The dynamic response of the asymmetric composite laminated beam carrying moving masses

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Abstract

In this paper the dynamic response of the asymmetric orthotropic composite laminated beam subjected to moving masses has been studied. Through a one-dimensional finite element having 24 degrees of freedom the effects of extension, bending, and transverse shear deformation are studied without losing the Poisson's effect. In order to preserve the characteristic coupling including bend-stretch, and bend-twist coupling the lateral strains and curvatures are presented on the basis of axial and transverse strains and curvatures. The dynamic responses of symmetric isotropic laminated beams under the action of moving masses have been compared to the experimental results. This study uses the higher order shear deformation theory that can be employed in the study of force and free vibration problems.
Keywords: moving masses, dynamic response, composite laminated beam, higher order shear deformation, rotary inertia.

1 Introduction

A lot of papers can be found on the theoretical [1-3], experimental [4-6], and numerical [7-13] analysis of traditional beams and plates under the action of moving masses, but on the study of the composite laminated beams under the action of moving masses few researchers have worked [14].

In this article to study the dynamic response of an orthotropic composite laminated beam under the actions of moving masses, a solution based on a finite element method has been developed. The algorithm also accounts for the shear deformation, the rotary and higher-order inertia effects, A conforming beam element based on Hermitian interpolation function that satisfies C^1 continuity condition, has been used. The time variable is evaluated by using the Newmark-method [15]. The algorithm presented in this paper can be applied to the moving masses with a constant speed motion or constant acceleration motion, for three deformation theories; Classical lamination theory (CLT), First shear deformation theory (FSDT) and Higher-order shear theory (HOST). As a part of this study, a computer code has been developed to analyze orthotropic unsymmetrical composite laminated beams under the action of moving masses that are more efficient than the other general purpose FEA codes for this specific problem.

2 Basic assumptions and governing equations

In the present study of CLT the small deflection theory of bending of thin laminate based on following assumptions are used:

a) The displacements of the midsurface are small compared with the thickness of the laminate and, therefore, the slope of the deflected surface is very small and the square of the slope is negligible compared to unity.
b) The Kirchhoff-Love assumptions are used, thus plane sections initially normal to the mid-surface remain plane and normal to the mid surface after bending.
c) The transverse normal stress is small compared with the other components therefore, can be neglected. There are a large number of plate theories that include transverse shear deformation. In present study two displacement-based theories, FSDT and HOST have been developed for consideration of shear deformation.

In the FSDT the assumption that mid-plane normal remains normal after deformation (assumption *b)* is relaxed to mid-plane normal remaining straight after deformation and need not be normal. In this theory the shear correction factor will be needed to satisfy the stress-free boundary conditions. Finally the higher-order shear deformation based on Reddy' s third-order shear deformation [16,17] not only includes transverse shear as in the case of the FSDT but also accounts for a parabolic variation of transverse shear through the laminate thickness, and hence there is no need to use the shear correction factor as in the FSDT. Also in the present study based on assumption a and *b*, further simplifying assumption given by $W= W_b+W_s$ where w is the transverse displacement, TD, of the mid-plane and W_b and W_s are its components due to bending and shear respectively, are made to Reddy's theory so that the number of variables reduced by one. Consider a laminated beam made of a number of layers with its computational coordinates (x,y,z) which are interactive by moving loads. The moving loads and beam are considered as a single system and the transverse inertia effects of moving loads are fully accounted for. Each lamina made of a unidirectional fiber-reinforced material is considered as a homogeneous orthotropic material. Orthotropic axes of symmetry in each lamina of arbitrary thickness and elastic properties are oriented at an arbitrary angle 9 to the beam axis. The moving masses travel at an equal constant velocity or an equal initial velocity and acceleration. Fig. 1 shows a composite laminated beam and the moving masses schematically where L is the beam length, b is the beam width, **t** is the total thickness of beam; i is the number of moving masses, E is the position of the first moving mass with respect to x-direction; L is the distance between the first moving mass with i-th moving mass along x-direction; m = mass of i-th moving mass; Y_o = position of moving masses with respect to y-direction (for this study y_o=0); n total number of moving masses; δ= Dirac Delta function. Applying the variational method for continuum media, the equations of motion, according to, the displacement field based on Reddy's third-order shear deformation, can be found as:

$$\frac{\partial N_1}{\partial x}+\frac{\partial N_6}{\partial y}=I_0\ddot{u}+\left(I_1-s\frac{4}{3h^2}I_3\right)\ddot{\psi}_x-s\frac{4}{3h^2}I_3\frac{\partial \ddot{w}}{\partial x} \tag{1a}$$

$$\frac{\partial N_6}{\partial x}+\frac{\partial N_2}{\partial y}=I_0\ddot{v}+\left(I_1-s\frac{4}{3h^2}I_3\right)\ddot{\psi}_y-s\frac{4}{3h^2}I_3\frac{\partial \ddot{w}}{\partial y} \tag{1b}$$

$$\frac{\partial Q_1^{**}}{\partial x}+\frac{\partial Q_2^{**}}{\partial y}+s\left(\frac{\partial^2 P_1}{\partial x^2}+2\frac{\partial^2 P_6}{\partial x\partial y}+\frac{\partial^2 P_2}{\partial y^2}\right)+\sum_{i=1}^{p}\delta\left[x-(\xi-L_i)\right]\delta(y-y_0)\left(F_i-m_i\frac{\partial^2 w_i}{\partial t^2}\right)$$

$$+q=I_0\ddot{w}-s\frac{16}{9h^4}I_6\left(\frac{\partial^2\ddot{w}}{\partial x^2}+\frac{\partial^2\ddot{w}}{\partial y^2}\right)+s\frac{4}{3h^2}I_3\left(\frac{\partial\ddot{u}}{\partial x}+\frac{\partial\ddot{v}}{\partial y}\right)$$

$$+s\frac{4}{3h^2}\left(I_4-\frac{4}{3h^2}I_6\right)\left(\frac{\partial\ddot{\psi}_x}{\partial x}+\frac{\partial\ddot{\psi}_y}{\partial y}\right) \tag{1c}$$

$$\frac{\partial M_1}{\partial x}+\frac{\partial M_6}{\partial y}-Q_1^{**}-s\left(\frac{\partial P_1}{\partial x}+\frac{\partial P_6}{\partial y}\right)=\left(I_1-s\frac{4}{3h^2}I_3\right)\ddot{u}+$$

$$\left(I_2-s\frac{8}{3h^2}I_4+s\frac{16}{9h^4}I_6\right)\ddot{\psi}_x-s\frac{4}{3h^2}\left(I_4-\frac{4}{3h^2}I_6\right)\frac{\partial\ddot{w}}{\partial x} \tag{1d}$$

$$\frac{\partial M_6}{\partial x}+\frac{\partial M_2}{\partial y}-Q_2^{**}-s\left(\frac{\partial P_6}{\partial x}+\frac{\partial P_2}{\partial y}\right)=\left(I_1-s\frac{4}{3h^2}I_3\right)\ddot{v}+$$

$$\left(I_2-s\frac{8}{3h^2}I_4+s\frac{16}{9h^4}I_6\right)\ddot{\psi}_y-s\frac{4}{3h^2}\left(I_4-\frac{4}{3h^2}I_6\right)\frac{\partial\ddot{w}}{\partial y} \tag{1e}$$

where :

$$Q_1^{**}=Q_1^{*}-sR_1 \quad \text{and} \quad Q_2^{**}=Q_2^{*}-sR_2 \tag{2}$$

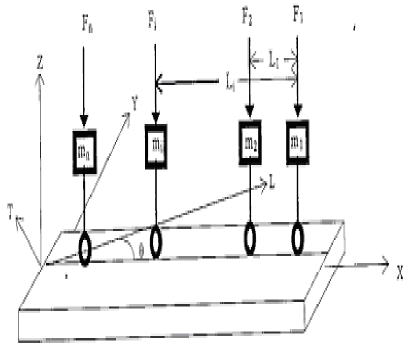

Figure 1: The composite laminated beam moving loads model.

In the above equations the coefficient *s* is a constant factor which by setting s=1 the displacement field of HOST can be achieved, and s=0 leads to the displacement field of FSDT and by setting $s = 0, \psi_x = -\frac{\partial W}{\partial x}, \psi_y = -\frac{\partial W}{\partial y}$ the displacement field of the CLT can be obtained. Also q is the distributed transverse load, m is the mass of i-th moving load, and Ni, N and Pi (1=1,2,6) are stress, moment and higher-order stress resultants, respectively which can be found in [18,19]. Q_i* and R_i (i=1,2) are the stress and higher-order shear stress resultants defined as follows:

$$(Q_1*, R_1) = \sum_{k=1}^{m} \int_{z_{k-1}}^{z_k} \left(\sigma_5^k, \frac{4z^2}{h^2} \sigma_5^k \right) dz, (Q_2*, R_2) = \sum_{k=1}^{m} \int_{z_{k-1}}^{z_k} \left(\sigma_4^k, \frac{4z^2}{h^2} \sigma_4^k \right) dz \tag{3}$$

In Eqs. (1), I_0,I_1 and I_2 are normal, coupled normal-rotary and *rotary* inertia coefficients and I_3,I_4 and I_6 *are* the higher-order inertia coefficients as defined in [18,19]. Using constitutive relation for a *composite* laminate, one *may* find the resultant *forces* and moments in terms of *displacements* for HOST as:

$$\{N_1, N_2, N_6, M_1, M_2, M_6, P_1, P_2, P_6\} = \begin{bmatrix} [A_{ij}] & [A_{ij}] & [A_{ij}] \\ & [A_{ij}] & [A_{ij}] \\ Sym. & & [A_{ij}] \end{bmatrix} \tag{4}$$

$$\left\{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, -\frac{\partial^2 w_b}{\partial x^2} - 2\frac{\partial^2 w_b}{\partial x^2}, -\frac{\partial^2 w_b}{\partial x \partial y}, -\frac{\partial^2 w_s}{\partial x^2}, -\frac{\partial^2 w_s}{\partial y^2}, -2\frac{\partial^2 w_s}{\partial x \partial y} \right\}^T$$

Matrices A,B, and D are extensional, bending-extensional (coupling) and bending stiffness coefficients and matrices E, F and H are higher-order stuffiness coefficients that can be observed in [18,19]. Similarly, the resultant shearing forces can be found by using the corresponding constitutive relation in terms of displacements:

$$\begin{Bmatrix} \ddot{Q}_2 \\ \ddot{Q}_1 \end{Bmatrix} = \begin{bmatrix} D_{44} & D_{45} \\ Sym & D*_{55} \end{bmatrix} \left\{ \frac{\partial W_s}{\partial y}, \frac{\partial W_s}{\partial x} \right\} \tag{5}$$

In Eq.(5) the components D_{ij}* of can be found as:

$$D_{ij}* = A_{ij}* + 2\, D_{ij} + F_{ij} \tag{6}$$

where:

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{m} Q_{ij}^k (z_k - z_{k-1}), \mathrm{D}_{ij} = \frac{-4}{3h^2} \sum_{k=1}^{m} Q_{ij}^k (z_k^3 - z_{k-1}^3), \\ F_{ij} &= \frac{16}{5h^4} \sum_{k=1}^{m} Q_{ij}^k (z_k^5 - z_{k-1}^5), \qquad (i, j = 4,5) \end{aligned} \tag{7}$$

In Eqs. (7), Q_{ij} are transformed reduced stiffness [20]. For beams, it can be assumed that the lateral strains are zero or lateral resultant forces are negligible. The first assumption is valid for beams with solid cross-section [21-22].

Second approach means lateral resultant forces are assumed to be zero:

$$N_2=0\ , M_2=0\ , P_2=0 \tag{8}$$

Combining Eqs. (4) and (8) and determining corresponding lateral strains in terms of other strains, Eq.4 may be simplified as

$$\{\bar{R}\} = [\bar{D}]\{\bar{\varepsilon}\}^T \tag{9}$$

The reader may refer to [13-14] for more details of driving this last equation.

By equating $Q_2^{**} = 0$, the above linearization procedure is completed and Eq. (5) is also simplified as follows:

$$Q_1^{**} = [D_{55}]\frac{\partial W_s}{\partial x}, [D_{55}] = \left[D*_{55} - \frac{(D*_{45})^2}{D*_{44}} \right] \tag{10}$$

Now by combining Eq. (9) and Eq. (10), the relation between resultant forces and generalized displacements may be written in its final form as:

$$\{\bar{R}\} = \begin{Bmatrix} R \\ Q_1^{**} \end{Bmatrix} = \begin{bmatrix} [\bar{D}] & \{0\} \\ \{\bar{\varepsilon}\}^T & [D_{55}] \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \dfrac{\partial w_s}{\partial x} \end{Bmatrix} = [D^{**}]\{\bar{\varepsilon}\} \tag{11}$$

The advantage of above approach is that a two-dimensional beam theory is reduced to a one-dimensional theory without ignoring the Poisson's effect.

3 Finite element formulation

In order to develop the finite element models of the laminated composite beam and moving loads, displacement models are used. The displacement finite element formulation of composite beams is based on the principle of virtual displacements where all governing equations are expressed in terms of displacements. In this section, a finite element model for HOST is developed by using Hermitian cubic interpolation function. Then finite element formulation for FSDT and CLT can be found as special cases. The field variables in the dynamic case for HOST can be represented as:

$$\begin{aligned} u &= \sum_{i=1}^{4} u_i H_i(x), \quad \beta_x = \sum_{i=1}^{4} \beta_{xi} H_i(x), \quad w_b = \sum_{i=1}^{4} w_{bi} H_i(x), \\ w_s &= \sum_{i=1}^{4} w_{si} H_i(x), \quad \lambda_b = \sum_{i=1}^{4} \lambda_{bi} H_i(x), \quad \lambda_s = \sum_{i=1}^{4} \lambda_{si} H_i(x), \end{aligned} \tag{12}$$

where $u_i, x_i, W_{bi}, W_{si}, \lambda_{bi}, \lambda_{si}$ denote the generalized nodal displacements and H1(x) are the Hermite interpolation polynomials. The element which is used for finite element procedures is a C conforming element, which has a total of twelve degrees of freedom per node. By eliminating generalized nodal displacement, x_i, this element can be used for FSDT and by eliminating generalized nodal displacement λ_{ij} and W_{si}, the element will have ten degrees of freedom per node, which is used for CLT in this study. Substituting Eq. (12) into Eq. (11) and using Hamilton variational principle, the element equations of motion are

$$[M^e]_T \{\ddot{q}^e\} + [M^e]\{q^e\} = \{F^e\} \tag{13}$$

The variables q and stiffness coefficients Kr (for α,β= 1,…,6) are defined by [13-14] and are not re-introduced here.

The overall mass matrix of the entire system at time *t* is given by:

$$\left[M^e\right]_T = \left[M^e\right] + \left[M^e_{mL}\right] \tag{14}$$

where $\left[M^e\right]$ is the element mass matrix and $\left[M^e_{mL}\right]$ is the element mass matrix due to the mass of moving loads.

For the i-th moving load with a concentrated mass m1, the elementary mass matrix can be obtained as:

$$\left[M^e_{mL}\right] = b\int_{l_0} \left[H\right]^T m_i \delta\left[x-(\zeta - L_i)\right]\left[H\right]dx = bm_i\left[\bar{H}\right]^T\left[\bar{H}\right] \tag{15}$$

where the bar symbol on H means that the term is evaluated in local coordinate of the specific elements where moving loads are located. The external forces due to transverse force q and moving load can be obtained by

$$\left\{F^e\right\} = b\int_{l_0}\left[H\right]^T q dx + b\int_{l_0}\left[H\right]^T F_i\delta\left[x-(\xi - L_i)\right]dx \tag{16}$$

The dynamic response of the composite laminated beam under the moving loads is investigated by a step-by-step method. At any instant of time t, the position of all moving loads are found and by using Eq. (15) and Eq. (16) the effects of moving loads are appeared on the elementary moving mass matrix $\left[M^e_{mL}\right]$ and nodal force vector $\left\{F^e\right\}$. It should be noted that all elementary moving mass matrices $\left[M^e_{mL}\right]$ and nodal force vector $\left\{F^e\right\}$ are equal to zero except that of the element on which the moving loads act.

4 Numerical results and discussion

4.1 Free Vibration of Symmetrically AS/3501-.6

Graphit-Epoxy Laminated Composite Beams for Various angle of layer. Numerical results have been presented for four symmetrical layer AS/3501-6 clamped-clamped graphit-epoxy beams (ɵ/-ɵ/-ɵ/ɵ). This example demonstrates the importance of the bend-twist coupling term and the Poisson-effect of angle-ply beams. Results obtained using the FSDT (with/without the bend-twist coupling and Poisson-effect), are compared to analytical results [23], where in their study they used first shear deformation and including the rotary inertia but neglecting bend-twist coupling and Poisson-effect. Beam width is taken as unity as mentioned in [23] and the material properties used in these examples are:
E_{LL}=144.8 GPa, E_{TT}=965.3 GPa, G_{LT}=413.7 GPa, G_{TT}=3.48 GPa, mass density ρ =1389.227 Kg/m^3, Poisson's ratio $\nu_{LT} = 0.3$.

Table 1 shows the non-dimensional fundamental frequencies $\varpi = \omega L^2\sqrt{\frac{\rho}{E_{LL}h^2}}$ of four layer symmetrical angle-ply beams for the clamped-

clamped boundary condition. In this table, the first row shows the results reported by [23] and in the second row the results of the present study used the FSDT without both the bend-twist coupling and Poisson-effect are shown (FSDT). In third row results of FSDT with consideration of the bend-twist coupling and neglecting Poisson's-effect are presented (FSDT) and finally, in forth row results of FSDT including both bend-twist coupling and Poisson effects are shown.

Table 1: Non-dimensional fundamental frequencies for the AS/3501-6 Graphit-Epoxy of (θ/θ-/θ-/θ) angle-ply Clamped-Clamped beams for slender ratio L/h=15.

Solution type	Non-dimensional fundamental frequency						
	0	15	30	45	60	75	90
Analytical	4.8487	4.6635	4.0981	3.1843	2.1984	1.6815	1.6200
FSDT**	4.8712	4.6835	4.1118	3.1908	2.2006	1.6814	1.6207
FSDT*	4.8712	4.1071	3.3806	2.6199	1.9611	1.6604	1.6207
FSDT	4.8629	4.0082	2.8762	1.9330	1.6290	1.6063	1.6161

The frequencies decrease with increase in fiber orientation. Also neglecting the bend-twist coupling and Poisson-effect may occurs overproduction of the fundamental frequency, specially for angle lay-out between 30° through 60°, for example, for the *(45/-45/-45/45)* clamped-clamped beam the fundamental frequency is 64.7% less than that reported in [23].

4.2 Forced vibration of isotropic simply supported beam with moving mass

In this example the algorithm were developed for the case of moving mass are studied for problem that has constant velocity. Since the exact solution for moving mass problem is not available, the results are compared with experimental work reported by [6]. In their investigation a uniform simply supported beam has been studied, where the dimensions and mechanical properties are as follows: length L=1.524m(60"), width b=0.1016m(4"), thickness h=0.476 *cm (0.1875")*, modulus of Elasticity E=206.8 GPa (30 MPsi), mass density μ.=7850.6 kg/rn^3, fundamental natural frequency ωf= 9.4π and mass ratio is Mml / Mb =0.5. The comparison of numerical method of present study and experimental res6lts are available for three different velocity ratios, $\frac{V}{L\omega_f}=\frac{1}{4\pi},\frac{1}{2\pi},\frac{3}{4\pi}$. Fig. 2 shows the time history of the dynamic TD of mass divided by static central TD of beam $ws=\frac{M_{mL}gL^3}{48EI}$ for the above three values of velocity ratio. As it can be observed, the agreement between the results of present numerical solution and those obtained by experimental is very good. In Fig. 3 shows the time histories of the TD of the beam divided by static central TD for a spectrum of velocity ratios are drown. As it is seen the maximum TD of the beam 11 under the action of moving mass occurs at lower velocity ratio ($V/L\omega_f$ $=0.5/\pi$) in comparing of moving force case. Another feature is that by increasing the velocity of moving mass the maximum displacement is shift to the right of centre of beam. Finally for relatively high velocity ratios the inertia of the beam is dominated by the dynamic effects of the mass. Fig. 4 shows a comparison of the TD of moving mass for various mass ratios when the velocity ratio is equal to one. As it is shown in this figure the motion of the mass would be like as a parabolic trajectory. Also by increasing the mass ratio the TD significantly decreases.

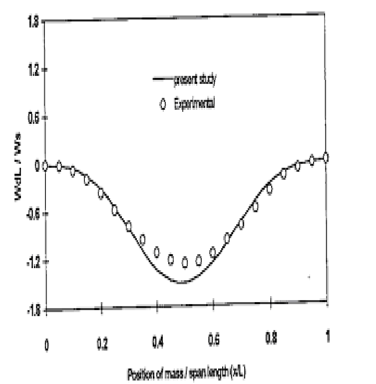

Figure 2: FEM vs. experimental TD of the moving mass $M_{mL}/M_b=1/2$, $V/L\,\omega_1=1/4\,\pi$.

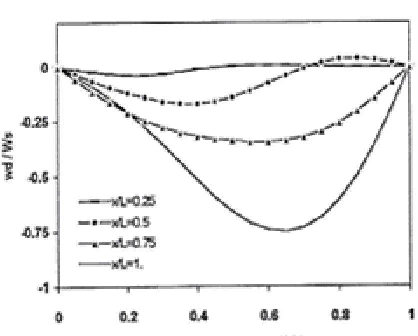

Figure 3: Time histories of the TD of the simply supported beams due to moving mass $V/L\,\omega_1=2/\pi$.

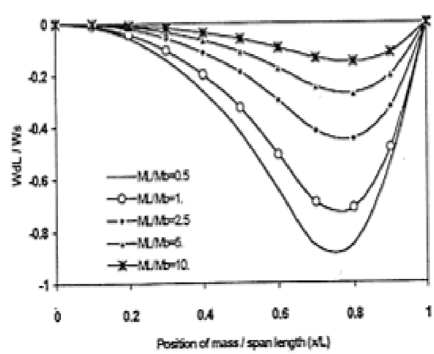

Figure 4: Comparison of TD of the moving mass due for several mass ratio using HOST with $V/L\,\omega_1=1/\pi$.

4.3 Forced vibration of the orthotropic composite laminated beam

We shall now solve an orthotropic simply supported composite laminated beam under the action of M=0.45 Kg moving mass. A symmetrical cross-ply laminated with four layer of equally thickness (0°/90°/90°/0°) for comparing the results is chosen. Each layer is a unidirectional fibre reinforced material with following properties; EL=144.8 GPa, E_T=9.65 GPa, G_{LT}=4.136 GPa, G_{TT}=3.447 GPa, μ=1389.297 kg/m^3, V_{LT}=0.25, V_{TT}= *0.25* where subscripts L and T are directions respectively parallel and perpendicular the fibers and V_{LT} is the Poisson's ratio measuring normal strain in the transverse direction under axial normal stress in L-direction. The laminated beam has 10.16 cm length, 0.635 cm width and a total thickness of h = *0.745* cm, with moment of inertia 1=218.8 1 mm^4 and fundamental period T_f=0.3187 ms. Fig. 5 shows the comparison of dynamic magnification factor of this Graphite-epoxy beam with that of the steel beam without considering the shear effect (CLT). As it can be observed, in spite of the fact that the total weight of composite beam is approximately *6.5* times less than the total weight of steel beam, the maximum dynamic response of centre of beam is approximately the same for both materials. Another important result of this investigation is that the critical velocity (V_c) of this Graphite- epoxy composite beam is approximately *2.5* times the critical velocity of traditional steel beam. Also in Fig.5, the dynamic magnification factor of composite beam is drawn for CLT, FSDT and HOST. It can be concluded that the shear deformation effect is very important in the strength analysis of composite laminated beams even if the slender ratio is not very low (L / h = 13.6).

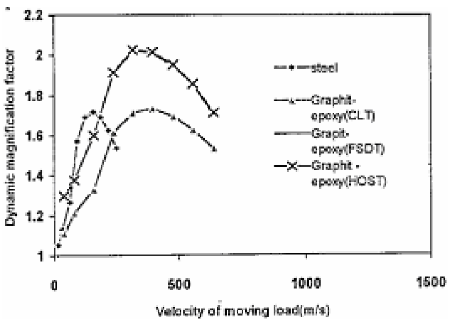

Figure 5: Comparison of Dynamic magnification factor of steel beam and Graphite-epoxy composite beam with respect to moving load velocity using CLT.

References

[1] Timoshinko, S.P. and Woinowsky, K., Theory of plates and shell, McGraw-Hill, 1959.

[2] Szilard, R., Theory and analysis of plates, classical and numerical method, Prentice-Hall, 1974.

[3] Raske, T.F. and Schiack, A.L., Dynamic response of plates due to moving loads, J. Acoust. Soc. Amer, 3, 42, 625-635, 1968.
[4] Smith, J.R. and Liigthholder, R.K., Moving load test on experimental prestressed concrete highway slab, Highway Res. Rec., 60, 59-76, 1963.
[5] Yoshizawa, M. and Takizawa, T., Vibration of a beam and a moving load with sprung and unsparing masses, Bulletin of ASME, 28, 239, 911-91S, 1985.
[6] Ayre, R.S., Jacobson, L.S. and Hsu, C.S., Transverse vibration of one- and two- span beam under the action of a moving mass load, *Proc. 1st U.S. Nat. Cong. App. Mech.*, 1951.
[7] Wu, I.S. and Dai, C.W., The dynamic response of the inspan-supported bridge due to moving loads, *Int. J. Num. Method Eng.*, 24, 743-762, 1987.
[8] Padovan, J. and Paramodilok, O., Generalized solution of time dependent traveling load problem via moving finite element scheme, *J. Sound Vibration*, 2, 91, 195-209, 1983.
[9] Hino, E., Yoshimura, T., Konishi, K. and Ananthanarayana, N., A finite element method prediction of the vibration of a bridge subjected to a moving load, *J. Sound Vibration*, 1, 96, 45-53, 1984.
[10] Mulcahy, N.L., Bridge response with tractor-trailer vehicle loading, *Earthquake Eng. St. Dynamics*, 11, 649-665, 1983.
[11] Nelson, H.D. and Conover, R.A., Dynamic stability of a beam carrying moving masses, *J. App. Mech.*, 93, 1003-1006, 1971.
[12] Taheri, M.R. and Ting, E.C., Dynamic response of plate to moving loads, *Computer Structures,* 34, 3, 509-521, 1990.
[13] Mohebpour, S.R. and Dehdashti, G., Finite element analysis of multi-span thin beam under the action of moving loads, J. M. S. Uni. Railway Eng., in press.
[14] Kadivar, M. H. and Mohebpour, S. R., Forced vibration of unsymmetric composite laminated beams under the action of moving loads, *Int. J. Com. Sc. Tec.*, 58, 10, 1675-1684, 1998.
[15] Bathe, K., *Finite element procedures in engineering analysis*, 1982.
[16] Reddy, J. N., A simple higher-order theory for laminated composite plates, *J. App. Mech.*, 106, 51, *745-752,* 1984.
[17] Reddy, I.N., Mechanics of laminated composite plates: theory and analysis, CRC, 1997.
[18] Reddy, I.N., *Energy and variational methods in applied mechanics*, 1984.
[19] Whitney, J.M., *Structural analysis of laminated anisotropic plates*, 1987.
[20] Jones, R. M., Mechanics of composite materials, NY, Hemisphere, 1975.
[21] Wu, X.X. and Sun, C.T., Vibration analysis of laminated composite bin-walled beams using finite elements, *A.IAA journal*, 29, 5, 736-742, 1991.
[22] Chen, A.T. and Yang, T.Y., Static and dynamic formulation of a symmetrically laminated beam finite element for a microcomputer, *J. Comp. Materials,* 19, 459-475, 1985.
[23] Chandrashekhara, K., Krishnamurthy, K., and Roy, S., Free vibration of composite beams including rotary inertia and shear deformation, *Comp. St.*, 14, 269-279, 1990.