Advanced Vector Analysis
for Scientists and Engineers

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Preface

Vector Analysis is one of the most useful branches of mathematics. It is used in practical problems arising in engineering and the applied sciences. The main purpose of this book is to illustrate the application of vector calculus to physical problems. This book is suitable for a one-semester course for senior undergraduate and graduate students in science and engineering. It is also suitable for scientists and engineers working in practical fields.

This book is the outgrowth of class notes used over years of teaching vector analysis in many universities including Dalhousie University. The theory is explained clearly and there is an abundance of worked examples throughout. The beauty of the book is its richness of examples. There are eight chapters, each one containing ample exercises at the end. There are also three important appendices at the end of the book. Solutions to selected exercises have been included in Appendix A. Appendix B contains a quick summary of important vector formulae, and Appendix C contains a brief historical background of vector calculus which was originally introduced in the days of Aristotle during the fifteenth century. A systematic development of the evolution of vectors with basic vector algebra is described. This appendix also describes the modern vector calculus and vector analysis due to the contributions of Josiah Willard Gibbs (1839–1903) and Oliver Heaviside (1850–1925). Later Maxwell contributed heavily in the electromagnetic theory applications.

Chapter 1 introduces the algebra of vectors. This chapter contains vector addition and subtraction, scalar and vector products and vector triple products. Representation of vectors by the coordinate system; equations of lines and planes using vector algebra are demonstrated. The chapter contains many applied exercises.

In Chapter 2, we consider vector functions of one variable to illustrate vector differentiation and integration along with the geometric interpretation of a position vector. Vector integration is applied to determine the length and arc length of a curve. This chapter also contains particle motion on a curve, its velocity and acceleration. Tangential and normal components of acceleration of a particle moving in space are discussed in a lucid manner. The chapter ends with a considerable number of practical exercises.

Partial derivatives of a scalar function for several independent variables are clearly defined in Chapter 3 and the idea is extended to vector functions. The functions of multiple variables and the concept of Jacobian are stated clearly with examples. The chapter is concluded with ample exercises of practical interest.

The Del ∇ operator (gradient operator) plays a very important role in practical problems arising in engineering and physical sciences. Chapter 4 is devoted to the precise definition of this important operator. With the help of this operator, a number of quantities such as gradient, divergence and curl are defined in the light of physical applications. Applied problems such as continuity of a fluid flow in incompressible and compressible fluids and also rotation of a fluid in the viscous case are very elegantly described with physical interpretation. We have also cited some formulae involving the Del operator with proofs. The Laplace operator ∇^2 plays a very significant role in physical problems and this is discussed very thoroughly. The chapter ends with exercises containing a number of important problems.

In Chapter 5, we have discussed the mathematical theory containing line, surface and volume integrals starting with the basics. A conservative vector field has been defined and illustrated with examples. Surface and volume integrals are also discussed with some practical examples. Triple integrals in cylindrical and spherical coordinates are discussed in this chapter. Practical situations are illustrated. Ample problems are included at the end of the chapter.

Chapter 6 contains some important and extremely useful integral theorems such as Green's Theorem, Stokes' Theorem and the Divergence Theorem. Many examples are solved to help the reader understand the practical use of these sophisticated theorems. They are very useful for the further study of research problems. The chapter concludes with ample exercises of academic and physical interest.

Orthogonal curvilinear coordinate systems are discussed in Chapter 7. The development of grad, div, curl are demonstrated clearly using the orthogonal curvilinear coordinate system. These important quantities are derived in the frames of Cartesian, cylindrical polar and spherical polar coordinates. Some practical examples are worked out. This chapter concludes with a number of practical exercises.

The last chapter (Chapter 8) of this book contains the applications of vector analysis. We have selected problems in fluid flow, electromagnetic theory with emphasis on Maxwell's equations. Some solution techniques are considered for Maxwell's wave and heat equations. A brief discussion on nonlinear ocean wave interaction has been presented at the end to show why vector analysis is important to the solution of physical problems. We have briefly demonstrated how to solve the nonlinear wave energy equation. The next section of the chapter deals with the numerical simulation of a vector field using Mathematica 3.0. The last section examines a practical problem of irrotational and inviscid fluid flow. The chapter concludes with a number of exercises of physical interest.

Chapters 1–5 contain elementary vector calculus and chapters 6–8 have dealt with advanced material of vector analysis. The book is developed sequentially starting with the elementary level and ending with the advanced level. It is our hope that the reader will find the book useful.

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