

Some improvements in minimum weight topology optimization with stress constraints

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Abstract

Topology optimization of continuum structures is a recent field in structural optimization. However, an increasing research activity in this area has been developed since the statement of the very first formulations. These formulations try to obtain the most adequate material distribution that satisfies the imposed structural limitations. The existence or absence of material in each part of the domain is usually defined by using a continuum variable (the relative density) in order to avoid dealing with a discrete optimization problem. This continuum approach of the material properties present important advantages since conventional optimization algorithms can be used. However, numerical models must be considered in order to develop the structural analysis for intermediate values of the relative densities.

In this paper, we present some improvements in a minimum weight approach of the structural topology optimization problem. The main goal of this paper is to present an improved formulation that tries to reach binary 0-1 material distributions by using a continuum approach of the design variables. Furthermore, a perimeter penalization is included in the objective function to simplify the solutions obtained. In addition, some computational aspects are considered in order to reduce the computational effort.

Finally, we compare the solutions obtained by using these formulations in two application examples.

1 Introduction

Topology optimization of continuum structures is a recent field in structural optimization. However, an increasing research activity in this area has been developed



since the statement of the very first contributions [1, 2]. The traditional statements of this problem try to obtain the optimal distribution of material that maximizes the stiffness of the solution by adequately distributing a predefined amount of material in a predefined domain [1–3]. Thus, the main objective is to determine the parts of the domain where material should exist or not. The existence or absence of material is usually defined by using a continuum variable (the relative density) in order to avoid dealing with a discrete optimization problem. This continuum approach of the material properties present important advantages since conventional optimization algorithms can be used to obtain the optimal material distribution. However, numerical models must be considered to develop the structural analysis for intermediate values of the relative densities. This fact is usually solved by using predefined material microstructure models like SIMP (Solid Isotropic Material with Penalty) [3], for example.

In this paper, we propose a different approach of the topology optimization problem that minimizes the weight of the structure with stress constraints [4–7]. This formulation presents important advantages versus the maximum stiffness approach since the most important instabilities associated to maximum stiffness approaches are avoided. On the other hand, the minimum weight approach with stress constraints requires much larger computing resources than the maximum stiffness approaches. Thus, computational aspects must be addressed in order to reduce the computing effort.

The main objective of this paper is to present an improved formulation that tries to reach binary 0-1 material distributions by using a continuum variable (the relative density). Binary 0-1 material distributions are obtained by using a modified objective function based on the weight of the structure. This approach presents two important advantages since the continuum approximation allows to deal with continuum design variables and binary solutions are finally obtained. Furthermore, a perimeter penalization is included to obtain solutions with a reduced number of trusses.

2 Topology optimization problem

The Topology Optimization problem with stress constraints can be formulated from a generic point of view as

$$\begin{aligned}
 &\text{Find} && \boldsymbol{\rho} = \{\rho_e\} \\
 &\text{Minimize} && F(\boldsymbol{\rho}) = \text{Cost}(\boldsymbol{\rho}) \\
 &\text{subject to:} && g_j(\boldsymbol{\rho}) \leq 0 && j = 1, \dots, m \\
 &&& 0 < \rho_{\min} \leq \rho_e \leq 1, && e = 1, \dots, N_e
 \end{aligned} \tag{1}$$

where $\boldsymbol{\rho}$ is the vector of design variables, g_j are the stress constraints, m is the number of constraints considered, N_e is the number of elements of the finite element mesh [7, 8] and ρ_{\min} is usually equal to 0.001.



The main issue of this kind of formulations consists in the definition of the most adequate objective function to obtain the benefits expected in the final solution and in the definition of the stress constraints in order to guarantee that the final design satisfies the limitations imposed.

3 Stress constraints

According to that, the authors have proposed in previous publications three different formulations to deal with stress constraints in the topology optimization problem: the local approach, the global approach and the block aggregation of stress constraints [4, 5, 8, 9].

The local approach of the stress constraints states one stress constraint in the central point of each element of the mesh by comparing a reference stress with the maximum elastic limit of the material being used [7, 8, 10, 11]. Thus, if we introduce the required modifications to deal with the singularity phenomena [10, 12], the local stress constraints can be stated as:

$$g_e(\boldsymbol{\rho}) = \left[\hat{\sigma} \left(\boldsymbol{\sigma}_e^h(\boldsymbol{\rho}) \right) - \hat{\sigma}_{max} \varphi_e \right] (\rho_e)^q \leq 0, \quad \varphi_e = 1 - \varepsilon + \frac{\varepsilon}{\rho_e}. \quad (2)$$

The exponent q can be used to deal with real stresses ($q = 0$) or to deal with effective constraints ($q = 1$) [6]. The reference stress $\hat{\sigma}$ is the Von Mises criterion and $\boldsymbol{\sigma}_e^h$ is the stress tensor of the material in the central point of element e . The relaxation factor ε usually takes the values (0.001, 0.1) [4].

The global approach of the stress constraints is defined to avoid the large computing resources required by the local approach of the stress constraints. Thus, the local stress constraints are aggregated in only one global function by using the Kreisselmeier-Steinhauser approach [4, 7, 13]. Thus, the global constraint can be stated as

$$G_{KS}(\boldsymbol{\rho}) = \frac{1}{\mu} \left[\ln \left(\sum_{e=1}^{N_e} \exp \mu (\hat{\sigma}_e^* - 1) \right) - \ln(N_e) \right] \leq 0, \quad (3)$$

being

$$\hat{\sigma}_e^* = \frac{\hat{\sigma} \left(\boldsymbol{\sigma}_e^h(\boldsymbol{\rho}) \right)}{\hat{\sigma}_{max} \varphi_e}. \quad (4)$$

The aggregation parameter μ must take the largest value that does not introduce numerical instabilities. These numerical instabilities are usually related to the high non-linearity of the aggregation function [4]. The most appropriate range of values of the aggregation parameter is (20,40). These values have demonstrated to work properly in the application examples solved [4, 5, 8, 9].

The block aggregation of the stress constraints is a more general method that includes both previous formulations. In this approach the domain of the structure is divided in a predefined number of groups of elements such that all the groups

contain approximately an equal number of elements. Thus, a global-type constraint can be imposed over the elements of each block of the mesh [5, 8]. Thus,

$$G_{KS}^b(\boldsymbol{\rho}) = \frac{1}{\mu} \left[\ln \left(\sum_{e \in B_b} \exp \mu(\hat{\sigma}_e^* - 1) \right) - \ln(N_e^b) \right] \leq 0 \quad (5)$$

where B_b is the set of elements contained in block b and N_e^b is the number of elements contained in block b .

4 Objective function

The aim of the Topology Optimization problem with stress constraints is to determine the minimum weight structure that supports the applied forces. According to that, the objective function of the optimization problem is the weight of the structure:

$$F(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \int_{\Omega_e} \rho_e \gamma_{mat} d\Omega \quad (6)$$

where γ_{mat} is the material density, ρ_e is the relative density of element e and ω is the domain occupied by element e .

This objective function introduces an unexpected phenomenon since the material distributions obtained present a large number of elements with intermediate values of relative density, which is an unwanted solution in practice. Thus, a different formulation of this objective function has been proposed in order to penalize the material distributions with intermediate values of the relative density. This formulation has been obtained by following the same idea introduced in the SIMP model of microstructure. This modified objective function can be stated as:

$$F_p(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \int_{\Omega_e} \Psi_p(\rho_e) \gamma_{mat} d\Omega, \quad \Psi_p(\rho_e) = \rho_e^{\frac{1}{p}}, \quad p \geq 1 \quad (7)$$

where the exponent p is the penalization factor and takes the values $p \geq 1$ (figure 1 left).

This formulation of the objective function has been widely analyzed in previous publications of the authors and some application examples have been studied [4–8]. However, this formulation does not guarantee a final solution with 0-1 material distribution. This fact can be easily understood if we consider that the derivatives of the objective function are always positive. Thus, a reduction of the design variable always introduces a reduction in the objective function. This fact is the expected one from a physical point of view. However, the solutions obtained with this approach are not 0-1 when we use small values of the relaxation parameter (e.g. $\varepsilon = 0.01$). Stress constraints usually impose lower limits of the relative densities. Thus, the lower limits of relative densities imposed by the stress constraints and the positive value of the derivatives of the objective function define an optimal solution that usually presents intermediate values of the relative densities.

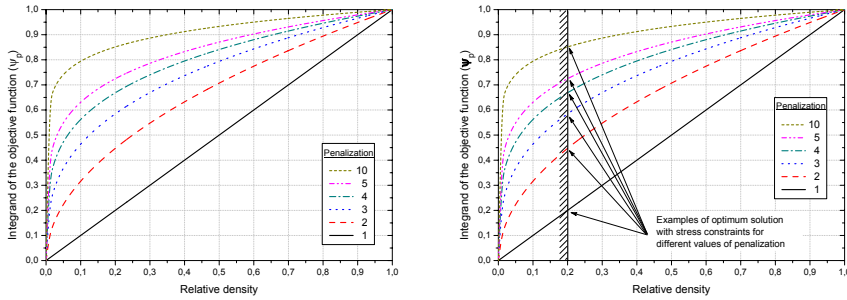


Figure 1: Integrand of the objective function ($\Psi_p(\rho_e)$) for different values of the penalization parameter (p) (left) and examples of optimum design variables when stress constraints are considered (right).

This issue can be explained by analyzing a simplified problem with one design variable and one stress constraint. The optimization algorithm tries to reduce the value of the design variable but the stress constraint avoids the reduction. The optimal solution is obtained for an intermediate value of the relative density although the penalization parameter has been used. This fact can be easily observed in figure 1 (right).

Thus, the proposed penalization of the intermediate relative densities does not guarantee 0-1 solutions when small values of the relaxation parameter are used. The values of the relaxation parameter must be as low as possible in order to avoid removing useful material in the final designs but, on the other hand, it must be large enough to avoid singularity phenomena.

The optimal solutions can be forced to reach 0-1 binary values by modifying the objective function proposed in (7). In this paper, we propose a modified objective function to obtain 0-1 optimal solutions as:

$$F_b(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \int_{\Omega} \Psi_b(\rho_e) \gamma_{mat} d\Omega, \quad (8)$$

$$\Psi_b(\rho_e) = \rho_e^{1/(1+\beta)^2} + \beta \exp^{-\beta \rho_e} \sin(\pi \rho_e), \quad (9)$$

This modification introduces aggressive changes in the mathematical function since the sign of the derivatives becomes negative in part of the range of relative density (see figure 2). This modification produces the expected benefits but it also introduces local minima and other unwanted phenomena. Thus, a whole procedure must be analyzed in order to include this modification in the optimization process.

Most of the initial solutions for the topology optimization problems use the maximum value of the relative density. Thus, if we use the modified objective function proposed in (8) the optimization algorithm will not modify the initial solution since the value of the objective function will rise for a reduction of the design variables. In this case, the optimum solution is the initial distribution and this is not the real

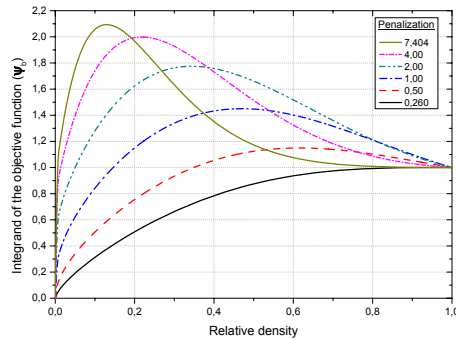


Figure 2: Integrand of the objective function ($\Psi_b(\rho_e)$) (8) for different values of penalization.

optimum solution. Thus, the modified objective function must be specifically used to obtain a final material distribution. The optimization process must be developed in two stages. First stage obtains an optimal material distribution with the objective function proposed in (7) by penalizing the intermediate densities. The second stage starts with the solution obtained in the first stage and incorporates the modified objective function in order to force the elements with intermediate densities to reach 0-1 values. This second stage must be developed by increasing progressively the value of the parameter β proposed in (8).

The values of the parameter β must be in the range $[0.260, 7.404]$ in order to obtain the expected solutions. Values smaller than, approximately, 0.260 introduce positive derivatives when $\rho \rightarrow 1$. Values higher than, approximately, 7.404 also introduces positive derivatives when $\rho \rightarrow 1$.

This formulation of the objective function does not produce minimum weight designs since the modifications introduced forces 0-1 material distributions. These solutions are not the optimal ones in minimum weight terms (when continuous design variables are considered) but they produce important benefits since no elements with intermediate relative densities appear. Thus, the optimal solutions obtained present the lowest cost in practice.

5 Parallel computing

The resulting optimization problem requires the use of efficient algorithms that allow to deal with a large number of highly non-linear stress constraints and a non-linear objective function. In this paper, we have used a Sequential Linear Programming algorithm with Quadratic Line Search proposed in [6, 7, 14].

This algorithm has demonstrated to produce optimal solutions for the optimization problems proposed. However, this algorithm and the sensitivity analysis involved [5, 8, 15] require large computing resources (computing time) when a large number of design variables and stress constraints is imposed. Consequently, it is necessary to propose computational techniques that allow to reduce the compu-

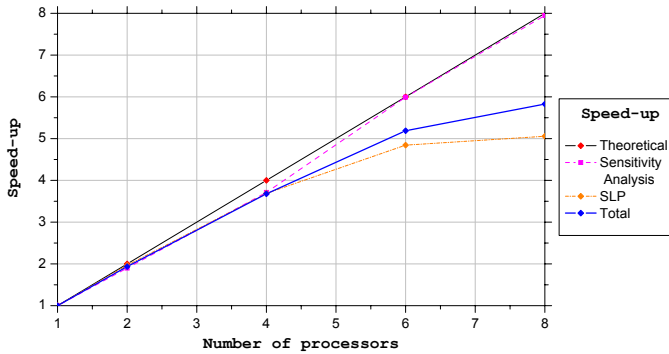


Figure 3: Speed-up obtained for the cantilever beam problem with 7200 design variables by using the local approach of stress constraints in a computer with 4 Intel Xeon 7120M Dual Core processors.

tational effort. According to this idea, parallelization techniques have been implemented in the sensitivity analysis and the optimization algorithm. First order derivatives can be computed independently for each stress constraint. Thus, the parallelization of this procedure is feasible and it produces suitable performance [8]. The Optimization algorithm can be also computed in parallel since the Simplex Algorithm involved develops a large number of matrix operations that can be computed in parallel. The performance of this parallelization is not as effective as in the sensitivity analysis computation since a number of sequential operations must be developed between each matrix modification. This fact can be observed in figure 3. However, the total speed-up of the optimization process allows to reduce the computing time about 6 times when 8 processors are used in a problem with 7200 design variables and constraints.

6 Perimeter penalization

The proposed objective function can be complemented with a perimeter penalization in order to obtain simplified solutions. This perimeter penalization produces optimal solutions with less structural elements (trusses) than the original statement of the objective function (7) and (8). This perimeter approximation is based on the total variation function proposed by Haber *et al.* [16]. Thus, the total variation of the perimeter is defined as:

$$TV(\rho) = \int_{\Omega \setminus \Gamma_J} \|\nabla \rho\| d\Omega + \int_{\Gamma_J} |\langle \rho \rangle| d\Gamma_J \quad (10)$$

where $\Omega = \bigcup_{\alpha} \Omega_{\alpha}$ and Ω_{α} is the set of disjointed regions (finite elements) that defines the whole domain Ω . The expression $|\langle \rho \rangle|$ indicates the absolute difference of relative density between two neighbour disjunct regions Ω_{α} (finite ele-

ments) and Γ_j represents the magnitude of the frontier between two contiguous elements (the length in a 2D problem).

In the topology optimization formulation, the relative density is uniform for each element and, consequently, the first term of equation (10) is null. Thus, the objective function including the perimeter penalization $\hat{F}(\rho)$ can be defined as:

$$\hat{F}(\rho) = F_p(\rho) + \eta \sum_{\Gamma_j} |\langle \rho \rangle| L_j \quad (11)$$

where L_j is the length of the frontier between two contiguous elements and η is the weight factor that relates the value of the perimeter to the value of the cost of the structure. This factor is determined by defining a reduced percentage of the ratio obtained by dividing the initial value of the weight by the initial value of the perimeter of the structure. This percentage usually varies from 1 % to 5 % in practical applications. High values of this percentage avoid the generation of trusses in the optimal solution since the minimum perimeter solution corresponds to an equal value of the relative density for all the elements. Thus, great areas with intermediate densities usually appear. On the other hand, low values of this percentage introduce an insignificant effect of the perimeter penalization.

7 Application examples

We present two structural problems frequently analyzed in the topology optimization field. These examples are 2D structures in plane stress. The examples proposed are analyzed by incorporating the perimeter penalization in order to show the effect of this penalization in the optimal solutions obtained. In addition, the effect of the modified objective function proposed in (8) can be also analyzed.

7.1 MBB beam

The first example corresponds to a classic MBB-beam with sliding supports [3]. Only the right half of the structure is analyzed due to symmetry. Figure 4 shows the dimensions of the domain and the position of the external forces (13.3 10³ kN). Self-weight is considered. The domain of the structure is discretized by using $N_e = 120 \times 40 = 4800$ eight-node quadrilateral elements. The material being used is steel with density $\gamma_{mat} = 76500 \text{ kN/m}^3$, Young's modulus $E = 2.1 \cdot 10^5 \text{ MPa}$, Poisson's ratio $\nu = 0.3$ and elastic limit $\hat{\sigma}_{max} = 230 \text{ MPa}$. The thickness of the structure is 1 m.

Figure 5 (up) shows the solution obtained by using the local approach of the stress constraints and the objective function proposed in (7). Figure 5 (down) shows the optimal solution obtained by using the local approach and the modified objective function proposed in (8) in order to obtain binary 0-1 solutions.

Table 1 shows the most important parameters involved in the optimum design formulation and the value of the final objective function as a percentage of the original weight of the structure. Note that the modified objective function does not



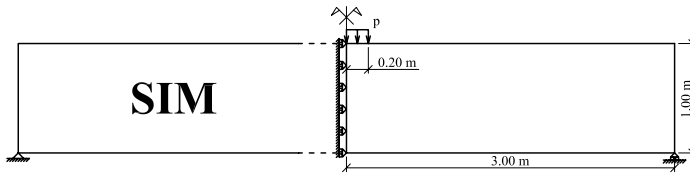


Figure 4: MBB-beam scheme (units in meters).

Figure 5: Optimal solution of the MBB-beam problem by using the local approach of the stress constraints (up) and by using the modified objective function (8) (down) ($\varepsilon = 0.01$, $q = 1$, $\eta = 0.03$).

Table 1: Summary of the most important parameters of the MBB-beam problem.

MBB BEAM	Local Approach (Fig. 5 up)	Modif. Objective F. (Fig. 5 down)
Number of elements	4800	4800
Penalization (p, β)	4	7
Final weight/Initial weight	15.43 %	19.60 %

produce minimum weight designs. However, the material distribution is essentially binary. A more accurate solution to minimum weight design with binary distribution can be obtained by using more refined finite element meshes.

7.2 Cantilever beam

The second example corresponds to a cantilever beam with no displacements allowed in the nodes of the left vertical edge. Figure 6 shows the dimensions of the domain and the position of the external load ($2 \cdot 10^3$ kN) applied on the border of 7 contiguous elements placed vertically. Self-weight is considered. The domain of the structure is discretized in $N_e = 120 \times 60 = 7200$ eight-node quadrilateral elements. The material being used is steel with density $\gamma_{mat} = 76500$ kN/m³, Young's modulus $E = 2.1 \cdot 10^5$ MPa, Poisson's ratio $\nu = 0.3$ and elastic limit $\hat{\sigma}_{max} = 230$ MPa. The thickness of the structure is 0.2 m.

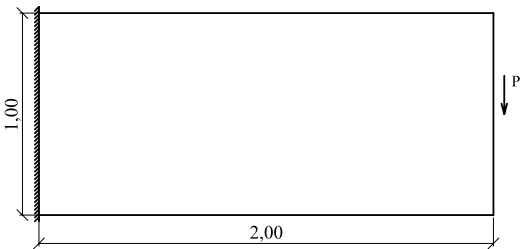


Figure 6: Cantilever beam scheme (units in meters).

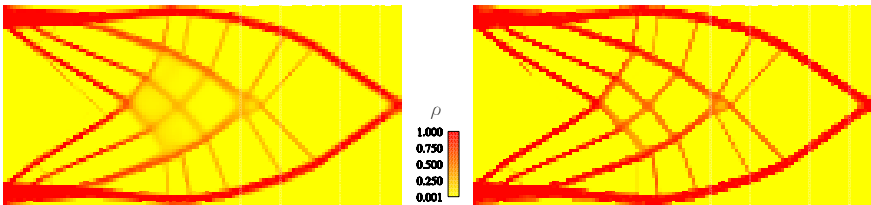


Figure 7: Optimal solution of the cantilever beam problem by using the block aggregation approach (left) and with the modified objective function (8) (right). ($\varepsilon = 0.01$, $\mu = 40$, $\eta = 0$, $N_e^b = 60$).

Table 2: Summary of the most important parameters of the cantilever beam problem.

CANTILEVER BEAM	Block Aggr. Approach (Fig. 7 left)	Modif. Objective F. (Fig. 7 right)
Number of elements	7200	7200
Number of constraints	120	120
Final weight/Initial weight	16.86 %	19.18 %

Figure 7 (left) shows the optimal solution obtained by using the block aggregation of the stress constraints. Figure 7 (right) shows the solution obtained by using the block aggregation of the stress constraints and a final stage with the modified objective function (8). Table 2 presents the most important parameters involved in the optimum design formulation and the value of the final objective function as a percentage of the original weight of the structure.

8 Conclusions

In this paper we present three different formulations to deal with stress constraints in structural topology optimization problems: the local approach of the stress con-

straints, the global approach of the stress constraints and a more general formulation that defines groups of elements and imposes one global constraint per group.

In addition, we introduce a modified objective function based on the weight of the structure that forces the solution to reach 0-1 values. Thus, elements with intermediate relative densities are completely avoided. This objective function allows to deal with continuum design variables during the optimization process but it produces final binary distributions of material.

We present a perimeter penalization in order to improve the results obtained and we develop the parallelization of the most expensive algorithms in order to reduce the computational effort.

Finally, some application examples are solved in order to verify the validity of the algorithms and formulations proposed in this paper.

Acknowledgements

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