Conservative averaging as an approximate method for solution of some direct and inverse heat transfer problems

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Abstract

The conservative averaging method was developed as an approximate analytical and/or numerical method for solving partial differential equation or its system with piece-wise constant (continuous) coefficients. The usage of this approximate method for separate relatively thin sub-domain or/and for sub-domain with al large heat conduction coefficient leads to a reduction of domain in which the solution must be found. To apply this method for all sub-domains of layered media, a special type of spline was constructed: the integral averaged values interpolating parabolic spline. The usage of this spline allows diminishing the dimensions of initial problem per one. It is important that in all cases the original problem with discontinuous coefficients from R^{n+1} transforms to problem with continuous coefficients in R^n. A method of conservative averaging for ill-posed inverse problems in some cases allows transforming them to well-posed inverse problems.

Keywords: heat transfer, piecewise constant (continuous) coefficients, conservative averaging, non-classical conditions, integral spline, mesh (dimension) reduction, direct problem, inverse problem.

1 Introduction

By modeling practically interesting processes, e.g. heat transfer processes in non-homogeneous media, very often we need to consider the situation, when the medium has an organized structure, i.e. it is not fully chaotic. For example it often has a layered structure. In addition some of these layers are relatively thin in comparison with adjacent layers and have strongly different physical properties.

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Mathematically speaking such a situation can be described by a partial differential equation (or its system) with piecewise constant/continuous coefficients, i.e. the domain in which the solution is defined, consists of several sub-domains. In each sub-domain the solution of the partial differential equation can be considered in the classical sense: the solution has continuous all highest partial derivations into the sub-domain. But it is not true on the contact surfaces S of adjacent layers: on these surfaces special additional conditions, which in the literature are often called conjugation or junction conditions, have been formulated: $\left[T\right]=0, \left[k\frac{\partial T}{\partial n}\right]=0$. Here $\left[T\right]=T\big|_{S^+}-T\big|_{S^-}$ is the difference of one-sided limit values (jump) of function T on surface S. In the case of non-ideal contact one of the conditions is the continuity of heat flux (energy conservation). The second junction or conjugation condition is usually written without detailed deduction. E.g., [1] mentions "temperature has discontinuity when passing through the boundary of non-ideal contact, with the height of the step being proportional to the heat flow, i.e.

$$[T]=\frac{k}{\alpha}\frac{\partial T}{\partial n},(x,y,z)\in S, \tag{1}$$

where the coefficient of contact heat transfer α is associated with the contact conditions". Similarly, in [1] the so-called "concentrated heat capacity" condition relation on the surface S is written as follows

$$\left[k\frac{\partial T}{\partial n}\right]=c_s\rho\frac{\partial T}{\partial t}. \tag{2}$$

Additionally, it commented that: "c_s is the lumped heat capacity of the contact".

In this paper we will show how these and other conditions and their generalizations can be obtained by our original method of conservative averaging (CAM) [2, 3]. This approach allows us to eliminate some separate sub-domains and reduce partial differential equations for these sub-domains to boundary conditions. This means that we reduce the definitions domain of problem for its analytical or numerical solution [4]. This means we can consider this approach using the mesh reduction method. To apply CAM procedure for several layers, it was necessary to construct a special type of spline: the integral averaged values interpolating parabolic spline [5]. In [6] such an approximation for convection-conduction heat transfer in a layered system was demonstrated. For the approximation of boundary layers we introduced rational spline [7] and in [8] we showed its effectiveness. It is important that in all the cases the original problem with discontinuous coefficients transforms from problem a in R^{n+1} to problem a in R^n with continuous coefficients. This method for ill-posed inverse problems in some cases allows transforming them to well-posed inverse problems, e.g. [9, 10].

2 The conservative averaging method for separate boundary sub-domain

Let us assume that in suitable Cartesian coordinates $x \in R, y \in R^n$ the domain of definition $\overline{D} \subset R^{n+1}$ of the solution is represented as consisting of two sub-domains: $\overline{D} = \overline{G} \cup \overline{G_0}$. Here the finite or infinite sub-domain G has form $G = \{(x, y) | x \in (0, \Delta(y)), y \in \Gamma(x)\}$ and the second sub-domain $G_0 = \{(x, y) | x \in (-\delta, 0), y \in \Gamma_0\}$ is a cylinder of finite height δ. The base of the cylinder $\Gamma_0 \in R^n$ is the bounded/unbounded domain in R^n under the condition $\Gamma_0 \subseteq \Gamma(0)$. The differential equation in G_0 has the form:

$$\frac{\partial}{\partial x}\left(k_0 \frac{\partial U_0}{\partial x}\right) + L_0(U_0) = -F_0(x, y), \tag{3}$$

where L_0 is the linear differential operator with respect to the argument y and coefficient $k_0 = k_0(y)$. The differential equation in G has a similar form:

$$\frac{\partial}{\partial x}\left(k \frac{\partial U}{\partial x}\right) + L(U) = -F(x, y), \tag{4}$$

but the operator L with respect to the vector argument y now in the general case can be non-linear and the heat conduction coefficient is $k = k(x, y)$. It should be mentioned that one of the y vector components may be time t, thus the equations (3) and (4) allow describing both the steady-state and transient processes. Let us denote by Γ_0^+ the part of the hyper-plane $x = 0$: $\Gamma_0^+ = \overline{G} \cap \overline{G_0}$. On this surface the conjugations conditions must be fulfilled:

$$U_0\big|_{x=-0} = U\big|_{x=+0}, \tag{5}$$

$$k_0 \frac{\partial U_0}{\partial x}\bigg|_{x=-0} = k \frac{\partial U}{\partial x}\bigg|_{x=+0} \tag{6}$$

On the second base $\Gamma_0^- = \{x = -\delta, y \in \Gamma_0\}$ of the cylinder, a typical boundary conditions for the heat transfer processes is given (we will specify it later). On the other hand we will not specify the conditions of the remaining parts of both sub-domains, because their form is not substantial for the description of the method.

We assume that the original problem (3)–(6) with all the necessary boundary or/and initial conditions have a unique and stable solution. In particular it should be emphasized that under the solution of this problem we imply a solution in a slightly revised classical sense: 1) it is continuous in closure of definition domain $\overline{D}$; 2) the solution has all necessary continuous highest derivations in open sub-domains G_0 and G; 3) the first derivative with respect to argument x of the solution has bounded one-sided limit values which fulfil the second junction condition (6) on surface Γ_0^+; 4) it fulfils all additional conditions on boundary of definition domain.

We start the description of the CAM by introducing the integral averaged value of the function $U_0(x,y)$ of the solution of the problem (3)–(6) on $\overline{G}_0$:

$$u_0(y) = \delta^{-1} \int_{-\delta}^{0} U_0(x,y)dx. \tag{7}$$

The integration of the differential equation (3) over the interval $x \in (-\delta, 0)$ and the utilization of second junction condition give us the basic relation

$$k\left.\frac{\partial u}{\partial x}\right|_{x=+0} - k_0\left.\frac{\partial U_0}{\partial x}\right|_{x=-\delta} + \delta L_0(u_0) = -\delta f_0(y). \tag{8}$$

Firstly, we denote the solution on sub-domain $\overline{G}$ by $u(x,y)$ instead of the function $U(x,y)$ because the solution of the new statement of the problem in general will differ from initial solution. Secondly, we denote by $f_0(y)$ according to (7) the averaged value of source function $F_0(x,y)$. A further transformation of the basic relation (8) depends on the type of boundary conditions on Γ_0^-. We will consider the boundary condition of second type (other types of boundary condition were considered in [2, 3]):

$$-k_0\left.\frac{\partial U_0}{\partial x}\right|_{x=-\delta} = \varphi^0(y). \tag{9}$$

Assuming the linear approximation for solution $U_0(x,y)$ for each fixed y (nevertheless it may be different for various y!) we easily find the representation for $u_0(y)$ by means of the junction condition (5) and condition (9):

$$u_0(y) = u(0,y) + \frac{\delta}{2}\frac{\varphi^0(y)}{k_0(y)}.$$

It remains to then insert this expression in (8), to use boundary condition (9) and we have obtained the non-classical boundary condition on boundary Γ_0^+:

$$k\frac{\partial u}{\partial x}+\delta L_0(u)=-\left[\varphi^0(y)+\delta f_0(y)+\frac{\delta^2}{2}L_0(\frac{\varphi^0}{k_0})\right] \tag{10}$$

for the solution $u(x,y)$ of the equation (4) on the reduced domain G :

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+L(u)=-F(x,y). \tag{11}$$

Of course all additional conditions of the original problem on the remaining part of boundary of domain G must be added to main differential equation (11) and new boundary condition (10) on the surface Γ_0^+. When this new problem is solved it is easy to obtain a posteriori error estimation $\Delta U_0(x,y)$ for the linearly approximate (relatively to x) solution $U_0(x,y)$ in following form:

$$\left|\Delta U_0(x,y)\right|\le\frac{|x|}{k_0}\times$$

$$\times\left|\left[k\frac{\partial u(0,y)}{\partial x}+\varphi^0(y)\right]+\frac{\delta}{2}\left[L_0(u_0)+\frac{\delta}{2}L_0(\frac{\varphi^0}{k_0})+\delta^2 f_0(y)\right]\right|$$

We can increase the order of approximation for solution $U_0(x,y)$ up to two by means of both junction conditions and condition (9). We get the representation:

$$U_0(x,y)=u(0,y)+\frac{k}{k_0}(1+\frac{x}{2\delta})x\frac{\partial u(0,y)}{\partial x}+\frac{x^2\varphi^0(y)}{2\delta k_0}. \tag{12}$$

We obtain from (12) following expression for the averaged integral value of the function $U_0(x,y)$:

$$u_0(y)=u(0,y)+\frac{\delta}{3k_0}\left[\frac{\varphi^0(y)}{2}-k\frac{\partial u(0,y)}{\partial x}\right]. \tag{13}$$

This means that the new boundary condition on surface Γ_0^+ consists of two equations. One of them is a consequence of equation (8):

$$k\frac{\partial u}{\partial x}\bigg|_{x=+0}+\delta L_0(u_0)=-\left[\varphi^0(y)+\delta f_0(y)\right], \tag{14}$$

the second one is expression (13). These two equations allow finding two unknown functions $u_0(y)$ and $u(0,y)$ on the boundary Γ_0^+. So the new problem consists of the main differential equation (11) together with a system of two non-standard boundary conditions (13), (14). The definition domain for the function $u(x,y)$ is sub-domain $\overline{G}$; the function $u_0(y)$ is given only on the surface $\overline{\Gamma}_0^+$.

So the CAM in this situation can be interpreted as the mesh reduction method. We finish this part of paper with some remarks. Quite often after the application of CAM with piecewise constant coefficients it is possible to solve the problem analytically. Our experience concerning the numerical methods for other problems with non-classical boundary conditions is as follows. Firstly, we have solved a large number of important practical problems (mostly by the method of finite difference). Secondly, in practice and in theory the stability criterion for classical boundary condition (9) is more stiff than for the non-classical boundary condition (10) (or for the system (13) and (14)). Unfortunately still we haven't succeeded in proving the solvability of problems with non-classical additional conditions in general (for general operators L_0 and L).

3 The conservative averaging method for separate inner sub-domain (layer)

Now we will consider the definition domain D which consists of three sub-domains: $\overline{D} = \overline{G} \cup \overline{G_0} \cup \overline{G_1}$, where finite or infinite sub-domain G_1 has the form $G_1 = \left\{(x,y) \middle| x \in (-\Delta_1(y), -\delta), y \in \Gamma_1(x)\right\}$. We add to equations (3) and (4) in sub-domain G_1 equation

$$\frac{\partial}{\partial x}\left(k_1 \frac{\partial U_1}{\partial x}\right) + L_1(U_1) = -F_1(x,y) \tag{15}$$

and to conditions (5) and (6) add further junction conditions on surface Γ_0^-:

$$U_1\big|_{x=-\delta-0} = U_0\big|_{x=-\delta+0},\ k_1 \frac{\partial U_1}{\partial x}\bigg|_{x=-\delta-0} = k_0 \frac{\partial U_0}{\partial x}\bigg|_{x=-\delta+0}. \tag{16}$$

The basic relation for this problem looks as follows:

$$k \frac{\partial u}{\partial x}\bigg|_{x=+0} - k_0 \frac{\partial u_0}{\partial x}\bigg|_{x=-\delta+0} + \delta L_0(u_0) = -\delta f_0(y). \tag{17}$$

To exclude thin interlayer – cylinder G_0 – and to obtain the new non-standard junction conditions on surface Γ_0^+ we shift the sub-domain G_1 to the right: $x \mapsto x + \delta$ and additionally assume the linear approximation for solution $U_0(x,y)$ for each fixed y. Then from (6) and (16) immediately follows the first junction condition on Γ_0^+:

$$k_1 \frac{\partial u_1}{\partial x}\bigg|_{x=-0} = k \frac{\partial u}{\partial x}\bigg|_{x=+0} . \quad (18)$$

The second junction condition on Γ_0^+ follows from (17) by taking into consideration linearity of function $U_0(x, y)$:

$$k \frac{\partial u}{\partial x}\bigg|_{x=+0} - \frac{k_0}{\delta}[u] + \frac{\delta}{2} L_0(u + u_1) = -\delta f_0(y) . \quad (19)$$

If we neglect in (19) both the operator L_0 and source term then this condition reduces to the non-ideal contact condition. In addition, we have explicated the expression for the coefficient of contact heat transfer α in (1) through physical and geometrical properties of the interlayer: $\alpha = \frac{k_0}{\delta}$. The "concentrated heat capacity" condition (2) follows from the basic relation (17) with operator $L_0 = c_0 \rho_0 \frac{\partial}{\partial t}$, $f_0 \equiv 0$ and equality (18). The explicit expression for the lumped heat capacity c_s derives from (17) and is as $c_s = c_0 \delta$.

4 The coefficient inverse one-dimensional problem for two-layer system

One of most popular experimental methods for thermal physical properties of homogeneous media (solids, fluids and gases) with low electrical conductivity is the transient hot strip (THS) method developed by Gustafsson [11]. Mathematically this method was formulated as coefficient inverse heat equation with constant coefficients for two-dimensional semi-bounded zone. In our publications [9, 10] we generalize this ill-posed problem for two-layers, solving it by use of Green's function and reducing it to a system of two transcendent equations. Nevertheless, the numerical solution of system of transcendental equations is an ill-posed problem. It would be important to offer some well-posed method for finding approximate values for coefficients as initial data for the iteration process. In this section we propose such an approach based on conservative averaging. The one-dimensional model for the THS method can be formulated as follows:

$$c_0 \frac{\partial U_0}{\partial t} = k_0 \frac{\partial^2 U_0}{\partial x^2} + f_0 \delta(0), \; 0 < x < H_0, \; 0 < t, \quad (20)$$

$$c_1 \frac{\partial U_1}{\partial t} = k_1 \frac{\partial^2 U_1}{\partial x^2}, \; H_0 < x < H_0 + H_1 = H, \; 0 < t \quad (21)$$

with homogeneous second type boundary conditions

$$k_0 \frac{\partial U_0}{\partial x}\bigg|_{x=0} = 0, \qquad k_1 \frac{\partial U_1}{\partial x}\bigg|_{x=H} = 0\,,$$

with conjugation conditions

$$U_0\Big|_{x=H_0-0} = U_1\Big|_{x=H_0+0}\,,\ k_0 \frac{\partial U_0}{\partial x}\bigg|_{x=H_0-0} = k_1 \frac{\partial U_1}{\partial x}\bigg|_{x=H_0+0}$$

and with homogeneous initial conditions

$$U_0\Big|_{t=0} = 0, U_1\Big|_{t=0} = 0\,.$$

Additional information is given:

$$U_0\Big|_{x=0} = T(t) \tag{22}$$

in the form $T(t_k) = T_k,\ t_k = k\Delta t,\ k = \overline{0,N}$ with $N >> 1$. The aim of the solution of the inverse problem is to find the unknown constants c_0, c_1 and functions $U_0(x,t)$ and $U_1(x,t)$.

Let us introduce the integral averaged values of functions $U_0(x,t)$ and $U_1(x,t)$:

$$u_0(t) = \frac{1}{H_0}\int_0^{H_0} U_0(x,t)dx\,, \qquad u_1(t) = \frac{1}{H_1}\int_{H_0}^{H} U_1(x,t)dx\,. \tag{23}$$

Further, we will approximate both unknown functions $U_0(x,t),\ U_1(x,t)$ according to argument x with expressions, which fulfil the boundary and conjugation conditions and equalities (23). Finally we have:

$$U_0(x,t) = u_0(t) - \frac{eG_0}{2(G_0+G_1)}\left(e^{\frac{x}{H_0}} + e^{-\frac{x}{H_0}} + e^{-1} - e\right)v(t)\,, \tag{24}$$

$$U_1(x,t) = u_1(t) + \frac{eG_1}{2(G_0+G_1)}\left(e^{\frac{x-H}{H_1}} + e^{\frac{H-x}{H_1}} + e^{-1} - e\right)v(t)\,, \tag{25}$$

where $v(t) = u_0(t) - u_1(t),\ G_i = H_i k_i^{-1}, i = 0,1$.

Now we integrate the main equations (20) and (21):

$$c_0 \frac{du_o}{dt} = G_o^{-1} \frac{\partial U_0}{\partial x}\bigg|_{x=H_0-0} + f_0\,, \quad c_1 \frac{du_1}{dt} = G_1^{-1} \frac{\partial U_1}{\partial x}\bigg|_{x=H_0+0}\,.$$

Finally we receive a system of two ordinary differential equations:

$$\beta_0 \frac{du_0}{dt} = -\alpha v(t) + f_0 H_0\,, \quad \beta_1 \frac{du_1}{dt} = \alpha v(t),$$

with homogeneous initial conditions $u_0(t) = u_1(t) = 0$.

Here $\alpha = \dfrac{e^2 - 1}{2(G_0 + G_1)}$ and $\beta_i = c_i H_i,\ i = 0,1$. The solution for $v(t)$ is:

$$v(t) = f_0 H_0 (\beta_0 \gamma)^{-1} \left(1 - e^{-\gamma t}\right),\ \gamma = \alpha\delta,\ \delta = \beta_0^{-1} + \beta_1^{-1}.$$

And the solution for $u_0(t)$:

$$u_0(t) = f_0 H_0 \beta_0^{-1} \left[(\beta_0 \gamma \delta)^{-1} \left(1 - e^{-\gamma t}\right) + \left(1 - (\beta_0 \gamma)^{-1}\right) t \right].$$

We have from additional information (22):

$$T_k = u_0(t_k) + \alpha_0 v(t_k),\ \alpha_0 = 2^{-1}(e^2 - 2e - 1) G_0 (G_0 + G_1)^{-1}.$$

The first finite difference may be written as follows:

$$\Delta T_k = f_0 H_0 (\beta_0 \gamma)^{-1} \left[(\alpha_0 + (\beta_0 \delta)^{-1}) \left(1 - e^{-\gamma \Delta t}\right) e^{-\gamma t_{k-1}} + \left(\gamma - (\beta_0)^{-1}\right) \Delta t \right]$$

and we have similar expression for the second finite difference:

$$\Delta^2 T_k = -f_0 H_0 (\beta_0 \gamma)^{-1} \left[(\alpha_0 + (\beta_0 \gamma)^{-1}) \left(1 - e^{-\gamma \Delta t}\right)^2 e^{-\gamma t_{k-1}} \right].$$

Finally, we obtain expression for the sum of the unknown coefficients:

$$\gamma = \frac{1}{\Delta t} \ln \frac{\Delta^2 T_{k-1}}{\Delta^2 T_k}. \tag{26}$$

Evidently we have found the unique solution of the inverse problem, but it is easy to see that solution (26) strongly depends on the errors of the measurements and in this sense we have unstable algorithm. But we can propose the modification of this algorithm by summing up the sub-sequences of the measurement data.

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