Navier–Stokes computation of hovering dynamics

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Abstract

A chronological overview of our research at the University of Colorado from 1985 to present, focused on the Navier–Stokes equation computational simulation of the aerodynamics which enables the hovering of certain insects and birds, will be given. This work was the first of its kind and provided an exceedingly good match of our computational vortex hovering dynamics to that observed in the laboratory and in nature. Then a reasonably good match of our computational lift coefficients to observed laboratory thrust coefficients was also obtained. Current issues will be discussed.

1 Introduction and summary

Recently my students and I [1–18] accomplished the first computational simulation, through the mathematical Navier–Stokes equations, of the rather complex and interesting vortex dynamics generated by hovering motions, such as the aerodynamics employed by dragonflies. It is the purpose of this review to give a unified account of this work, especially inasmuch as our results were published in several different venues, e.g. in separate mathematics, physics, and engineering literatures. It is our hope that in this way our findings may become more widely available to the engineering community and in particular to the biology community, where it seems our work is little known.

This research came about when we were approached in 1985 by Prof. Peter Freymuth who wondered if we could validate his physical laboratory results [19–24] by means of mathematical computations based solely upon the Navier–Stokes partial differential equations for incompressible fluid flow. Physical measurements, at the University of Colorado in the early 1980s [25], from biological specimens had revealed that some species of dragonflies can generate aerodynamic forces permitting them to lift up to 15 times their weight. We (my two PhD students R. Leben and J. McArthur and I) were extremely fortunate to find ourselves thereby mathematically juxtaposed to the experimental state-of-the-art laboratory results of Freymuth [19–24] along
with the state-of-the-art biological results of Luttges [25] and co-workers, all of us here at the University of Colorado at that time.

There will be no attempt at any comprehensive discussion of all aspects of this particular kind of unsteady aerodynamics. These may be found in [1–25], elsewhere in this volume, and in the wider literature. Similarly, the general field of computational fluid dynamics (CFD) is huge and will not be discussed here. However, later in this paper I will mention some critical issues that are intrinsically linked to how one decides to discretize the mathematical Navier–Stokes equations and how one treats the accompanying boundary conditions in order to obtain accurate hovering simulations.

Let me close this short introduction by briefly summarizing in chronological fashion our contributions in [1–18]. Further details may be found in those papers. Some further discussion will be given in the following pages, especially emphasizing certain issues/results not yet in the open literature.

Professor Freymuth’s overture to us in 1985 came as we were exploring multigrid methods for computing the fine vortex structure of fluids [1–3]. We reported those results to the international CFD communities in [4–6]. Thereafter we specialized in hovering aerodynamics [7–9]. By this time we had obtained excellent agreement of our mathematically generated hovering vortex dynamics with those of the physical laboratory [19, 20], and reasonable agreement of our computed near-wing lift coefficients with the laboratory downstream measured thrust coefficients [21] (See Figs 1 and 2). Throughout this period 1985–1992 we enjoyed excellent collaboration with Professor Freymuth and this is summarized in [22].

Due to other research interests, the second half of 1993 to the present has been less active. In [10, 11] a mathematical analysis and explanation of the cause of numerical instabilities in our alternating direction implicit (ADI) method was performed. Let me quickly summarize here. Upwinding, i.e. the use of an implicit scheme, is widely employed in CFD as the Reynolds number increases and numerical instabilities begin to be noticed. For us such instabilities occurred in the region of the steep gradient hovering-induced reverse Karman street jet which is comprised of counter-rotating vortices as they are shed alternately from leading and trailing edge of the airfoil in hovering motion. Briefly, we employed [10, 11], see also the book [14, Part I, Section 3.3], the Baker–Campbell–Hausdorff (BCH) theory of matrix exponentials to analyze this situation. To my knowledge this was an original idea which can be used elsewhere for optimizing algorithms for other flow problems and other applications. From this point of view, stencil-splitting then becomes the moving between sums and products in exponentials.

I do not like upwinding, sometimes the reasons are subtle. See my treatment in [14, Part I, Section 3.1]. The point is that one can lose secondary, tertiary, or other fine vortex structures. Nonetheless, we put in an ‘upwinding switch’ to turn on and off a small amount of upwinding when necessary to traverse the vortex jet while we were computing the hovering motions. This was done only when computing the vortex dynamics at higher Reynolds numbers. See [10, 11, 14] for more details.

In [12, 13] the hovering aerodynamics problem was analyzed within the larger context of dynamical systems. Because this work seems to be virtually unknown to the engineering and biology communities, I will summarize it later below.

In [14–16] I formulated and discussed an issue of proper far-field boundary conditions for better resolution of the full downstream vortex history. In our near-wing hovering dynamics and lift computations we did not care much about the far-field so long as it did not interfere with the near-field dynamics. Our approach made sure that it did not, by setting vorticity to 0 at \(\infty\), reflecting an assumption of irrotational flow at infinity. Such ansatz was naturally compatible with our mapping techniques (see Fig. 3) and was one of the great advantages of the infinite domain
method [26] which we employed. Moreover it provided a simple far-field boundary condition for the computational domain shown in Fig. 3. Our analysis [14–17] provides an improved far-field boundary condition, but this has not yet been implemented and tested.

Another issue was the exact hover-induced downstream jet direction. As mentioned above, we needed [10, 11, 14] to determine this part of the flow in order to know when to switch on and off the upwinded numerical methods in order to cross the hover-jet in stable fashion. Later we found [10, 18] that our computationally determined jet angle independently corroborated, on a preliminary basis, a similar important laboratory finding by Freymuth [20] (see Fig. 4(a)). However, a full investigation of this ‘2 for 1’ rule for determining the exact angles needed for optimal hovering and its further possible ramifications has not yet been carried out.

In connection with our studies of numerical stability issues related to crossing the reverse Karman jet region of the downstream flow, it was natural to also obtain instantaneous fluid dynamic force details, in particular, instantaneous time-dependent $C(t)$ lift plots, and also qualitative dynamical signatures such as phase plots and power spectra (see Figs 5 and 6).
Figure 2: Laboratory downstream thrust coefficients (solid line) versus Navier–Stokes computed near-wing lift coefficients (dotted line) at several angles of attack and plunge amplitudes. Top (a) is for Mode 1 and (b) is for Mode 2. Limited Mode 3 (oblique) computations produced lift values $C_L \approx 2$, comparable to those found in the laboratory. Adapted from [9].

Section 2 gives the essentials of the modeling and numerical methods we chose to successfully enable the first Navier–Stokes validation of the laboratory hovering vortex dynamics and thrust findings. Section 3 presents our analysis and proposed improved boundary condition for the far field vorticity. Section 4 discusses our preliminary findings for the direction of the hover-jet
Figure 3: (a) Sample grids which are near-orthogonal in the physical domain and uniformly rectangular in the computational domain. When the airfoil is an ellipse, the auxiliary domain is exactly circular. (b) Schematic representation of the model employed which takes an airfoil physical exterior domain to a Navier–Stokes rectangular computational domain, via an intermediate near circular auxiliary domain. Adapted from [6].

and related considerations. Section 5 addresses some associated mathematical issues which are of interest in their own right and which will have a bearing on future investigations. Section 6 contains further remarks.

2 Navier–Stokes computation of hovering dynamics

With reference to Fig. 3, we used a body-centered, body-concentrated grid scheme for the computation of these highly oscillatory hovering motions. Details may be found in [4–7] beyond the brief description below. For airfoils we used NACA airfoil profiles but when proceeding to the hovering simulations we reverted to an elliptical airfoil of 15% thickness. This rounding of the trailing edge did not seem to be critical to the accuracy of our simulations. A novel feature of our approach was to conformally map the domain exterior to an airfoil into a near-circular interior auxiliary domain which is then mapped to an aspect-2 computational cavity. See Fig. 3. In this way we were able to avoid the need to specify outflow boundary conditions at the far-field boundary, which has been effectively moved to $\infty$. However, we did set vorticity $\omega$ to zero on the far-field boundary ($\xi = 0$) of the computational domain. I will return to this point in the next section.

The modeling of the physical laboratory setup flow dynamics by the mathematical Navier–Stokes equations on computer was accomplished as follows. See [22] for further details, especially as concerns the exact physical laboratory set-up. We recall that in the physical laboratory, the airfoil plunges and pitches in a still-air environment, thereby doing work on the fluid. In our computational
setup (Fig. 3), the fluid environment translates and rotates about the fixed airfoil, thereby doing work on the airfoil. In this situation, the appropriate Navier–Stokes equation becomes

\[ \frac{d^2 \mathbf{r}}{dt^2} + \frac{d\Omega}{dt} \times \mathbf{r} + \Omega \times \mathbf{r} + 2\Omega \times \mathbf{v} + \frac{d\mathbf{v}}{dt} = - (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{Re} \nabla^2 \mathbf{v} - \nabla p \]

translational, Eulerian, centrifugal, Coriolis, acceleration

advection, diffusion, pressure gradient.

Figure 4: (a) Laboratory-measured downstream thrust jet angle (dotted line) versus Navier–Stokes computed (near-airfoil) average jet angle (solid line) for several angles of attack (average pitch angle). (b) Absolute value of lift versus the mean pitch angle. (c) Lift dependence on Reynolds Number. (d) Coefficient of lift versus solver order. Adapted from [10, 18].
It can be shown that in two dimensions all of these apparent body forces can be taken into an ‘effective’ pressure $p$, except for the Eulerian term, and that in our hovering motions the latter is a pure curl. In our simulations, we work with the ‘apparent’ vorticity $\omega^* = \omega_{\text{lab}} + 2\Omega(t)$ where $\Omega(t)$ is a nondimensionalized angular rotation rate of the airfoil. We also work with the perturbation...
Figure 6: Phase portraits and power spectra for hovering lift dynamics. (a) Tuned Mode 2, \( \text{Re} = 100, \ \alpha = 25^\circ, \ h_a/c = 0.5, \ f = 1.8 \text{ Hz.} \) (b) Tuned Mode 2, \( \text{Re} = 400, \ \alpha = 25^\circ, \ h_a/c = 0.5, f = 1.8 \text{ Hz.} \) Adapted from [12, 13].

stream function \( \psi^* = \psi - \psi_\infty \), where \( \psi_\infty \) is the stream function due to the flow at infinity and includes the effects of the angular rotations and translations of the environment about the fixed airfoil. By the use of the noninertial rotating coordinate system and apparent velocities in that system, the Euler term accelerations are accounted for, and dropping all asterisks (*) for convenience, our flow equations then become the stream function equation

\[
\nabla^2 \psi = -\omega
\]

and the vorticity transport equation

\[
\frac{\partial \omega}{\partial t} + \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial \xi} \left( h_2 \mu \omega \right) + \frac{\partial}{\partial \eta} \left( h_1 v \omega \right) \right] = \frac{L}{R} \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial \xi} \left( \frac{h_2}{h_1} \frac{\partial \omega}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{h_1}{h_2} \frac{\partial \omega}{\partial \eta} \right) \right],
\]

where

\[
\nabla^2 \psi = \frac{1}{h_1 h_2} \left( \frac{\partial}{\partial \xi} \left( \frac{h_2}{h_1} \frac{\partial }{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{h_1}{h_2} \frac{\partial }{\partial \eta} \right) \right) \psi.
\]

See [7] or [22] for more details. We recall that in (2) \( u \) and \( v \) are the \( \xi \) and \( \eta \) velocity components, \( L \) is a characteristic length (related to airfoil chord length), \( R \) is the effective (accelerative) Reynolds number, and \( h_2 \) and \( h_1 \) are grid-stretching factors in the \( \xi \) (perpendicular to airfoil surface) and \( \eta \) (parallel to airfoil surface) directions, respectively. We employed the grid-generation scheme of [26] to create a near-orthogonal body-centered, body-concentrated grid upon the physical domain, by using the method of [26] to find a near-polar grid in the auxiliary domain corresponding to a
uniform grid in the computational domain, as shown in Fig. 3. To do so, we used the so-called weak constraint method from [26], in which the distortion function \( f(\xi, \eta) = h_2/h_1 \) inside the auxiliary domain may be chosen to obviate the effect of mapping singularities occurring when infinite domains are mapped to finite domains. In particular, we chose the simple weak constraint interpolation formula \( f(\xi, \eta) = \xi f(1, \eta) \) in the auxiliary domain to assure that the distortion function \( f \to 0 \) as \( \xi \to 0 \). The price one pays for this is the anisotropy evident in the coefficients on the right-hand side on (2). See [5] for further details.

The stream function equation was solved by a multigrid method at each time step. The vorticity transport equation was solved using an ADI (alternating-direction-implicit) method. Each dimensionless time step is decomposed into two successive time steps in which one employs only a three-point stencil first in the \( \xi \) direction, then in the \( \eta \) direction. I mention in editorial interjection here that the ADI methods remain very competitive even today and are hidden within some of the current NASA flow codes.

At \( \infty \) a uniform freestream flow \( \psi \) at angle of attack \( \alpha \) is specified. On the airfoil surface we used the impermeable body condition \( \psi = 0 \). On the airfoil surface we used a first order vorticity boundary condition inasmuch as many studies have shown this to be superior to higher order formulations. At \( \infty \) we simply set vorticity \( \omega = 0 \) (see [7, 22] for more details). As Figs 1 and 2 illustrate, our approach was successful for the ends we sought: accurate simulation of the hovering motion vortex dynamics, and accurate airfoil lift computation for several hovering modes.

To briefly recapitulate what was involved in producing Fig. 1, following Freymuth’s [20, 21] laboratory apparatus set-up, in our computational model a plunge plane (e.g. an inclined \( x \) axis) is chosen and an elliptical airfoil with preset eccentricity and with center restricted to the plunge plane at all times is specified. Then the hovering motion is generated by the airfoil being driven to a continuous sinusoidally periodic motion governed by the plunge amplitude \( h(t) = h_0 \sin(2\pi ft) \) and the pitch angle of attack \( \alpha(t) = \alpha_0 + \alpha_\xi \sin(2\pi ft + \phi) \). The plunge \( h(t) \) is always taken to mean the center of the moving airfoil at time \( t \), and the pitch \( \alpha(t) \) is also specified to be at the center of the (plunging) airfoil. The preset parameters \( h_0 \) and \( \alpha_\xi \) govern the plunge and pitch amplitudes, respectively. The wing beat frequency \( f \) establishes an effective Reynolds number according to \( R_e = 2\pi h_0 c/v \) based on maximum plunge speed, \( c \) being an airfoil length parameter and \( v \) the kinematic viscosity. The parameter \( \alpha_0 \) is variously called the average (over one full hovering cycle) angle of attack, the mean pitch angle, the stroke plane angle, the tilt angle. The phase angle \( \phi \) determines the relationship between the plunging and pitching motions. For the Mode 1 hovering shown in Fig. 1(a) one takes \( \alpha_0 = 0 \) and \( \phi = -\pi/2 \).

For Mode 2 hovering shown in Fig. 1(b) one takes \( \alpha_0 = \pi/2 \) and \( \phi = \pi/2 \). One can determine the angle at which the motion starts simply by setting \( t = 0 \). As illustrated in Fig. 1, in Mode 1 the leading and trailing edges change at the end points of the plunging motion, whereas in Mode 2 the leading and trailing edges retain their roles throughout the hovering motion. The Mode 1 motion of Fig. 1(a) results from parameter settings \( \alpha_\xi = 66^\circ, h_0/c = 1.5, f = 1.0 \text{ Hz}, R_e = 340 \). The time increment between pictures is 1/64 second and each represents many more (about 25) Navier–Stokes calculation increments. For the Mode 2 motion of Fig. 1(b) we took \( \alpha_\xi = 33^\circ \), \( h_0/c = 1.0, f = 1.3 \text{ Hz}, R_e = 300 \). There the time separation of the frames is 1/16 s.

The computation of lift \( C_L \) is more involved and I defer to [9, 14] for the details, and as well to [21, 22] for the details of downstream thrust \( C_T \) measurement in the laboratory. The basis for comparison is a belief that the calculation of the average thrust coefficient \( C_T \) from the time averaged far-field momentum excess should be approximately the same as an average of the instantaneous lift coefficient computed mathematically by an integration of the airfoil surface pressure and friction forces perpendicular to the stroke (i.e. plunge) plane. Figure 2 represents our parametric study for Mode 1 and Mode 2 motions at different plunge amplitudes \( h_0/c \) and...
different maximum pitch angles $\alpha$, as indicated there. These were the identical parameters used in the laboratory studies of Freymuth [21]. We also of course used his Reynolds number $R_f = 1700$ and $f = 1.8$. The Mode 1 agreement shown in Fig. 2 is remarkable. That our Mode 2 results, although qualitatively in agreement with Freymuth’s, differ quantitatively (i.e. are smaller), is not fully understood.

Some comment can be supplied here, however. First, it should be noted that true lift $C_L$ should be calculated from pressures and friction ‘on the airfoil surface’ whereas due to our imposed no-slip boundary condition on the surface, we computed $C_L$ by an integration around the first grid contour, which is very close to the airfoil by our airfoil-centered grid scheme. Second, our procedure was to start up the motion, go through several (e.g. four) initial hovering cycles to establish the near-periodic motion, then the coefficient of lift $C_L$ was computed from a twice averaged instantaneous lift data set, the first being an integration around the airfoil to obtain the ‘surface’ pressures and frictions and resultant instantaneous lift data for each time-step, and then we took the average of those values over the next four full cycles to produce $C_L$. As I mentioned, the agreement with the Mode 1 $C_T$ values was very rewarding. Even for Mode 2, for a given angle of attack both laboratory and computational calculations predict the same plunge amplitude for peak lift. Third, there may be effects on $C_L$ which are dependent on airfoil thickness. We used a thickness of 15%, which corresponds to ellipse eccentricity of 0.998. This would seem a rather ‘thin’ airfoil but Freymuth’s (rounded) rectangular airfoil had only a 6% thickness. Since Mode 2 retains the same leading edge throughout its motion, such a profile difference may be having its effect on $C_L$.

3 Vorticity in the far field

Our interest [4–9] was in hovering vortex-shedding dynamics and near-wing lift computation for comparison to the physical laboratory results [19–23]. Thus we concentrated attention and computation to the near field. Let us now come to (2) in the physical far-field. In the near physical field, the grid-stretching ratio favors $h_1$, because we concentrate the finite difference grid near the airfoil. Stated another way, we want high aspect ratio grid rectangles, $h_1 \gg h_2$ layered along the airfoil surface. On the other hand, in the physical far field in our scheme this means that $h_2 \gg h_1$. There, the radically high aspect ratio grid rectangles elongate out toward infinity. This follows because the distortion factor $f = h_2/h_1$ within the interior auxiliary domain decreases from its basic airfoil-set aspect ratio, which depends upon the coarseness or fineness of mesh employed there, to zero as the grid tends outward toward physical infinity. To better understand (2) in the physical far field, taking into account the grid-stretching factors, we now treat $u$ and $v$ as fixed (or more generally we could let them move in bounded intervals) so that we may simplify the analysis of (2). Then the second advection derivative in (2) becomes

$$\frac{\partial}{\partial \eta} (h_1 v \omega) \approx v \omega \frac{\partial}{\partial \eta} (h_1) + v h_1 \frac{\partial}{\partial \eta} (\omega).$$

But in the far field, $h_1 \approx (\xi \cdot \text{some factor})$, and in these radially elongated grid rectangles that factor is relatively unvarying with respect to $\xi$, so that we conclude that the first term of (3) is $\approx v \omega$ (factor $\partial \xi / \partial \eta \approx 0$, because the grid has been constructed near-orthogonal. This is the basic idea which makes the following analysis possible.
Thus the accelerative and transport left-hand side (LHS) of (2) becomes, using for efficiency the subscript notation for the derivatives, approximately

\[ \omega_t + \frac{\mu \omega}{h_1 h_2} (h_2^2) + \frac{u h_2}{h_1 h_2} (\omega_\xi) + \frac{v \omega}{h_1 h_2} (h_1 \eta) + \frac{v h_1}{h_1 h_2} (\omega_\eta) \approx \omega_t + 0 + \frac{u}{h_1} \omega_\xi + 0 + \frac{v}{h_2} \omega_\eta, \]

\[ (4) \]

and thus

\[ (\text{LHS}) \approx \omega_t + \frac{u}{h_1} \omega_\xi, \]

\[ (5) \]

where the term \( \frac{\nu \omega_\eta}{h_2} \) was dropped because \( h_2 \) is so large in the physical far field. This is the second idea/assumption in our analysis.

The diffusive right-hand side (RHS) of (2) becomes by this analysis

\[ \frac{R}{L} h_1 h_2 (\text{RHS}) \approx \omega_\xi \frac{\partial}{\partial \xi} \left( \frac{h_2}{h_1} \right) + \frac{h_2}{h_1 h_2} \omega_\xi + \omega_\eta \frac{\partial}{\partial \eta} \left( \frac{h_1}{h_2} \right) + \frac{h_1}{h_2} \omega_\eta \eta \]

\[ \approx \omega_\xi h_2 \left( -\frac{1}{h_1^2} \right) + \frac{h_2}{h_1} \omega_\xi + \omega_\eta h_1 \left( -\frac{1}{h_2} \right) + \frac{h_1}{h_2} \omega_\eta \eta \]

\[ (6) \]

so that

\[ (\text{RHS}) \approx \frac{L}{R} \left[ \omega_\xi \left( -\frac{1}{h_1^2} \right) + \frac{1}{h_1^2} \omega_\xi + \omega_\eta \left( \frac{1}{h_2} \right) + \frac{1}{h_2^2} \omega_\eta \right] \]

\[ (7) \]

and thus

\[ (\text{RHS}) \approx \frac{L}{R} \left[ -\frac{1}{h_1^2} \omega_\xi + \frac{1}{h_1^2} \omega_\xi \right], \]

\[ (8) \]

where the terms \( \frac{\nu \omega_\eta}{h_2^2} \) and \( \frac{\nu \omega_\eta}{h_2^2} \) were dropped because \( h_2 \) is so large in the far field.

Therefore, we may conclude that the effective vorticity boundary condition in the far field is

\[ \omega_t + \frac{u}{h_1} + \frac{1}{h_1^2} \omega_\xi \]

\[ \omega_t + \frac{1}{h_1} \omega_\xi = 0 \]

\[ (9) \]

This is a one-dimensional (in outgoing \( \xi \) spatial direction) advection–diffusion equation in which the far field rectangle grid width \( h_1 \) (rotative \( \eta \) scale factor) enters in three different powers. Since \( u = \frac{v}{h_2} (\partial \psi / \partial \eta) \) vanishes in the very far field as \( h_2 \) becomes very large, we may arrive at the simpler equation

\[ \omega_t + \frac{1}{h_1} \omega_\xi = \frac{1}{h_1} \left( \frac{L}{R} \right) \omega_\xi = 0 \]

\[ (10) \]

as the simplified far field vorticity boundary equation.
We may also present a scale analysis [17] to provide a theoretical corroboration of the grid-based analysis above. It turns out that such conventional scale analysis provides slightly less detail than the grid-based analysis, but on the other hand it applies of course to more general situations.

Therefore consider boundary-fitted coordinates in an exterior physical domain such as that of Fig. 3. Let $\xi$ be the radial direction coordinate and $\eta$ the angular coordinate in a polar coordinate system with origin placed somewhere upon the body. Consider the Navier–Stokes equation (2) with $h_1 \sim O(1)$ and $h_2 \sim O(r)$ as $r \to \infty$. Let

$$\epsilon \equiv h_1/h_2 \sim 1/r \ll 1.$$  \hfill (11)

Following the rules of scale analysis as presented for example in [27, pp. 18–21], we first note the spatial extent of the region of interest to be $R \sim 1/\epsilon$ unbounded. Let us assume a bounded flow, i.e. $u, v, w$ all $O(1)$ in that region, i.e. we assume that no flow singularities develop in the far-field flow. Then we have the scaling order of magnitude relations

$$\xi \sim R \sim 1/\epsilon \quad \text{unbounded}$$

$$u, v, w \sim O(1) \quad \text{bounded}$$

$$h_2 \sim 1/\epsilon, \quad h_1 \sim O(1)$$

$$\frac{\partial}{\partial \xi} \sim 1/R \sim \epsilon, \quad \frac{\partial}{\partial \eta} \sim \frac{1}{R}.$$  \hfill (12)

Thus the scaling of eqn (2) must satisfy

$$\frac{\partial \omega}{\partial t} + \frac{1}{h_1 h_2} \frac{\partial}{\partial \xi} (h_2 u \omega) = \frac{L}{\text{Re}} \frac{1}{h_1 h_2} \frac{\partial}{\partial \xi} \left( h_2 \frac{\partial \omega}{\partial \xi} \right).$$  \hfill (13)

which becomes

$$\frac{\partial \omega}{\partial t} + \epsilon [1 + \epsilon] = \frac{L}{\text{Re}} \epsilon \left[ \epsilon + \epsilon^3 \right].$$  \hfill (14)

By the sum and product scaling rules [27] when $\epsilon \to 0$ relation (14) requires that the $\partial(h_1 \omega)/\partial \eta$ and $\partial((h_1/h_2)\partial \omega/\partial \eta)/\partial \eta$ terms in (2) be dropped in the physical far field. Thus one has a far field vorticity equation

$$\frac{\partial \omega}{\partial t} + \frac{1}{h_1 h_2} \frac{\partial}{\partial \xi} (h_2 u \omega) = \frac{L}{\text{Re}} \frac{1}{h_1 h_2} \frac{\partial}{\partial \xi} \left( h_2 \frac{\partial \omega}{\partial \xi} \right).$$  \hfill (15)

If we now hold $u$ constant this provides a far field vorticity boundary equation

$$\omega_t + \frac{u}{h_1} \omega_\xi - \frac{1}{h_1^2} \left( \frac{L}{\text{Re}} \omega_{\xi \xi} \right) = 0$$  \hfill (16)

comparable to (9) and (10).

Much interest was attached to the issue of appropriate outflow boundary conditions in the 1980s. A discussion and comparison of those of Enquist, Agarwal, Glowinski, Lugt and Haussling, Mehta and Levan, Telionis, Ghia, Zakharenkov, Wang, and others, may be found in [17, 18]. Some worked well. Most were based upon physical reasoning. A main point was to ‘let the vortices out’ and
we also did that. However, none of the earlier analyses were as mathematically rigorous as our discussion above. Future testing of our formulation is envisioned, especially as it will apply to better downstream full hover-jet simulation.

**Remark.** This section presents the analysis of [14, Part I, Section 3.2, pp. 32–34] but with the $h_2$ and $h_1$ scale factors interchanged and thus corrected. The confusion in the presentation of [14] arose from interpreting the distortion function $f(\xi, \eta) = h_2/h_1 = \xi f(1, \eta)$ in the physical exterior domain rather than in the interior auxiliary domain where it is originally defined. Indeed the above analysis and that of [14] employ a physical domain far-field grid-aspect ratio $h_2/h_1 \to \infty$. But those scale factors are the images under the Joukowsky map of a grid-aspect ratio $h_2/h_1 \to 0$ in the interior auxiliary domain. The same remark/correction applies to [16, Section 3, p. 173]; eqn (2) there should be replaced with eqn (10). Similarly, in the stability analysis of [14, p. 36], in eqns (3.2.11) and (3.2.12) there one should replace $\eta$ and $h_2$ by $\xi$ and $h_1$ to keep vorticity bounded in the far field for the standard explicit upwinded convection-centered diffusion scheme used there for illustration. It should be emphasized that these notational corrections in no way diminish the analyses and conclusions of [14, 16].

### 4 The direction of the hover-jet

Another current and related issue concerns the direction of the hover-jet, i.e. the downstream shed reverse Karman sheet of counter-rotating vortices which make up the principal aerodynamic thrust generated by the insect hovering dynamics. For a fine laboratory downstream vortex development, see [22, figure 6.3]. As I already mentioned above, the most difficult part of the flow forms to resolve numerically was the traversing of the jet.

On the other hand, in nature, it is the vectoring of the hover-jet that enables all of the dragonfly’s maneuvers, hence its survival. This brings us to a discussion of Mode 3, in which the mean pitch angle is oblique to the plunge plane. This is the hovering mode which best describes dragonfly aerodynamics. For dragonflies as observed in nature, the mean stroke (i.e. plunge) plane is forward and downward at a 60° angle with respect to a horizontal body axis. Then the required mean pitch angle to enable hovering over a jet directed vertically downward is $\alpha = 120°$. See [24, especially figure 1 there] for more details.

In [8, 9] we verified these findings by computational Navier–Stokes simulations of Mode 3. Moreover we were able to attain rather good vortex dynamics pattern matches to Freymuth’s laboratory experiments. Note [9, figure 5] that the oblique hover-jet direction is approximately 30° to the stroke plane, in accordance with the discussion above. Our (limited) Mode 3 computations showed lift coefficients $C_L \approx 2$ for angles $\alpha_0 = 30°, 45°, 60°$, in rough agreement with Freymuth’s laboratory findings [21]. These computational results are shown here in Fig. 4(b). Mode 3 approaches Mode 1 and Mode 2, respectively, as $\alpha_0$ approaches 0° and 90° respectively, and Fig. 4(b) bears that out. To obtain the values there we tried to optimize the flow parameter settings for each $\alpha_0$, i.e. the lift values shown are best cases. It should be mentioned that for most cases the lift-coefficient for Mode 3 with $\alpha_0 = 45°$ was much lower, $C_L \approx \pm 0.5$. These lift studies were incomplete. Another incomplete study was the relation of lift to the Reynolds number. Preliminary results are shown in Fig. 4(c), the computations performed on identical parameter sets except for the Reynolds number. Intuition would suggest that lift should increase with the Reynolds number up to some point, provided that the other flow parameters are tuned to maintain a ‘clean’ Karman vortex street, but that thereafter turbulence would win and increased Reynolds numbers would just produce more disarray. Our flow codes indeed produced highly disordered pictures as we pushed the Reynolds number too high, although it is not clear if this was physical
or numerical instability. One difficulty is that to adequately resolve higher Reynolds flows, one must use finer grids and many more time steps.

Returning to the issue of the direction of the hover-jet, more generally Freymuth’s laboratory studies found the following important experimental result. Specifically, Freymuth found that [20, p. 239], ‘Crudely speaking, for every degree of change in angle of attack $\alpha_0$ there is a $2^\circ$ change in the direction of the hover jet.’ We [18] call this the ‘2 for 1’ rule. I am able to visualize it in an optical way by imagining a light ray reflecting off a mirror as I slowly change the angle of the mirror from the vertical. There are other ways to intuitively justify it.

Interestingly, and as I mentioned above, we were independently concerned computationally with the same question, but for completely different reasons. Therefore, we [10, 11, 14, 18] have computationally already inadvertently roughly corroborated Freymuth’s hypothesis. This work has not been published in the open literature so let me present the main result here [10, 18]. It is useful to remember that the ‘2 for 1’ rule is the hypothesis that (roughly) $\alpha_{\text{jet}} = 2\alpha_0 - 90^\circ$ in terms of angle of attack $\alpha_0$. In Fig. 4(a) (from [10, 18]) for Mode 2 (figure eight) hovering one finds our computed jet angle plotted against the average angle of attack $\alpha_0$, as compared to Freymuth’s laboratory findings [20, 21]. It should be recalled that the average angle of attack for such a pitching-plunging mode is taken to mean the average angle of attack with respect to the stroke plane over a full hover cycle, i.e. it is the mean pitch angle. For more details see Freymuth’s paper [24, figure 1] in this volume and his discussion there.

Now, some caveats about these preliminary findings. We differed from the exact ‘2 for 1’ rule somewhat. For example, in Mode 2 hovering we computed (see Fig. 4(a)) a jet angle which had an average departure from the vertical of $7.5^\circ$. Physical intuition would indicate that this value should be $0^\circ$. Our jet angles were computed as follows: at each time step all angles were inspected to determine the angle of maximum instantaneous lift. This value $\alpha_{\text{jet}}(t)$ was computed and stored. Then in post-processing the average jet angle was computed for each cycle. Each data point in Fig. 4(a) represents the jet angle averaged over 4 hovering cycles. Those data points were then fit by least squares. The resulting linear fit was $\alpha_{\text{jet}} = 1.92\alpha_0 - 89.44$ in degrees, which contains the small deviation from physical intuition mentioned above.

It is useful to ask what does it mean to say ‘the jet angle’? Presumably one envisions some average linear direction extending from the airfoil outward through the center of the Karman vortex street. This contrasts somewhat from our mathematical determination of the $\alpha_{\text{jet}}$ direction which we computed just from $C_L(t)$ values on the grid contour closest to the airfoil. Clearly one needs to do the same for downstream grid contours. Or one could work from a sequence of downstream vortex pair centers to arrive at an approximate jet direction. Or even calculate a Navier–Stokes-based thrust from downstream velocities at approximately the laboratory pitot tube locations. To do so one would need to employ a much finer far-field grid. Also then the issue of the most appropriate far-field vorticity computational boundary condition which I discussed in the previous section comes into play.

5 Associated mathematical matters

Extrapolating from Fig. 4(b), it is interesting [10, 18] to consider analytically $C_L$ as a continuous function of mode, i.e. as a continuous function of $\alpha_0$. In this connection, one has considerable freedom to reconfigure phase angles $\phi$, e.g. for Fig. 4(b) we used phase $\phi = -90^\circ$ for Mode 2 to match that of Mode 1. The physical effect of such a phase sign change is to create a jet in the downward direction for Mode 2 and the upward direction for Mode 1. Then intuitively a Mode 3 flow with $\alpha_0 = 45^\circ$ might seem to have a horizontal jet but we did not get $C_L$ equal to zero, instead we obtained the small values $\pm 0.5$ mentioned above.
In any case, by analytically treating $C_L$ as a continuous function of mode, and Mode 1 and Mode 2 as special cases of Mode 3, we showed [10, 18] that for airfoil eccentricity equal to zero, Modes 1, 2, and 3 are equivalent for the same pitch and plunge amplitudes and wing beat frequencies. This implies that the different dynamics observed between Mode 1 and Mode 2 are strongly dependent upon the eccentricity of the airfoil as the airfoil departs from a circular profile. This mathematical conclusion is consistent with the computational experiments of Loe [28] for a pitching and plunging cylinder. Furthermore, it would indicate that our computational simulations should be repeated with airfoil eccentricity as a variable parameter for better comparison to laboratory experiments, and especially with very large eccentricities, i.e. very thin airfoils, for better comparison to nature. The very thin airfoils will cause some deterioration of the grid near-orthogonality that we try to maintain in our model.

Let me turn next to numerical matters, those being the essential core producing our flow results. Because there are so many such considerations one could discuss, I will concentrate on our ADI-based solver for the vorticity transport equation of (2). There are two innovations unique to what we did: the use of BCH theory for devising our discrete operator splitting methods, and the implementation of an upwinding switch (only when necessary).

Before describing those two innovations, it should be recalled that in most CFD computations involving discrete elliptic Laplacians and parabolic advection equations, one prefers to use tridiagonal linear solvers wherever possible. The efficiency gained by doing so far outstrips many other considerations. ADI is based upon this fact and therefore alternates between three-point stencils in the chosen spatial directions. Also the multigrid smoothing steps, using Gauss–Seidel or some other relaxation method, also employs tridiagonal solvers. We employed a two-step relaxation procedure, periodic red–black line relaxation in the $\eta$ direction followed by red–black line relaxation in the $\xi$ direction. The coefficients of the matrices in the $\eta$ direction are larger and the periodicity requirement (to maintain continuity at the cut about the airfoil) needs two tridiagonals for each line relaxation. Thus a complete line relaxation makes use of six tridiagonal calls, two for each of the red and black periodic $\eta$ sweeps, 1 for each of the red and black $\xi$ sweeps. For example, using a multigrid scheme with five $V$ cycles, on a $65 \times 65$ grid, our loops called the tridiagonal solver $2 \times 64$ (red, black, periodic) plus $2 \times 32$ (red, black $\xi$) = 192 calls/iteration. The number of iterations is three per multigrid grid change (two before the coarse grid correction, one after). Thus we use 2880 tridiagonal solver calls just to do the stream function update, for each time step. So even with the most efficient methods (multigrid, ADI), one wants to keep the grid not overly refined and the numerical schemes as low order as possible. To get better resolution by means of just increasing the number of grid points (the ‘brute-force’ approach), for example to increase the number of grid points to $257 \times 257$, one would need seven $V$ cycles and a similar operation count reveals 11,520 tridiagonal solver calls per time step.

Such considerations coupled with the physical fact that hovering dynamics principally involves vortices with large spatial extent, allowed us to get early good results on relatively coarse $(33 \times 33, 65 \times 65)$ grids. We also concentrated our (previous) grid points near the airfoil, by our infinite-domain mapping method. However, for better far-field flow resolution, we needed finer grids or better numerical analysis.

Our innovation toward the latter stems from the fact that most advection solver schemes may be seen (e.g. see [3]) as approximations to exponential expressions. For example the famous (unconditionally stable but inefficient) Crank–Nicolson scheme for the simple heat equation $\omega_t = \omega_{xx} = -A \omega$ where $A$ is a discretization matrix and $\omega$ is the numerical (discretized, in vector form) solution, becomes the iteration

$$\omega^{n+1} = e^{-t/2A}e^{-t/2A}\omega^n \approx \left(I + \frac{t}{2}A\right)^{-1}\left(I - \frac{t}{2}A\right)\omega^n. \quad (17)$$

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With more complications the Navier–Stokes vorticity transport equation of (2) may be viewed as a problem of exponentiating a discretized operator equation of the (simplified a bit here) type

$$\omega_t = A\omega + B\omega$$

(18)

to a flow solution $\omega(t) = e^{(A+B)t}\omega(0)$. To better understand the splitting errors involved in splitting $e^{A+B}$ into $e^A e^B$ advection–diffusion factors in the various ways to discretize the problem without asking that $A$ and $B$ commute (which is physically out of the question), we used the Lie theory in which for a given $e^X$ and $e^Y$ we are assured of an operator $Z$ such that the exponentially split $e^X e^Y = e^Z$, where

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] - \frac{1}{24}[X, [Y, [X, Y]]]$$

$$- \frac{1}{720}[[[X, [Y, Y]], Y], Y] - \frac{1}{720}[[[Y, X], X], X] + \cdots,$$

(19)

where $[X, Y]$ denotes the commutator $XY - YX$ and where one may go to as high an order as one wants. Our analysis [10, 11, 14] established (for the first time, to our knowledge) that: most CFD schemes create symmetry breakings; in particular, numerical-operator splittings may be interpreted as symmetry breakings; from such matrix exponentiations a qualitative change from hyperbolic to trigonometric matrix elements gives a new explanation of numerical oscillation (e.g. instability) onset; the in principle Lie theory provides an exact error analysis and a means of scheme correction. Thus, in particular, given a chosen advection–diffusion split operator scheme $X, Y$, using (19) we can compute the matrix $Z$ which represents the actual stencil being used. Then given such $Z$ and any other chosen $X$, we can use BCH backwards to any desired level of accuracy (up to the accuracy preserved in $Z$ in the expansion (19), carried out to more terms if needed) to determine $Y$ such that $e^X e^Y \approx e^Z$.

We then employed this theory to set our ADI upwinding switch when needed when traversing the vortex-jet region of the flow. ADI methods offer increased temporal stability, rapid convergence rates, and computational efficiency (e.g. tridiagonal matrices, even in two space-dimensional solvers such as we needed). The implicit sweeps are unconditionally stable and provide adequate relaxations to keep the explicit direction sweeps at least bounded. The ADI theory in higher dimensions, for noncommuting operators, is not complete, but a general rule of thumb is that one should at least keep matrix diagonal dominance to ensure that the computational matrices remain positive definite.

Our studies revealed that rather than just employing a more standard advection–diffusion splitting scheme, it was advantageous to devise instead a directional splitting. Even when the $u$ and $v$ velocity profiles were such that the directionally split operators no longer commuted, we still obtained a smaller first-order error for our directionally split scheme than for advection–diffusion splittings. Others doing advanced CFD simulations have also employed directional splitting for various reasons. Our BCH methods could be seen in the wider perspectives as reaching the same (lower order errors) conclusions by a more rigorous mathematical analysis.

Within any given flow application, no matter what your approach to error analysis may be, the critical issue becomes how to best remove (unwanted) numerical oscillations without removing the (wanted) physical oscillations. An original (new) analysis of this general issue in computational fluid dynamics may be found in the book [14, Part I, Chapter 3, Section 3.1]. This new theory led me to formulate the (new) notion of numerical rotational release. Apparently this notion remains relatively unknown, even though in some applications, its importance is as fundamental as the well-known notion of numerical viscosity. The point is that upwinding often ‘numerically releases’ the (wanted) secondary and tertiary (smaller) vortices.
For hovering motions the smaller vortices do not seem to play such an important role since the Karman street large-vortex dynamics dominate the qualitative development of the flow. Nonetheless upwinding usually hides a decreasing of the effective Reynolds number. Also, upwinding truncates or clips fluid structures in regions of large gradients. As a compromise with these countervailing issues, we replaced the usual two-step ADI with a three-step ADI, thus gaining an additional order of accuracy in time. Second, we implemented a local velocity based upwinding switch for the advection operator to be triggered in steep gradient regions, switching back to the more accurate centered schemes elsewhere. The amount of upwinding to be included was determined by minimizing the first order correction term in the BCH expansion of the directionally split operator for the magnitude of velocities and velocity gradients anticipated for a given flow problem. Our overall ansatz was to retain diagonal dominance, hence positive definiteness, hence stability, for the iteration matrix.

Figure 4(d) shows results of the first order two time step ADI solver as compared to the second order three time step ADI solver. The Mode 1 run was for parameter values $\alpha = 66^\circ$, $f = 1.8$ Hz, Reynolds number $R_f = 1700$. The largest lift variation was less that ten percent, although the second order solver gave slightly larger lifts. Although not shown, this was due to slightly higher vorticity capture in each vortex region. The general closeness of the first- and second-order solver lift coefficients can be attributed to the fact that the larger (signed) vorticity extrema tended to cancel each other out due to the total airfoil integration in the $C_L$ computation. Both solvers placed the vortical fluid structures in the same locations relative to the airfoil. The message should have been expected: low-order solvers are often sufficient, and more efficient. On the other hand, it may be noted (not shown, but see [11]) that the second-order solver better resolved features in the far field. In this discussion it should be recalled that ADI is second order in space and that is usually sufficient.

6 Further remarks and outlook

Our two-dimensional fundamental hovering vortex dynamics and lift findings were later confirmed in three dimensions by Liu and Kawachi [29]. They used a different computational scheme and primitive variables $u$ and $p$, which further corroborates the validity of our results. The issue of far-field vorticity boundary condition is not considered in [29]. There, the far-field computational boundary conditions are specified to be: Neumann boundary condition $\partial u/\partial n = 0$ on velocity, Dirichlet boundary condition $p = 0$ on pressure, on the exterior grid boundaries.

Because the subject is so fascinating, the topic of the aerodynamics of hovering insects is often rediscovered. This causes disparate independent studies by biologists, engineers, mathematicians, each somewhat unaware of the others. In some sense this volume testifies to that diversity. As one example we cite the recent study of insect flight [30]. To quote: 'Until recently, however, an embarrassing gap has marred our understanding of insect flight: scientists have had a difficult time explaining the aerodynamics of how insects generate the forces needed to stay aloft.' Suffice it to say that we [1–23] felt confident of our explanation. And it was the first, from a Navier–Stokes computational point of view. Of course, there are always further issues and improvements to be made.

Other instances of this cultural separation between researchers in the biological, engineering, and mathematical communities, may be seen in the separate journals in which their results are presented. For example, in the recent paper [31] it is stated that we did not provide any details of the instantaneous fluid dynamical forces. But we did, in [13], a rather mathematically dedicated journal far from the typical AIAA audience. To that end I show in Fig. 5 a typical Mode 2 lift
coefficient waveform at Reynolds number $R_f = 400$. One goal of [13] was to bring the dragonfly hovering dynamics into the general context of dynamical systems, e.g. into the context of chaos research, the control of large dimensional dynamical systems by their low-order qualitative behavior determining manifolds, issues of the existence of strange attractors, period doubling, and so on. Figure 6 uses methods of dynamical systems research, the phase portrait and the power spectra, to further analyze the time dependent instantaneous lift. Due to the fact that $C_{L}(t)$ is an averaged entity, these dynamical signatures concentrate the power in lower frequencies. As the Reynolds number is increased from 100 to 400, a wider band of higher frequencies is activated. I refer the reader to [13] for more details. I also presented those results in [12] but the audience was mostly physicists. One concept I wanted to emphasize in both [12] and [13] was that of ’structural stability’, a topological notion from the theory of nonlinear differential equation systems. From this viewpoint I conjecture the existence of an effective inertial manifold for the hovering dynamics. Biologically this would be evidenced by the fact that dragonflies and hummingbirds fly rather well even when their hovering flight parameters are not optimally tuned. Another message of [12] was that such structural stability only exists in certain parameter intervals, which have been determined by evolution.

A second goal of [12] was to attempt for the first time a synthesis of other key subsystems, beyond just the aerodynamical subsystem, which critically affect the overall success of hummingbird, dragonfly and similar hovering species. Thus I would like to see a total dynamical systems detailed mathematical theory and model which integrates the aerodynamical, thermodynamical, structural dynamical, neurodynamical, perhaps other key subsystems, into one encompassing theory. For example, the beautiful hovering aerodynamics is of little use for some envisioned applications if so much fuel is consumed (enter the thermodynamics) that flight over sustained time intervals cannot be planned. A lot of scattered information about the thermodynamical, structural dynamical, and neurodynamical subsystems has been accumulated over the years by the biological community, some of it well represented by some of the authors of this book. The outlook I am sketching here would comprise two principal tasks. One would be an integrated model, from the engineering viewpoint. This would be constructed by a small team using existing software such as Simulink or similar packages. The second task would be a more rigorous mathematical theory from the dynamical systems viewpoint. This would take a bit longer.

As we [23] and Freymuth [24] point out, Wang [32] obtained results for jet angle significantly different from ours. We conjecture that the discrepancy could be at least partially due to the use of an incorrect far-field boundary condition. The recent investigations [31] obtained lifts different from ours. The earlier investigations [29] corroborated our computed lifts. It is beyond the scope of this paper and this investigator to try to arbitrate all such issues, even though they be important.

However, a few words may be said. Our studies [13] tentatively concluded that lift discrepancies may be due to laboratory measuring setups and airfoil scalings. The qualitative match of our lift curves to those of the laboratory, and the qualitative match of our computational vortex patterns to those of the laboratory, could indicate scaling inconsistencies. See [13] for more details. Then (as I have mentioned earlier) there is the issue of airfoil thickness as a factor. High eccentricity means low drag and low eccentricity means high drag, and in measuring lift computationally we have ignored the amount of drag within the total generated aerodynamical force. Also in our computational scheme we have assumed that the far-field vortices do not need to be considered in our near-field lift computation, even though one knows that solutions to elliptic PDEs (e.g. the stream function equation) are always global.

We have not discussed the so-called [20] Modes 4 and 5. However, as a general statement, we see no reason why our Navier–Stokes based simulations could not be applied to all such hovering motions. Also with sufficient resources we could model caudal fin-fish propulsion as discussed
elsewhere in this book by Freymuth [33]. For example, note that our Navier–Stokes computational $C_T$ results in the left side of Fig. 4(b) here agree quite closely to Freymuth’s $C_T$ results [33, figure 3], even though his Reynolds number was higher. Propeller efficiencies are typically no more than 40% whereas 86% efficiency has been achieved in airfoils with properly tuned plunge and pitch motion. Experimentally it has been found that highest propulsive efficiency in fish occurs for foil excursions to angles of attack between 15° and 25° [34]. It would be desirable to model and validate by unsteady Navier–Stokes computation such reported propulsive efficiencies. It should be mentioned that our scheme need not be sinusoidally driven, any pitch-plunge periodically matched driving action could be entertained.

In closing, I wish to emphasize that I have in no way attempted to state all known results. Rather, here I have given an account of our own work. Finally let us note and acknowledge the wonderful classic books [35–37], and hope that this present volume joins them.

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