_1'(0) &= 0, & \psi_1''(0) &= 0, & \psi_1'''(0) &= 0, \\
\psi_2(0) &= 0, & \psi_2'(0) &= 1, & \psi_2''(0) &= 0, & \psi_2'''(0) &= 0, \\
\psi_3(0) &= 0, & \psi_3'(0) &= 0, & \psi_3''(0) &= 1, & \psi_3'''(0) &= 0, \\
\psi_4(0) &= 0, & \psi_4'(0) &= 0, & \psi_4''(0) &= 0, & \psi_4'''(0) &= 1.
\end{align*}
\]  

(44)

Because of the boundary conditions (43)

\[ C_1 = 0, \quad C_2 = 0. \]

Consequently, taking \( C_3 = 1 \) without any loss of generality, we have

\[ \varphi(y) = \psi_3(y) + C_4 \psi_4(y). \]  

(45)

Substituting eqn (45) in eqn (35) we pose two initial-value problems, which are

\[
(U - c)(\psi_3'' - \alpha^2 \psi_3) - U'' \psi_3 = \frac{-i}{\alpha R_e} (\psi_3''' - 2 \alpha^2 \psi_3'' + \alpha^4 \psi_3),
\]

\[
\psi_3(0) = 0, \quad \psi_3'(0) = 0, \quad \psi_3''(0) = 1, \quad \psi_3'''(0) = 0,
\]  

(46)

\[
(U - c)(\psi_4'' - \alpha^2 \psi_4) - U'' \psi_4 = \frac{-i}{\alpha R_e} (\psi_4''' - 2 \alpha^2 \psi_4'' + \alpha^4 \psi_4),
\]

\[
\psi_4(0) = 0, \quad \psi_4'(0) = 0, \quad \psi_4''(0) = 1, \quad \psi_4'''(0) = 1.
\]  

(47)

Equations (46) and (47) have complex coefficients and should yield nontrivial solutions because of the inhomogeneous initial conditions. At \( y = \delta \),

\[ \varphi(\delta) = \psi_3(\delta) + C_4 \psi_4(\delta), \]

which when substituted in \( \zeta = 0 \) of eqn (42) gives

\[
C_4 = \frac{-\psi_3''(\delta) + \zeta \psi_3'''(\delta) - \alpha^2 \left\{ \psi_3'(\delta) + \zeta \psi_3'(\delta) \right\}}{\psi_4''(\delta) + \zeta \psi_4'''(\delta) - \alpha^2 \left\{ \psi_4'(\delta) + \zeta \psi_4'(\delta) \right\}}.
\]
We now pose eqn (35) as an initial-value problem under the initial conditions:

\[ \phi(0) = 0, \quad \phi'(0) = 0, \quad \phi''(0) = 1, \quad \phi'''(0) = C_4. \]

The complex solution \( \phi(y) \) is continued to \( y = \delta \) and the values of \( \phi(\delta), \phi'(\delta), \phi''(\delta), \phi'''(\delta) \) are used in \( Z = 0 \) of eqn (42) to solve for \( c \) and \( \alpha \) from

\[ Z_r = 0, \quad Z_i = 0. \]

For further discussion refer to Warsi [12]. The preceding methodology is one of the many developed over the years. Most of the methods are of the finite-difference approximation type for eqn (35). One method developed by Orszag [29] depends on the expansion of \( \phi(y) \) in Chebyshev polynomials and the coefficients are determined by a QR matrix eigenvalue algorithm. The Chebyshev approximation is highly accurate in comparison with the finite-difference approximation. Orszag [29] has used the above method for plane Poiseuille flow in which

\[ U(y) = 1 - y^2, \quad -1 \leq y \leq 1 \]

and has obtained the unstable mode for \( \alpha = 1, R_e = 10000 \) as

\[ \omega = \omega_r + i\omega_i = 0.23752649 + 0.00373967i. \]

By using a computer program based on the Chebyshev approximation and with double-precision arithmetic, the present author has obtained

\[ \omega = \omega_r + i\omega_i = 0.23750577 + 0.00372402i. \]

These values are very close. Orszag [29] has also obtained the critical Reynolds number for plane Poiseuille flow as

\[ R_{cr} = 5772.22. \]
The temporal stability curves for the boundary layer on a flat plate as given in Cebeci and Bradshaw [28] are shown in Fig. 5. Here

\[ \alpha_{r1} = k_1 \delta^*, \quad R_{\delta^*} = \frac{U_\infty \delta^*}{\nu}, \]

where \( \delta^* \) is the displacement thickness. The importance of this figure is that it shows a comparison of theory with the experimental data of Schubauer and Skramstad [30] shown by small circles.

### 4.4 Inviscid laminar flow

In the inviscid case \( R_e = \infty \) and eqn (35) reduces to a lower-order equation

\[ (U - c)(\phi'' - \alpha^2 \phi) - \phi U'' = 0, \quad (48) \]

with boundary conditions:

\[ y = y_1 : \phi = 0; \quad y = y_2 : \phi = 0. \quad (49) \]

A great deal of work for this case is available in Lin [9]. Here we shall explore some of the basic results that are important for high Reynolds number case, i.e., \( R_e \rightarrow \infty \). It must be noted that the solution of eqn (48) under eqn (49) will not yield \( R_{cr} \). The basic results referred to above are in the form of two theorems due to Rayleigh (1880), (refer to [1, 7]).
(i) Rayleigh’s first theorem

“The existence of a point of inflection in the velocity profile $U(y)$ is a necessary condition for instability”.

A point of inflection $U(y)$ means that $U''(y) = 0$ for some $y$ in $y_1 \leq y \leq y_2$. Let $\phi^*$ be the complex conjugate of $\phi$. Thus the two equations for real $\alpha$ (temporal instability) are

$$\phi'' - \alpha^2 \phi - \frac{U''}{U-c} \phi = 0,$$

(50a)

$$\phi''^* - \alpha^2 \phi^* - \frac{U''}{U-c^*} \phi^* = 0.$$

(50b)

Multiplying the first equation by $\phi^*$ and the second by $\phi$ and subtracting, we get:

$$\frac{d}{dy} (\phi^* \phi' - \phi \phi'^*) = \frac{2ic_i U''}{|U-c|^2} |\phi|^2.$$

(51a)

Integrating from $y_1$ to $y_2$ while using the boundary conditions, we get:

$$2ic_i \int_{y_1}^{y_2} \frac{U''}{|U-c|^2} |\phi|^2 dy = 0.$$

(51b)

For temporal instability $c_i > 0$, and therefore for the satisfaction of eqn (51b) we must have $U''(y) = 0$ at some point in the range $y_1 \leq y \leq y_2$, thereby proving the theorem.

(ii) Rayleigh’s second theorem

“The velocity of propagation of neutral disturbances ($c_i = 0$) is smaller than the maximum of the laminar flow velocity”.

The value of $y = y_c$ where $U = c_r$ lies in a layer called the critical layer. Note that in eqn (51a),

$$\phi^* \phi' - \phi \phi'^*,$$

is purely imaginary. We therefore define a real quantity $W$ as

$$W = \frac{-i}{2} (\phi^* \phi' - \phi \phi'^*),$$
so that from eqn (51a)

\[
\frac{dW}{dy} = \frac{c_i U''|\varphi|^2}{(U - c_i)^2 + c_i^2}.
\] (52)

Let \( c_i = 0 \), then \( W \) is constant everywhere except at a point where \( U = c_r \). We denote this point as \( y = y_c \). In the latter case, we integrate eqn (52) between \( y_c - \delta \) and \( y_c + \delta \) where \( \delta \) is arbitrary. Then the change in \( W \) is

\[
W = \int_{U_1}^{U_2} \frac{c_i U''|\varphi|^2}{U'[U - c_r]^2 + c_i^2} \, dU,
\]

where

\[
U_1 = (y_c - \delta), \quad U_2 = (y_c + \delta).
\]

Since the purpose is to find the limiting value

\[
\lim_{c_i \to 0} [W],
\]

we use the formula (based on the Dirac delta function)

\[
\lim_{\lambda \to 0} \int_{\xi_1}^{\xi_2} \frac{\lambda f(\xi)}{(\xi - x)^2 + \lambda^2} \, d\xi = \pm \pi f(x),
\]

to have

\[
\lim_{c_i \to 0} [W] = \pm \pi \left( \frac{U''}{U''_c} \right) |\varphi_c|^2.
\] (53)

Equation (53) describes the discontinuous behavior of \( W \) near the critical point. However, in any event the total change in \( W \) must be zero, which leads one to conclude that \( U''_c = 0 \). Thus, in the neutral case \( (c_i = 0) \) the velocity of wave propagation is exactly the velocity of laminar flow at the point of inflection, i.e.,

\[
U(y_c) = c_r.
\]

Since \( U(y_c) < U_{\text{max}} \), this proves the second theorem. This theorem can also be proved for those profiles for which \( U(y) \) has no point of inflection, i.e., \( U''(y) < 0 \) for every point in \( y_1 \leq y \leq y_2 \). This can be shown by multiplying eqn (50a) by \( \varphi^* \)
and eqn (50b) by $\phi$, adding and integrating from $y_1$ to $y_2$. Besides the result that $U(y_c) < U_{\text{max}}$ and $U(y_c) = c_r$, we also note that for the integral to exist

$$\text{Lim}_{y \to y_c} \left[ \frac{U''}{U - c_r} \right]$$

must exist, which is impossible since $U''$ is not zero in the range $y_1 \leq y \leq y_2$. This points to a discontinuity across the critical layer giving rise to oscillations. In real flows this wavy behavior for velocity profiles with or without inflection is termed a Tollmien–Schlichting (TS) wave.

From the two theorems of Rayleigh we conclude that the most attractive features of the inviscid stability theory are: (1) the velocity profiles with a point of inflection are bound to become unstable for $Re \to \infty$ and, (2) the existence of a critical layer $y = y_c$. From the second conclusion we find that a general solution of eqn (50a) will be of interest only near $y = y_c$. Since eqn (50a) is of the second order, then we seek a solution near the critical point in the form of a series expansion as:

$$\phi = a_o \left( y - y_c \right)^\lambda + a_1 \left( y - y_c \right)^{\lambda+1} + \cdots,$$

where $\lambda$ is a parameter to be determined. Furthermore, the equation is homogeneous and linear, so that any constant multiple of $\phi$ is also a solution. We therefore take $a_o = 1$. Near the critical point we also expand $U$ in powers of $y - y_c$, which is

$$U = c_r + (y - y_c)U'_c + \frac{1}{2} (y - y_c)^2 U''_c + \cdots.$$

On substitution in eqn (50a), we have the indicial equation:

$$\lambda(\lambda - 1) = 0,$$

or

$$\lambda = 0, \quad \lambda = 1.$$

The two independent solutions are

$$\phi_1 = (y - y_c) + a_1 (y - y_c)^2 + \cdots,$$

$$\phi_2 = 1 + b_1 (y - y_c) + \cdots + \frac{U''}{U'_c} \phi_1 \ln(y - y_c).$$

The logarithmic singularity in $\phi_2$ for those velocity profiles for which $U''_c \neq 0$ is a consequence of neglecting viscosity. The above solution is used in providing the matching conditions for the Orr–Sommerfeld for the case $Re \to \infty$. According to
Instability of Flows

Lin [9], the upper and lower branches of the neutral stability curve in Figure 4 have the following behaviors.

For \( R_e \to \infty \):

Upper branch: \( \alpha_r = \left( \frac{601.2116}{R_e} \right)^{\frac{1}{11}} \), \( c_r = 0.2667 \alpha_r^2 \).

Lower branch: \( \alpha_r = \left( \frac{211.7088}{R_e} \right)^{\frac{1}{7}} \), \( c_r = 0.661 \alpha_r^2 \).

4.5 Anomalies in small perturbation theory

After reviewing the fundamental aspects of the existence of instability in the flow of fluids and then the classic theory of small perturbations, it is appropriate to point out that there are two well-known problems in which the small perturbation theory does not work. Two cases that do not give the critical Reynolds number \( R_{cr} \) by using the small perturbation theory are the plane Couette flow and the Hagen–Poiseuille flow in a circular pipe. That is, the flow in both cases remains stable for all Reynolds numbers. Betchov and Criminale [31] have advanced a reason that in both cases the vorticity production term is zero. In the case of plane Couette flow \( U''(y) = 0 \) for all \( y \) and in pipe flow

\[
\frac{d}{dr} \left( \frac{1}{r} \frac{dW}{dr} \right) = 0,
\]

where \( W \) is the axial velocity and \( r \) is the radial coordinate, and both are the vorticity production terms. Other authors, e.g., Joseph [18] have noted that in these cases the nonlinearity (finite disturbances) has to be considered that will lead first to instability and then transition to turbulence. Trefethen et al. [32] have advanced the idea that the linear eigenvalue analysis of the Orr–Sommerfeld equation from and operator-theoretic view point must be conducted for each case. An operator whose eigenfunctions are orthogonal is said to be normal. The linear operators that arise in Rayleigh–Benard and circular Couette flow problems are normal but the operators that arise for the plane Couette flow and the pipe flow are far from normal. Reference [32] must be consulted for further details. The anomalies stated here point out the limitations of the linear theory to seemingly basic flows that respond to perturbations in a nonlinear manner.

5 An introduction to nonlinearity

According to Drazin and Reid [26] the foundations of the theory of nonlinear hydrodynamic stability were laid by Landau in 1944, Landau and Lifshitz [11]. According to Landau a nonsteady flow field is established for \( R_e = R_{cr} \) and the linear perturbation of this new field will give rise to another that will be more complicated than the previous one. Furthermore, a succession of instabilities will
give rise to transition and turbulence. To describe this scenario Landau takes the next perturbation as

\[ u' = A(t) f(r), \]

where \( A(t) \) is a complex amplitude. The complex frequencies are \( \sigma = \sigma_r + i \sigma_i \). If \( \sigma_i < 0 \) then \( R_e < R_{cr} \) while for \( R_e > R_{cr} \), \( \sigma_i > 0 \) for some values of \( \sigma_r \). For small values of \( t \)

\[ A(t) = \text{const} \cdot e^{\sigma_r t - i \sigma_i t}, \]

which is the same result as previously obtained. In this situation

\[ \frac{d}{dt}|A|^2 = 2 \sigma_i |A|^2. \]

Following a series of arguments and for \( R_e - R_{cr} \) very small Landau obtains the next approximation as

\[ \frac{d}{dt}|A|^2 = 2 \sigma_i |A|^2 - \alpha |A|^4, \tag{54} \]

where \( \alpha \) is a positive or negative constant and is called the Landau's constant. If \( |A_o| = |A(0)| \) is given, then eqn (54) can be solved easily by writing

\[ y = \frac{1}{|A|^2}, \]

as

\[ \frac{y}{y_o} = \frac{\alpha}{2 \sigma_i y_o} + \left(1 - \frac{\alpha}{2 \sigma_i y_o}\right) e^{-2 \sigma_i t}, \tag{55} \]

where \( y_o = \frac{1}{|A(0)|^2} \). From eqn (55) we find that the maximum amplitude of the next perturbed flow is

\[ |A|^2_{\text{max}} = \frac{2 \sigma_i}{\alpha}. \]

Since \( \sigma_i \) is a function of \( R_e \) and \( \sigma_i(R_{cr}) = 0 \), the expansion of \( \sigma_i \) in powers of \( R - R_{cr} \) is possible for \( R - R_{cr} \) very small. On these and other aspects of the nonlinear theories refer to [11, 33–35]. It must, however, be pointed out that Landau's
equation, eqn (54), may also be deduced from eqn (11) if its right-hand side is assumed to be expanded in powers of $|A|^2$.

6 Conclusions

The preceding sections have reviewed the basic idea of instability and the classic theories of the effects of small perturbations on laminar viscous and inviscid flows. The utility of these results lies in the observation that in real flows a number of disturbing sources always exist and are practically unavoidable. Though the theory of small perturbations does not provide an answer to the problem of transition from laminar to turbulent flow, the results of linear theory provide some qualitative indications toward transition.

References


