Stochastic equilibrium assignment with variable demand: literature review, comparisons and research needs

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Abstract

The aim of this paper is threefold: (1) propose a detailed literature review; (2) compare existing approaches and (3) discuss the main research needs. The literature review is structured with respect to (a) solution approaches: optimization, variational inequalities and fixed-point (internal and external approaches). (b) choice dimensions taken into account, (c) solution algorithms. The comparison is aimed at highlighting the transportation problems that each approach is able to address and to highlight the most effective approach within a specific problem and/or the most performing solution algorithm within the same approach. Finally, a sort of research "road-map" is drawn in order to emphasize research needs and research perspectives.
Keywords: traffic assignment, variable demand, state of art, research perspectives.

1 Introduction

Most modelling approaches to equilibrium assignment assume that origin-destination demand flows are known hence path choice is the only behaviour explicitly modelled. Such assignment models, known as assignment model with constant (rigid) demand, do not consider the role that other choice dimensions (such as trip production, and/or choice of departure time slice, destination, transport mode, parking type and area) may have on equilibrium configuration (and more broadly on the transport system evolution). In such a context, many government agencies and transport analysts point out the need for assignment

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models with variable (elastic) demand since demand elasticity may be relevant for urban planning over a medium-long term horizon. To estimate the effects of actions on a transport system the use of models describing the interaction between demand and supply based on the consolidated (steady state) equilibrium approach is consolidated. Two ways are generally considered to deal with this approach: (a) a fixed demand approach (Fig. 1), in which path choice is assumed variable (respect to congested costs) and the remaining choices that typically characterize a trip are considered invariant (frequency, destination, time slot, mode of transport, etc..); (b) a variable demand approach (Fig. 2), in which, in addition to path choice, other choice dimensions are assumed variable.

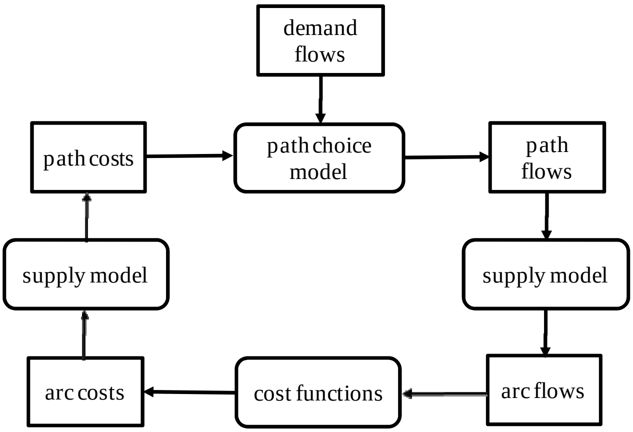

Figure 1: Scheme of the fixed demand approach.

The choice between the two approaches depends on operational and/or methodological considerations. From an operational point of view, the variable demand approach is necessary when it is reasonable to imagine that interventions on the transport system will cause substantial changes in the characteristics of mobility in addition to the path choice [31]. From a methodological point of view, even in the absence of operational reasons, the variable demand approach is necessary when demand is unknown and must be estimated by a set of models.

Although the topic has been investigated since the 1970s, application are mainly based on heuristic methods, based on a very single application of the external approach, on deterministic path choice behaviour and other simplifying assumptions.

In this context, the aim of this paper is threefold:

1. propose a detailed literature review;
2. compare existing approaches;
3. discuss the main research needs.

The literature review is structured with respect to:

a. solution approaches: optimization, variational inequalities and fixed-point (internal and external approaches);
b. choice dimensions taken into account;
c. solution algorithms.

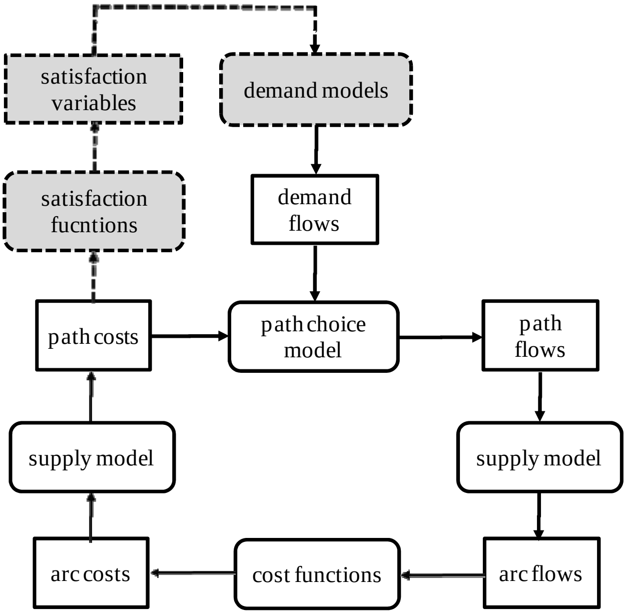

Figure 2: Scheme of the variable demand approach.

The comparison is aimed at highlighting the transportation problems that each approach is able to address and to highlight the most effective approach within a specific problem and/or the most performing solution algorithm within the same approach. In particular, which kind of cost functions can be considered (separable or not separable), which kind of choice models can be implemented (which theoretical paradigm; which mathematical formulation), which kind of travel demand dimension can be embedded.

Finally, with respect to literature evidence and with respect to the most effective and efficacy approaches, a sort of research "road-map" is drawn in order to emphasize research needs and research perspectives.

2 Literature review

2.1 Introduction

The assignment problem has been the subject of extensive research for several decades. Exhaustive analyses of the state of the art of the models (and the algorithms) for uncongested network and user equilibrium assignment are reported in the books by Sheffi [1], Thomas [2] and Patriksson [3], the latter being mainly devoted to deterministic assignment models. For deterministic assignment models, the paper by Florian and Hearn [4] can also be referred to while a state of the art for stochastic assignment models is described in Cascetta and Cantarella [5].

Deterministic User Equilibrium (DUE) models with separable cost functions and Systems Optimum models were formulated with optimization models analogous to those described in the mid-1950s in the pioneering work of Beckman *et al*. [6], based from the enunciation of Wardrop principles [7]. But it was not until the 1970s, with the increasing availability of computing power, that assignment problem received continuous theoretical attention and a number of applications.

The extension of the optimization model to symmetric deterministic equilibrium and the formulation of asymmetric deterministic equilibrium with variational inequality models, together with existence and uniqueness conditions, are dealt with in the work of Dafermos [8–11] and Smith [12]. These papers also describe first extensions of DUE models to variable demand and multi-class assignment. Variable demand assignment problem has aroused concern since the deterministic user equilibrium was introduced by Wardrop [7] and, successively, formalised through optimization models in the pioneering work of Beckman *et al*. [6]. From Beckmann work the topic has received much attention at the end of the 1970s till the mid-1980s and, again, in the middle of the 1990s. The main contributions may be classified depending on:

i. the approaches to analyze and solve the equilibrium problem.
ii. the choice dimensions considered variable respect to the link costs (path costs)
iii. the hypothesis on the mutual influence between different transport modes that share the same infrastructure.

2.2 Approaches and models

As regard the approaches, optimization models and fixed-point models are the most used ones. Optimization models and their extensions (variational inequalities) allow a compact formulation, can rely on several algorithms and can be applied to large scale case study. On the other hand they require simplistic hypothesis on cost functions (separable vs. not separable), on the demand functions, on the route choice models and on the mutual influence between different transport modes. In fact, optimization models require separable cost functions, whereas variational inequality models allow us to comply with not-separable cost functions, but not with stochastic route choice models.

Fixed point models have a simpler mathematical formulation, comply with less binding hypothesis, but cannot rely on effective algorithms that can be easily extended to large scale applications.

A further problem is related to the simulation of the interaction between different modes using the same infrastructure. This issue, not allowing rigorous mathematical formulations using optimization models, determines the adoption of variational inequalities models [11] and fixed point models [13, 14]. The fixed-point approach allows us to use both deterministic and probabilistic choice models.

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2.3 Choice dimensions

As regards choice dimensions, assuming the demand divided in four main choice dimensions (trip production, destination, transport mode), the main contributions can be classified respect to the choice dimensions which are supposed variable: only destination, only transport mode, destination and transport mode.

Concerning the mutual influence between transport modes, it is useful to distinguish between two cases: (i) single-mode assignment, there is one mode for which link costs depend on flows, and either the demand elasticity does not depend on modal split at all or link costs for all other modes are not congestion-dependent; (ii) multi-modal assignment, there is more than one mode with link costs depending on flows (congested modes). In this case, the cost attributes of congested modes cannot be known before the solution of the assignment model, and it is necessary to solve the equilibrium assignment problem simultaneously.

Respect to the above classification the literature can be split in contributions oriented to a theoretical formalization of the problem and contributions that apply consolidated models.

One of the first tempt to formalize the problem is proposed by Florian *et al.* [15] that analyze the destination and assignment problem and propose an optimization model with linear constraints, and continuous and separable cost functions. The model is solved through an algorithm proposed by Tomlin [16] and derived from Frank and Wolfe algorithm. In the same year Evans [17] faces the same problem through an optimization models and proposes an algorithm based on partial linearization method. For the first time, in 1977, Florian takes into account the mode choice dimension and explicitly simulates the interaction between different transport modes without capacity constraints. The problem is decomposed in sub-problems solved through known techniques, the resulting algorithm converges, as demonstrated in Fisk and Nguyen [18].

In the same year, Erlander [19] copes with the distribution and assignment problem proposing an optimization model with entropy maximizing formulation. The same problem is faced by Abdulaal and LeBlanc [20]. Among three different approaches the most interesting one introduces a link cost function for car and bus, the function depends on the total flow on each link and capacity constraints are taken into account. An optimization model and an algorithm similar to Frank and Wolfe are proposed. Florian and Nguyen [21] analyse the combination of destination choice and mode choice and propose an optimization model where the transport modes are independent. Two algorithms are suggested: a modification of Frank and Wolfe and the one by Evans [22]. A further generalization is reported in Safwat and Magnanti [23], they present an optimization model that includes the trip production, the destination choice and the mode choice. The Frank and Wolfe algorithm is used. LeBlanc and Farhangiam [24] discuss different algorithms, mainly based on the Frank-Wolfe technique, for solving variable demand traffic assignment problems and modal split-assignment problems.

In 1982, Dafermos [11] presents the first variational inequality model applied to deterministic user equilibrium with variable demand. The main hypotheses are

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very general: links cost depend on the flow of the other transport modes and the demand functions depend on the systematic utility of all the possible destinations and of all transport modes available. If cost functions and demand functions are continuous and their feasibility sets are compact, it is demonstrated that the solution exists and it is unique. The solution algorithm solves iteratively a mono-modal equilibrium problem with variable demand. In 1983 Florian and Spiess [25] consider a two mode (bus and transit) equilibrium assignment model which incorporates a zonal aggregate mode choice model, and formulate the problem as a variational inequality model. Moreover, they propose an equivalent optimization formulation. Nagurney [26] test the efficiency of the relaxation and projection methods for calculating the traffic equilibrium in multimodal networks. She tested three different classes of monotone travel cost functions and found that the form of the travel cost functions affects the performance of both methods.

For the first time, fixed-point models for SUE assignment were introduced by Daganzo [27], who also analyzed variable demand assignment (with the hyper-networks approach) and multi-class assignment.

Sheffy [1] proposes a systematic overview for all choice dimensions, he follows the optimization approach and, through the definition of an augmented network model, he demonstrates that the equilibrium with variable demand can be solved through algorithms similar to those used to solve the equilibrium with fixed demand. In the proposed model the transport modes do not influence each other.

Lam and Huang [28] present a combined trip distribution and assignment model with multiple user classes. The entropy-type trip distribution submodel is used and the link travel time is assumed to be similar for all traffic flows sharing the same link. Unsymmetrical link cost functions are used by conversion into symmetric forms through a ‘normalization’ procedure. Two different algorithms based on the Frank-Wolfe’s and Evans’ are developed and their computational results on test networks are reported.

Patriksson [3] gives an exhausting framework of models and methods to solve the deterministic equilibrium with variable demand problem. He investigates several approaches founded on: optimization models, variational inequalities or methods that use augmented network models.

In Oppenheim’s book [29] optimization models and solution algorithms are proposed as the choice dimensions involved change. The most interesting model involves destination choice and mode choice, and distinguish the following contexts: transport modes not scheduled and link costs of one mode independent from link flows of other modes, transport modes scheduled with link costs independent, both type of transport modes and link costs dependent from link flows of all modes. The algorithms proposed are mainly founded on partial linearization method.

In 1997 Cantarella [13], starting from fixed-point models introduced by Daganzo [27], develops a general treatment with fixed-point models of multi-modal/multi-class variable demand equilibrium assignment also for pre-trip/en-route path choice behaviour, including stochastic as well as deterministic user equilibrium. The existence and the uniqueness of solution are demonstrated and simple convergent algorithms based on Method of Successive Averages (MSA) are proposed.

Bell and Iida [30] face both stochastic and deterministic user equilibrium with variable demand. All the choice dimensions are taken into account, optimization models and variational inequalities models are proposed. For the combination of destination choice and route choice the authors refer to the model and to the algorithm proposed by Evans [22]. In presence of stochastic route choice models the approach proposed by Lam and an algorithm based on MSA method are suggested. Ferrari [32, 33] proposes a single-mode equilibrium assignment model with variable demand which takes into account the presence of capacity constraints. The model follows the variational inequality approach, and assumes non-separable cost functions and variable demand up to mode choice.

Bellei *et al.* [34] introducing a network pricing optimization problem, propose a fixed point model to deal with multi-user multimodal assignment problem with variable demand. The demand is supposed variable up to trip production, the multimodal network is modelled through hypergraph and asymmetric arc cost functions are used. Bar-Gera and Boyce [35] present a fixed point formulation for combined models and convergent algorithm for combined models is presented. In 2005 García and Marín [36] develop an approach to the multi-modal assignment problem with combined modes based on the variational inequality approach. The model explicitly takes into account the choices of route, mode and transfer node, may be used with fixed and variable demand and may deal with deterministic and stochastic traffic assignment models. Salis [37] and Smith [38] introduce two-direction methods for finding traffic equilibrium. The first paper shows that the common algorithm used for estimating variable demand may give unrealistic results. The second paper deals with the solution of variable demand equilibrium models with and without signal control. Finally, Cantarella and Cascetta [39] propose a systematic framework founded on fixed point modes for the stochastic equilibrium and on variational inequalities for the deterministic equilibrium.

Alternative and interesting approaches are presented by Wong [40] and Wichiensin *et al.* [41]. The former presents a continuum equilibrium model to simulate the assignment of traffic flows from central business districts to the possible destinations over the city. The flows from one origin to the destinations are considered as one commodity and the interaction of the traffic flows among different commodities is governed by a cost–flow relationship. A finite element method is proposed to solve the continuum problem. The latter proposes a Bertrand-Nash equilibrium model that incorporates mode choices and fare settings. Besides car, two public transport modes are considered and the road congestion affects the modes sharing the same infrastructure. In 2009, Liu *et al.* [42] introduce the method of successive weighted averages to solve stochastic user equilibrium problem in order to enhance solution convergence. They propose an algorithm in which higher weights are given to the auxiliary flow patterns from the later iterations, moreover develop a self-regulated averaging method, in which the step sizes are varying, rather than fixed, depending on the distance between intermediate solution and auxiliary point.

2.4 Applications

Starting from the described contributions, a relevant part of literature has been devoted to application issues oriented to simulate the effect of different strategies to real case studies. Irwin and Von Cube [43], Manheim and Ruiter [44], Wigan and Bamford [45], although without any theoretical background, propose the first applications that take into account the interaction between different transport modes and its effect on modal split or trip distribution. Since then, many contributions and applications to transportation planning scenarios have been proposed, from software packages (SATURN and SATCHMO [46]) to specific modelling specifications. The most interesting one are summarized below.

Fernandez *et al*. [47] deal with the assignment problem with combined transport modes; Nagurney *et al*. [48] propose an optimization model oriented to evaluate mitigation strategies of pollutants emission, the solution algorithm is based on the modified projection method introduced by Korpelevich [49]. Abrahamsson and Lundqvist [50] take into account destination choice and mode choice, and assume that transport modes do not interact with each other. They propose an optimization model solved through the partial linearization technique implemented by Boyce *et al*. [51]. May and Milne [52] evaluate the effects of alternative road pricing systems using SATURN software and SATEASY routine. De Cea *et al*. [53] present a model where two choice dimensions (destination and transport mode) are considered variable respect to link costs, different users classes are considered and the transport mode share the capacity supplied by each link. The assignment model is based on variational inequalities and it is solved through the diagonalization method. In 2001, Clegg *et al*. [54] present a multicopy approach, whereas Yildirim and Hearn [55] extend the first best toll pricing framework for the fixed demand traffic equilibrium model to the variable demand traffic assignment problem. They formalize the system problem and the user problem through a variational inequality model. Following the fixed point approach introduced by Daganzo [27] and Cantarella [13], D'Acierno *et al*. [56] propose an optimization model for urban parking pricing problem. The assignment model assumes that the mode choice is elastic and that the link costs depend on the flows of all transport modes that share the same link. Solution algorithm is based on MSA framework.

Florian *et al*. [57] report, with reference to a large scale application (case study of Santiago de Chile), the application of equilibrium with variable demand considering in a multi-class and multi-modal context. The mode choice model is represented by a logit hierarchized structure and the destination choice by means of an entropic model. The variational inequalities capture all the components of the model in an integrated form. The algorithm adopted is the MSA ("method of successive weighted averages") and the numerical results are obtained by the software EMME/2 [58]. Maher [59] demonstrates the possibility of resolving the "Stochastic User Equilibrium with elastic demand" (SUEED) using the "Balanced Demand" algorithm in alternative to MSA, checking its faster computational performances, as well as its applicability to SUE Probit and Logit.

From this experience, he draws some theoretical considerations that SUEED formulation does not appear to be much more complex than SUE, and the possibility to extend both the formulation of the objective function that the applicability of the algorithm to the multi-class case.

Montella *et al.* [60] deal with the problem of the variable demand considering costs of the individual and transit mode are both function of the total flows that affect their respective arcs. Three different algorithmic approaches are considered: external, internal and hyper-network. The multi-modal assignment problem is formulated as a fixed point by means of two sub-problems: a fixed point problem for individual transport and a stochastic loading problem for the transit system. Theoretical properties related to the existence and uniqueness of the solution are investigated.

Lo *et al.* [61] propose a methodology to build a multimodal network that allows us to exploit the mathematical and algorithmic formulations of assignment problem consolidated over the last decade. The work draws on three aspects: the growing interest on multimodality; the need to realistically simulate the phenomenon of transfer in terms both of the number of transfers and of types of transhipment; the need to implement non additives costs in public transport assignment problem.

Nagurney and Dong [62] suggest an alternative formulation to the existing models for the multi-modal and multi-criteria simulation for the equilibrium of transport networks. They assume that travellers perceive the disutility of the generalized cost as a combination of travel time and travel cost (each one depending on flow). Weights depend both on the class and on the arc. The formulation is based on the theory of variational inequalities.

Some algorithmic issues on variable demand assignment in the case of transit are made by Lei and Chen [63]. They propose a variational inequalities model of transit assignment with variable demand and capacity constraint. The model is able to estimate the flow of passengers in case of congested network. Moreover, they suggest, on the basis of the method of the penalty function, a resolutive algorithm. Yildirim and Hearn [55] discuss the extension of schematization for first best toll pricing from the case of the constant demand to the case of variable demand. They formalize the problem in terms both of system and of user, through a variational inequalities model.

On the basis of the approach introduced by Daganzo [27] and Cantarella [13], D'Acierno *et al.* [56, 64], Gallo *et al.* [65, 66] propose different contributions based on optimization models to solve pricing and transit network design problems.

3 Comparisons

The assignment with variable demand can be carried out through two main approaches: the internal or the external one (Fig. 2). The former relies on formal modelling frameworks, consolidated algorithms and formal demonstration of their convergence. The external approach can address any kind of assignment problem (any type of cost function, any type of travel demand model), but neither a model's formalization may be derived (solution's existence and uniqueness) nor a solution algorithm can be demonstrated to converge. In this framework, the external approach may be interpreted more as a heuristically procedure rather than a robust approach to the assignment problem with variable demand.

From an interpretative point of view it is possible to distinguish two approaches: a first one that incorporate all choice dimensions (departure time, destination, mode, path) within a global equilibrium problem and a second one which aims to study the most interesting choice dimension for operational purposes: modal choice. The second approach is adopted in most practical applications, while the simulated modal context is typically mono-modal.

Within the internal approach, scientific literature points out that three main methods have been adopted for addressing variable demand assignment problems (Tab. 1): optimization (opt), variational inequality (VI) and fixed-point (f-p). The former, if on the one hand can count on a larger set of solution algorithms, on the other hand can only address mono-modal assignment problem, with separable cost functions and with stochastic path choice models. The VI's approach allows us to address the assignment problem with not-separable asymmetric cost functions. In this case if a formal model can be derived, solution algorithms convergence can be demonstrated for symmetric cost functions only. Fixed-point approach relies on a very small set of solution algorithms, even not much efficient, but it can cope with a wider variety of operational issues: separable, not separable and asymmetric cost functions; mono-modal and multi-modal assignment problems; different travel demand models (continuous and invariant); stochastic and deterministic path choice models; mono-user and multi-user contexts. Moreover, f-p's solution algorithms converge in the asymmetric cost function assignment problem, if based on arc-costs.

Finally it should be noted that in some (but not all) cases, assignment methods with constant demand can be extended to cope with elastic demand through hyper-network approach.

4 Conclusions and research needs

The static equilibrium with variable demand appears as the more robust, effective and efficient approach both from a point of view of interpretation of the phenomenon, and from a theoretical and operational point of view. Within the internal approach and the fixed-point method, several research needs seem worthy of interest. At this aim a sort of "road map" may be drawn.

1) As concerns cost functions, deeper convergence analyses should be carried out for not-separable asymmetric cost functions;
2) As regards travel demand choice dimensions, it should be tested the effect on solution efficiency of choice models different from those usually adopted [70, 71] and/or of intelligent information systems [72, 73];

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Table 1: Literature framework.

		DEMAND DIMENSIONS					ASSIGNMENT MODEL			SOLUTION
		trip freq	*dest*	*mode*		*path*	*approach*	*variables*	*notes*	
				mono	multi					
SUPPLY	*Separable or not-separable symmetric cost functions*					d	opt	flows		c
		✓	✓			d	opt	flows	nat	c
		✓	✓	✓		d	opt	flows	nat	c
				✓		logit	opt	flows (arc and path)		c
				✓		s	opt	flows		c
				✓		s	f-p	flows		c
		✓	✓		✓	s	f-p	flows/costs		c
		✓	✓		✓	s	f-p	flows/costs		c
	asymmetric cost functions	✓	✓	✓		d	VI	flows	nat	c-sym normally applied
		✓	✓		✓	d	f-p	flows		c-sym
		✓	✓		✓	s	f-p	flows		c-sym
		✓	✓		✓	s	f-p	costs		c

List of symbols: s = stochastic; d = deterministic; nat = not any type of travel demand models may be used;
c = convergent; c-sym = convergent with symmetric cost function only.

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3) As regards solution algorithms, further enhancement of the Method of Successive Averages are necessary. Different averaging schemes, different averaging variable (one vs. two), different convergence tests.
4) As regards more theoretical issues
   - formal proof for convergence conditions of internal algorithms with non-separable demand functions;
   - more general uniqueness conditions including the case of arc cost functions not necessarily monotone on the feasibility set;
   - more general convergence condition for internal algorithms;

Finally, another relevant research perspective concerns model calibration. Indeed several groups of parameters are to be estimated to fully specify a model for assignment with *variable demand*; yet it is common practice to estimate them separately. Pursuing such an approach incoherent estimations are obtained. In fact, travel costs used to estimate model parameters are different from those obtained by assigning the travel demand to the network. In this case parameters estimation should be carried out simultaneously with the assignment procedure with variable demand.

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