STRUCTURAL TOPOLOGY OPTIMIZATION WITH HIGH SPATIAL DEFINITION BY USING THE OVERWEIGHT APPROACH

DIEGO VILLALBA, JOSÉ PARÍS, IVÁN COUCEIRO & FERMÍN NAVARRINA
GMNI – Group of Numerical Methods in Engineering, E. T. S. de Ingeniería de Caminos, Canales y Puertos, Universidade da Coruña, Spain

ABSTRACT
The first formulation of topology optimization was proposed in the 1980s. Since then, many contributions have been presented with the purpose of improving its efficiency and expanding its field of application. The aim of this research is to develop a structural topology optimization algorithm considering minimum weight and stress constraints. Structural topology optimization with stress constraints has been previously formulated with several different approaches, mainly: local stress constraints, global stress constraints or block aggregation of stress constraints. In this research the overweight approach, an improvement of the so-called damage approach, is used. In this method, a virtual relative density (VRD) is defined as a function of the violation of the local stress constraints. VRD is increased as the stresses exceed the maximum allowable value, with the exception of the areas with the minimum value of the relative density, since full-void solutions are intended. The distribution of the material in the domain is modelled using two different approaches: a uniform relative density within each element of the mesh and a relative density defined by means of quadratic B-splines. For this reason, the structural analysis is performed by means of the finite element method (FEM) and the isogeometric analysis (IGA) respectively. The optimization is addressed by means of the sequential linear programming algorithm (SLP), which is driven by the information provided by a full first order sensitivity analysis extension of both FEM and IGA formulations. Finally, the overweight approach is tested by means of some two dimensional problems. The domain has been divided in an elevated number of elements to attain high spatial definition solutions. The results show that the overweight approach is a feasible alternative for the damage approach and the stress constraints aggregation techniques to solve the topology optimization problem. A comparison between both formulations of the material distribution is included.
Keywords: topology optimization, structures, stress constraint, overweight approach, finite element method, isogeometric analysis, aggregation techniques, high spatial definition.

1 INTRODUCTION

One of the first works about structural topology optimization of continuum structures was submitted by Bendsøe and Kikuchi in 1988 [1]. These works supposed the establishment of the basis of this new field. Since then an important number of contributions has been made. Although an important number of different kind of problems have been formulated, only two of them have been deeply analysed: maximum stiffness problem and minimum weight with stress constraints problem. On the other hand, the variety of ways developed to solve these problems is extremely high due to the drawbacks emerged during the solution process. Nevertheless, the attention in this publication will be only focused in the second one because of its high interest since the engineering point of view. The different alternatives used to solve the minimum weight with stress constraints problem are related with the different ways to define the material layout in the domain and the different ways to impose the stress constraints in the topology optimization problem. While the first aspect has a high influence in the quality of the solution regarding to the spatial definition, the second one impacts in the CPU time used to solve the problem. First, regarding to the way to define the material layout in the domain, the approaches developed until now are: the homogenization techniques [2]–

WIT Transactions on The Built Environment, Vol 209, © 2022 WIT Press
www.witpress.com, ISSN 1743-3509 (on-line)
doi:10.2495/HPSU220071

[4], the solid isotropic material with penalty (SIMP) [5], [6], the multimicrostructural approach [7]–[9], the level set methods [10]–[12], the bubble method [13], the phase field approach [14], and more recently the isogeometric analysis [15]–[17]. On the other hand, with regards to the imposition of the stress constraints, the strategies used to impose them until now are: local stress constraints [18], [19], global stress constraints [20], [21], block aggregation of stress constraints [22], [23] and more recently the damage approach [24]. In this research for defining the material layout in the domain the isogeometric analysis will be used. However, the overweight approach, an alternative formulation of the damage approach previously developed, is established to impose the stress constraints in the topology optimization problem.

2 STRUCTURAL ANALYSIS

The structural analysis used in the topology optimization problem proposed is the classical method of the finite element method. In this approach several hypotheses must be considered: small displacements, small displacement derivatives and elastic and linear material. On the other hand, this formulation is also used with the isogeometric analysis. Since the design variables used in the topology optimization problem define the material layout, it is necessary to consider the effect of the relative density in the finite element formulation. For this reason, the classical finite element method formulation has to be modified in order to include the design variables of the optimization problem. This issue has been addressed by using a numerical formulation that includes the effect of the relative density [22], [25], [26]. This formulation is equivalent in a certain sense to the SIMP method, and only requires to introduce slight modifications in the classical formulation. These modifications are reduced to consider the effect of the relative density in the integrals of the contribution of each finite element or knot span of the mesh, in case of using the finite element method or the isogeometric analysis respectively. Therefore, the structural analysis can be obtained by solving the system of linear equations:

$$\boldsymbol{K}\boldsymbol{\alpha} = \boldsymbol{f}, \tag{1}$$

where in this case:

$$\boldsymbol{K} \equiv \boldsymbol{K}(\boldsymbol{\rho}), \quad \boldsymbol{\alpha} \equiv \boldsymbol{\alpha}(\boldsymbol{\rho}), \quad \boldsymbol{f} \equiv \boldsymbol{f}(\boldsymbol{\rho}). \tag{2}$$

The matrix $\boldsymbol{K}$ is the structural stiffness matrix, $\boldsymbol{\alpha}$ is the nodal displacement vector and $\boldsymbol{f}$ is the applied loads vector. Each one of the terms $\boldsymbol{K}_{ji}$ of the stiffness matrix $\boldsymbol{K}$ is obtained as:

$$\boldsymbol{K}_{ji} = \textstyle\sum_{e=1}^{N_e} \boldsymbol{K}_{ji}^{e}, \quad j = 1, \ldots, N, \quad i = 1, \ldots, N, \tag{3}$$

where N_e is the number of elements or knot spans, depending on the method used to compute the structural analysis, finite element method or isogeometric analysis, respectively; and N is the number of the nodes of the mesh. At this point, it is important to remark that the number of points required to compute the structural analysis is different depending on the method used, since the way to define the material layout will be also different. Except for particular cases, isogeometric analysis needs fewer nodes than the finite element method. The elemental stiffness matrixes $\boldsymbol{K}^e$ must be computed as usual by multiplying the corresponding integrand times the relative density of element or knot span e [25], [26]. On the other hand, the applied forces vector $\boldsymbol{f}_j$ can be obtained as usual in a conventional finite element formulation by considering that the contribution of the forces per unite of volume must be multiplied times the relative density [25], [26]. The whole finite element formulation including the relative density effect has been specifically addressed in previous papers by Navarrina et al. [25] and

Paris et al. [26]. The linear equation system proposed in eqn (1) is solved by means of a Cholesky factorization algorithm since it allows to compute additional systems of equations with small effort. These additional linear equation systems are required in the attainment of the sensitivity analysis. Finally, if a comparative analysis between the use of the finite element method or the isogeometric analysis is made, it is expectable that the CPU time requirements will be lower in the case of the isogeometric analysis, since the size of the equation system is lower due to the number of points required to compute it.

3 OPTIMIZATION PROBLEM

The optimization problem can be formulated in different ways; however, its more typical formulation is:

$$\begin{array}{lll} Calculate & \rho = \{\rho_i\}, & \mathrm{i} = 1, \dots, \mathrm{n} \\ which \left(\dfrac{minimize}{maximize}\right) & F(\rho) & \\ verifying & g_j\left(r_j^0, \rho\right) \leq 0 & j = 1, \dots, m \\ & h_l(r_l^0, \rho) = 0 & l = 1, \dots, p \\ & \rho_{min} \leq \rho_i \leq \rho_{max} & i = 1, \dots n, \end{array} \tag{4}$$

where $\boldsymbol{\rho}=\{\rho_i\}$ is the design variable vector, $F(\rho)$ is the objective function, g_j are the inequality constraints of the problem and, finally, h_l are the equality constraints of the problem. Furthermore, in the optimization problem, n is the number of design variables, while m and p are, respectively, the number of inequality and equality constraints. Finally, as far as optimization problem concerns, it will be also necessary to establish the side constraints of the design variables, which are respectively ρ_{min} and ρ_{max}.

3.1 Objective function

In the topology optimization problem developed in this publication, the objective function of the problem is the structural weight whose value will have to be minimized. The formulation of the objective function will be, therefore:

$$F(\rho) = \sum_{e=1}^{N_e} \int_{\Omega_e} \rho \, d\Omega_e, \tag{5}$$

where N_e is the number of elements or knot spans in which the domain is discretised, Ω_e represents the domain of the element or knot span e, and ρ is the relative density. At this point, the methods used to define the material layout are briefly explained. First, each design variable will represent the relative density in each element of the mesh in the finite element formulation, however, in the isogeometric analysis formulation, the scheme used to define the density relative coincides with the scheme used to compute the structural analysis, for this reason, it is necessary to use the same B-splines, since the design variables are in this case, the value of the relative density in the control points. Finally, a factor which penalizes intermediate relative density values "p" is introduced in the formulation since the attainment of full-void solutions is intended. This factor "p" is introduced in the objective function, and as a result, the objective function of the topology optimization problem will be:

$$F(\rho) = \sum_{e=1}^{N_e} \int_{\Omega_e} \rho^{1/p} d\Omega_e. \tag{6}$$

3.2 Overweight constraint

In the topology optimization problem developed in this publication, there will be only one inequality constraint. This constraint will be known as overweight constraint, since it is obtained through the application of the overweight approach. The overweight approach is used as an alternative way to impose the stress constraints in the topology optimization problem. In this approach, two different models that describe the same mechanical body will have to be defined. They will be known hereinafter as original model and overweight model. The overweight model will be defined from the original one doing a sequence of steps. First, the structural analysis of the original model is computed. Then, the structural stresses in the points where they have to be checked can be computed. Finally, an overweight is introduced in the model in the areas where stresses exceed their maximum allowable value. Since the relative density is the property modified in the original model to create the overweight model, the relative density in the overweight model will have to satisfy these conditions:

$$\begin{cases} \tilde{\rho}(x) > \rho(x) & \forall x \in \Omega_\sigma := \{x | \sigma(x)| > \sigma_{lim}\} \\ \tilde{\rho}(x) = \rho(x) & \forall x \in \Omega \backslash \Omega_\sigma, \end{cases} \tag{7}$$

where ρ is the relative density in the original model and $\tilde{\rho}$ in the overweight model. Finally, the overweight constraint imposed in the topology optimization problem is a comparison between the structural weight of both models:

$$g(\rho) = \frac{\widetilde{W}}{W} - 1 \leq 0, \tag{8}$$

where W is the structural weight of the original model and $\widetilde{W}$is the structural weight of the overweight model and they can be computed as follows:

$$W(\rho) = \int_\Omega \rho \, d\Omega \qquad \widetilde{W}(\rho) = \int_\Omega \tilde{\rho} \, d\Omega \, . \tag{9}$$

3.2.1 Overweight model

The formulation of the relative density in the overweight model can be defined by different ways. A similar scheme that in Verbart et al. [24] is used to define the relative density in the overweight model, this relative density is increased in the overweight model. Furthermore, the relative density of the overweight model can be computed as:

$$\tilde{\rho} = \rho_{min} + \beta(\rho - \rho_{min}) \qquad where\ \beta(\sigma; \sigma_{lim}) \geq 1, \tag{10}$$

where β is the overweight function. This function has to satisfy two conditions: to be at least first order differentiable and to be monotonically crescent when stresses exceed its allowable limit.

3.3 Side constraints

The side constraints of the design variables are the only term of the topology optimization problem which has not been defined yet. While the upper limit of the design variables is obvious since its value will be equal to 1, what represents the area full of material, the lower limit of the design variables will not be null, since this circumstance can produce inherent problems in some parts of the topology optimization algorithm, like the singularity of some matrixes. For this reason, the value of this lower limit will have to be a value slightly higher to zero; in this publication it will be equal to 0.001.

4 OPTIMIZATION ALGORITHMS

The solution of the topology optimization problem (4) will be made by means of the use of different algorithms. The algorithms used to solve the problem in this publication are: the steepest descent method, the sequential linear programming algorithm based on the simplex algorithm and a, so called, back to the feasible region algorithm based on constraint derivatives. In the first place, the steepest descent method will be only used when the value of all the structural stresses considered are inferior to a certain percentage of the maximum allowable value. This circumstance only happens in the first iteration of the topology optimization problem, since the structural domain is full of material in the first iteration. For this algorithm, it is only necessary to compute the objective function sensitivity analysis. On the contrary, the back to the feasible region algorithm is used when the structural weight of the overweight model exceeds a certain percentage of the structural weight of the original one. In this case, the solution is far from the feasible region, and the simplex algorithm does not run properly. For this reason, it is only necessary to compute in this case the overweight constraint sensitivity analysis. Finally, in the intermediate situations the simplex algorithm is used. This situation will be more common during the optimization procedure since the solution tend to be near to the border of the feasible region. Therefore, in this case it is necessary to compute both sensitivity analyses: objective function and overweight constraint. The main reason to use these algorithms is to compute the best improvement direction at each iteration depending on the situation of the previous solution. Therefore, once the improvement direction has been computed, the next step is to establish the most appropriated value of the improvement factor which will determine the solution of this iteration. For this purpose, the first order Taylor series will have to be calculated. On the other hand, a linear search will be made in order to avoid the calculus of directional derivatives, since the first order directional derivatives can be calculated directly with the information used with the previous algorithms. Lastly, once the improvement direction and the improvement factor are known, the next step is to update the solution of the topology optimization problem, and to repeat the optimization process until convergence.

5 SENSITIVITY ANALYSIS

The solution of the topology optimization problem requires to compute the derivatives of the objective function and the overweight constraint. The process of computing the derivatives is known as sensitivity analysis. This procedure is the other key point in the solution of the topology optimization problem apart from the structural analysis. In this publication only a first order sensitivity analysis will be developed. On the other hand, it is necessary to use two different approaches for doing the first order sensitivity analysis due to the different characteristics of the objective function and the overweight constraint. First, the sensitivity analysis of the objective function can be computed analytically, since the structural weight depends directly on the design variables. On the contrary, the calculation of the sensitivity analysis of the overweight constraint require the use of more complex procedures, since the relationship between the damage constraint and the design variables is not as straightforward as in the objective function. On the other hand, there are other approaches more computationally efficient than the direct differentiation, if the number of constraints is considerably lower than the number of design variables. This circumstance happens in this problem where only one constraint is defined. For this reason, the approach used in this publication to calculate the sensitivity analysis of the overweight constraint will be the adjoint variable approach [27]. Although, the directional derivatives of the objective function and the overweight constraints are necessary for the optimization algorithm, their calculation can be made directly from the first order derivatives of them, by means of the multiplication of the first order derivative vector times the improvement direction obtained with the algorithms previously introduced. Finally, it is important to remark an important difference between the use of the finite element method and the isogeometric analysis regarding to the way that each design variable have influence in one or more elements or knot spans. While for the finite element method used in this publication, only one design variables has influence in each element of the mesh, for the isogeometric analysis proposed in this publication, this number will be equal to NINE. These numbers depend on the interpolation scheme used with the isogeometric analysis. Consequently, if the interpolation scheme is changed, the number of design variables which has influence in each element or knot span can be not only increased but also reduced. Because of this circumstance, an increase of the computational requirements and CPU time will be obtained in case of using the isogeometric analysis instead of the finite element method. Furthermore, this situation supposes a contradiction with respect to the structural analysis, where the optimal method to do it was the isogeometric analysis.

6 APPLICATION EXAMPLES

In this paper, only one example is shown. This example is used to test the performance of the overweight approach technique as an indirect way to impose the stress constraints. This example is solved both with the finite element method and the isogeometric analysis. The numerical example solved in this paper is the optimum design of a cantilever beam. The proposed example corresponds to a 2D structure in plane stress. Finally, a comparative analysis of the results with both methods is made, not only regarding to the quality of the solutions but also the computational requirements used to obtain them. This analysis let establish if one of the methods is highly advisable to solve the topology optimization problem or if there are not important differences between them.

6.1 Optimum design of a cantilever beam

The example studies the optimum design of a cantilever beam with null displacements in the left edge and with a vertical force applied in the middle of the right edge. Fig. 1 shows the dimensions of the domain and the position of the vertical force applied. Self-weight of the structure is included as a structural load in this example. The domain of the structure has been discretized by using a homogeneous mesh with $160 \times 80 = 12{,}800$ 8-node quadrilateral elements in case of the finite element method and quadratic knot spans in case of the isogeometric analysis. The thickness of the structure is 0.2 m. The external force applied (1×10^3 kN) has been distributed in eight contiguous elements or knot spans in order to avoid stress accumulation phenomena. The material being used in this problem is steel with density $\gamma_{mat} = 7{,}850$ kg/m^3, Young's Modulus $E = 2.1 \times 10^5$ MPa, Poisson ratio $\nu = 0.3$ and elastic limit $\sigma_{max} = 230$ MPa. Fig. 2 shows the optimal solution of the problem for the finite element method and the isogeometric analysis. The optimal solution is represented by means of its material distribution in the domain with the value of the relative density at each point. This relative density defines the percentage of the material phase. The structural weight of the optimal solution is 136.75 kg in case of the finite element method and 130.02 kg in case of the isogeometric analysis, this means the 17.42% and the 16.56% of the structural weight of the initial solution (all the domain is full of material, structural weight of the initial solution: 785 kg).

Fig. 3 shows the normalized stress state for the cantilever beam problem obtained by using the overweight approach for both methods. This normalized stress σ_e* has been computed by

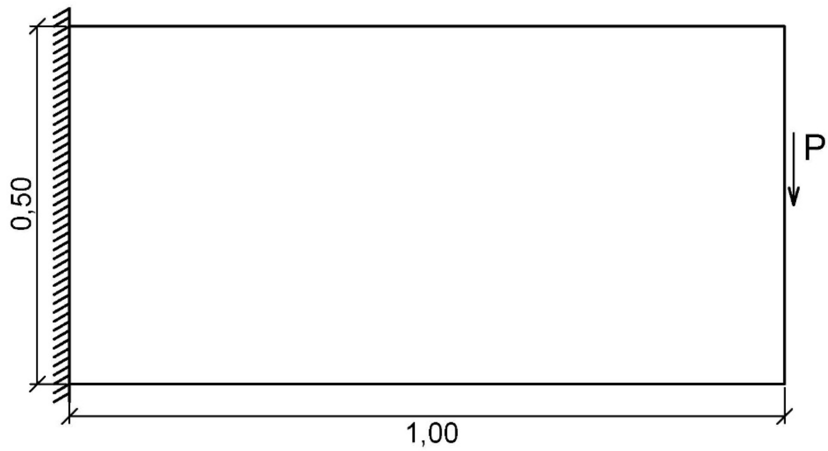

Figure 1: Scheme of the cantilever beam problem (units in m).

means of the maximum allowable value of the stress $\sigma_{max,}$ the relaxation parameter of stresses φ_e which increases the value of the maximum allowable stress when the relative density is lower than 1 and the Von Mises stress $\sigma_{VM,e}$, which is the most typical failure criterion when the steel is the structural material.

The optimal solution of the topology optimization problem has been obtained after 2250 iterations with both methods: finite element method and isogeometric analysis. Finally, the distribution of the computing time per algorithm of all the iterations computed in the optimization process for both methods: finite element method and isogeometric analysis is shown in Tables 1 and 2, respectively.

6.2 Comparative analysis

In the first place, the way to define the relative density in the domain plays an important role in the obtained results. In general, the results obtained with the isogeometric analysis proposed are better in terms of quality in comparison with the obtained with the finite element method proposed, regarding to the spatial definition. In other words, the number of design variables used to obtain solutions with the same spatial definition is lower with the isogeometric analysis proposed. Nevertheless, this circumstance can be solved, if an alternative way to define the material layout is established in case of the finite element method. However, the topology of the results obtained with both methods is the same, since the optimal solution consists on a set of bars that coincide with the isostatic lines. In the second place, regarding to the CPU time, the critical algorithm of both methods does not coincide. While the structural analysis is the critical step of the finite element method proposed, since it supposes the 75% of the time required to solve the topology optimization problem, the sensitivity analysis is the critical part of the isogeometric analysis, since it supposes the 60% of the time used to solve the topology optimization problem. On the one hand, the CPU time required to compute the structural analysis is related with the number of points used to solve it, since this number is related with the size of the structural problem. On the other hand, the CPU time required to compute the sensitivity analysis is related with the number of design variables which have influence in each element of the mesh or knot span

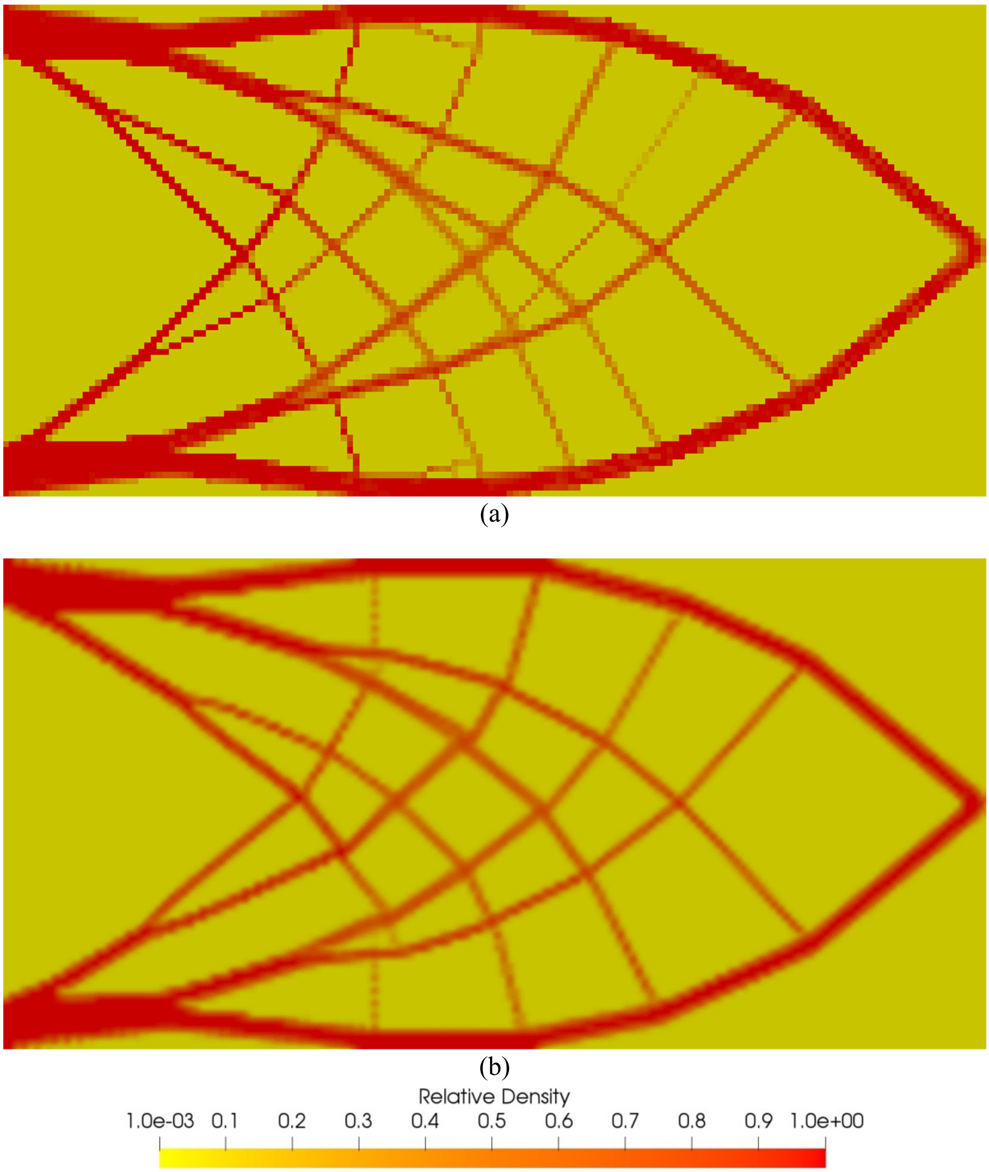

Figure 2: Optimal solution of the cantilever beam problem by using overweight approach. (a) Finite element method; and (b) Isogeometric analysis.

of the patch, since this number is related with the way to define the material layout in the domain. Finally, other important aspect that does not suppose an important difference between both methods, is the appearance of stresses whose value is higher than their maximum allowable value. This circumstance happens in the areas where there is a stress concentration phenomenon, and in areas whose relative density is close to their minimum allowable value.

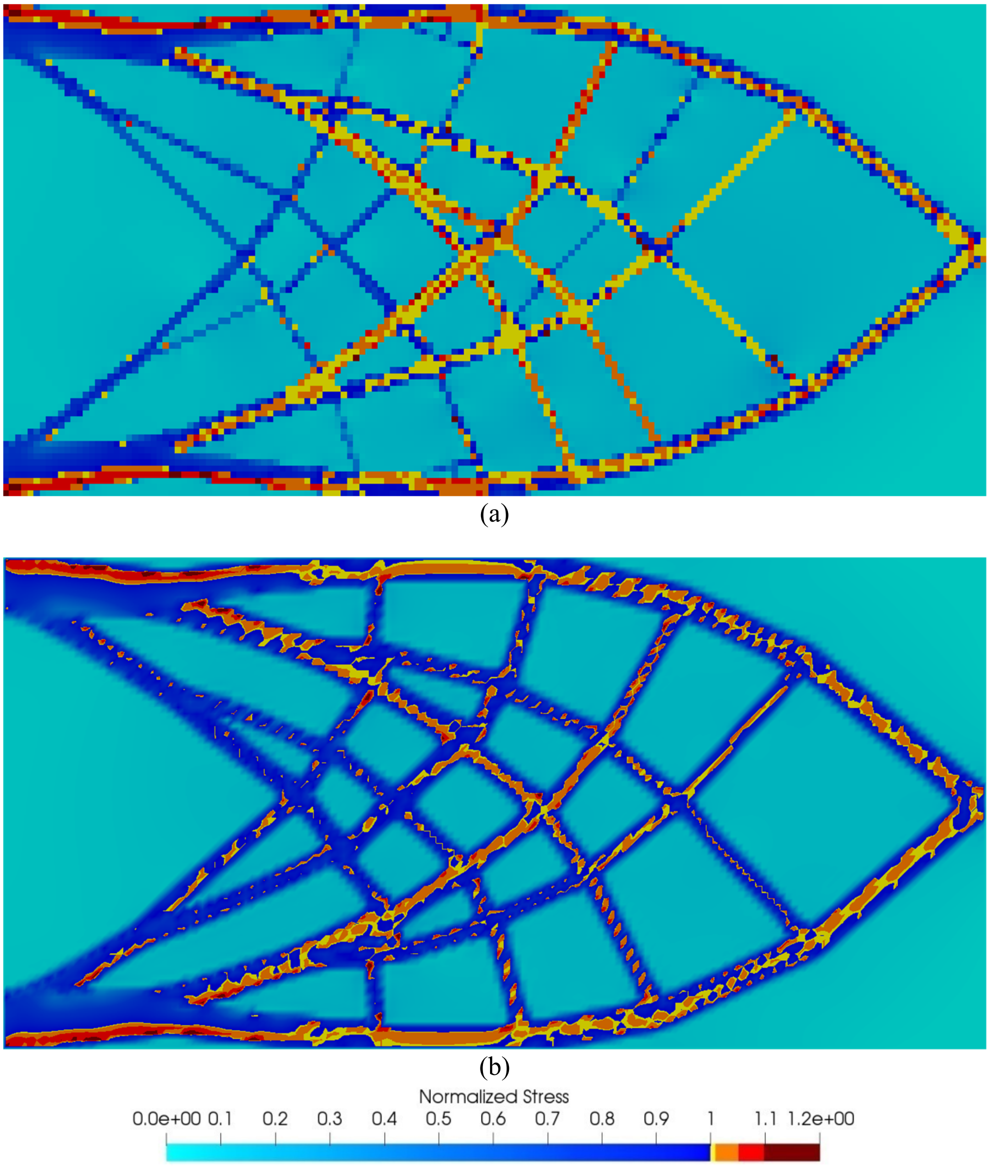

Figure 3: Normalized stress state of the cantilever beam problem by using the overweight approach (σ_e*= $\sigma_{VM,e}$/(φ_e $\sigma_{max,}$)). (a) Finite element method; and (b) Isogeometric analysis.

7 CONCLUSIONS

This paper introduces a new method to solve the topology optimization of structures problem with minimum weight and stress constraints. The attainment of results with high spatial definition makes necessary to consider a large number of stress constraints and to use a large

Table 1: Distribution of CPU time per algorithm in the solution of the cantilever beam problem with the finite element method by using only one processor.

Algorithms	Time (s)	% Total time
Structural analysis	17,235	73.01
Sensitivity analysis	5,546	23.50
Optimization algorithm	539	2.29
Rest of the process	283	1.20
Total CPU time	23,603	

Table 2: Distribution of CPU time per algorithm in the solution of the cantilever beam problem with the isogeometric analysis by using only one processor.

Algorithms	Time (s)	% Total time
Structural analysis	4,608	27.41
Sensitivity analysis	10,418	61.97
Optimization algorithm	702	4.18
Rest of the process	1,083	6.44
Total CPU time	16,811	

number of design variables. For this reason, new methods which can combine the effect of all these stress constraints in a general constraint have to be developed. In this paper, an overweight constraint is used. While the critical step of the classical algorithms used to solve this problem was the search of the improvement direction since all the active stress constraint derivatives have to be considered. This drawback disappears with the overweight approach, since only two derivatives have to be considered in this step: the objective function and the overweight constraint. On the other hand, the isogeometric analysis is introduced as an alternative to the classical finite element method, not only as a way to define the material layout, but also to compute the structural analysis. Moreover, the amount of CPU time required to solve the example proposed in this paper is lower in case of the isogeometric analysis. This is important since, for the same problem, the isogeometric analysis also provides solutions with high spatial definition in comparison with the finite element method. In other words, the attainment of solutions with the same spatial definition requires to use less design variables and CPU time in case of isogeometric analysis in comparison with the finite element method. In conclusion, the use of the Overweight Approach reports important benefits from a computational point of view in topology optimization problems with stress constraints, and the results obtained are similar in terms of quality with the obtained with the classical approaches. The efficiency of the isogeometric analysis in comparison to the finite element method has been also clearly demonstrated, since solutions with high spatial definition are obtained in less CPU time.

ACKNOWLEDGEMENTS

This work has received financial support from the Xunta de Galicia (Secretaria Xeral de Universidades) and the European Union (European Social Fund (ESF)) through “Grants for funding predoctoral stages in universities, public research organisms and other R&D entities of Galicia” ED481A-2016/387. This work has also been partially supported also by FEDER funds of the European Union, by the “Ministerio de Economía y Competitividad” of the Spanish Government through grants DPI2015-68431-R and RTI2018-093366-B-I00 and by the “Conselleria de Educación, Universidade e Formación Profesional” of the Xunta de

Galicia, through grants for the consolidation and structuring of competitive research units of the Galician University System: GRC2014/039 and GRC2018/41.

REFERENCES

[1] Bendsøe, M.P. & Kikuchi, N., Generating optimal topologies in structural design using a homogenization method. *Computer Methods in Applied Mechanics and Engineering*, **71**, pp. 197–224, 1988.
[2] Murat, F. & Tartar, L., Calcul des variations e homogénéisation, les méthodes de l'homogénéisation théorie et applications en physique. *Collegue Dir. Etudes et Recherches EDF*, **57**, pp. 319–369, 1985.
[3] Suzuki, K. & Kikuchi, N., A homogeneization method for shape and topology optimization. *Computer Methods in Applied Mechanics and Engineering*, **93**(3), pp. 291–318, 1991.
[4] Allaire, G., Jouve, F. & Maillot, H., Topology optimization for minimum stress design with the homogenization method. *Structural and Multidisciplinary Optimization*, **28**, pp. 87–98, 2004.
[5] Rossow, H.P. & Taylor, J.E., A finite element method for the optimal design of variable thickness sheets. *AIAA Journal*, **11**, pp. 1566–1569, 1973.
[6] Bendsøe, M.P. (ed.), *Optimization of Structural Topology, Shape and Material*, Springer-Verlag, 1995.
[7] Rodrigues, H., Guedes, J.M. & Bendsøe, M.P., Hierarchical optimization of material and structure. *Structural Multidisciplinary Optimization*, **24**, pp. 1–10, 2002.
[8] Nakshatrala, P.B., Tortorelli, D.A. & Nakshatrala, K., Nonlinear structural design using multiscale topology optimization. Part I: Static formulation. *Computer Methods in Applied Mechanics and Engineering*, **261**–**262**, pp. 167–176, 2013.
[9] Sivapuram, R., Dunning, P.D. & Kim, H.A., Simultaneous material and structural optimization by multiscale topology optimization. *Structural Multidisciplinary Optimization*, **54**, pp. 1267–1281, 2016.
[10] Sethian, J.A. & Wiegmann, A., Structural boundary design via level set and immersed interface methods. *Journal of Computational Physics*, **163**, pp. 489–528, 2000.
[11] Wang, M.Y., Wang, X. & Guo, D., A level set method for structural topology optimization. *Computer Methods in Applied Mechanics and Engineering*, **192**, pp. 227–246, 2003.
[12] Amstutz, S. & Andrä, H., A new algorithm for topology optimization using a level-set method. *Journal of Computational Physics*, **216**, pp. 573–588, 2006.
[13] Eschenauer, H.A. & Schumacher, A., Bubble method: A special strategy for finding best possible initial designs. *Proceedings of the ASME Design Technical Conference 19th Design Automotion Conference*, **65**, pp. 437–443, 1993.
[14] Wang, M.Y. & Zhou, S., Phase field: A variational method for structural topology optimization. *Tech Science*, **6**(6), pp. 547–566, 2004.
[15] Hassani, B., Khanzadi, M. & Tavakkoli, S.M., An isogeometrical approach to structural topology optimization by optimality criteria. *Structural Multidisciplinary Optimization*, **45**, pp. 223–233, 2012.
[16] Qian, X., Topology optimization in B-spline space. *Computer Methods in Applied Mechanics and Engineering*, **265**, pp. 15–35, 2013.
[17] Lieu, Q.X. & Lee, J., A multi-resolution approach for multi-material topology optimization based on isogeometric analysis. *Computer Methods in Applied Mechanics and Engineering*, **323**, pp. 272–302, 2017.

[18] Duysinx, P. & Bendsøe, M.P., Topology optimization of continuum structures with local stress constraints. *International Journal of Numerical Methods in Engineering*, **43**, pp. 1453–1478, 1998.

[19] Navarrina, F., Muiños, I., Colominas, I. & Casteleiro, M., Minimum weight with stress constraints topology optimization. *Proceedings of 7th US National Congress on Computational Mechanics*, 2003.

[20] Martins, J.R.R.A. & Poon, N.M.K., On structural optimization using constraint aggregation. *VI World Congress on Structural and Multidisciplinary Optimization*, 2005.

[21] Poon, N.M.K. & Martins, J.R.R.A., An adaptive approach to constraint aggregation using adjoint sensitivity analysis. *Structural Multidisciplinary Optimization*, **34**, pp. 61–73, 2007.

[22] Paris, J., Navarrina, F., Colominas, I. & Casteleiro, M., Block aggregation of stress constraints in topology optimization of structures. *10th International Conference on Computer Aided Optimum Design in Engineering, OPTI 2007*, 2007.

[23] Paris, J., Navarrina, F., Colominas, I. & Casteleiro, M., Advances in the statement of stress constraints in structural topology optimization. *4th International Conference on Advanced Computational Methods in Engineering, ACOMEN 2008*, 2008.

[24] Verbart, A., Langelaar, M. & Van Keulen, F., Damage approach: A new method for topology optimization with local stress constraints. *Structural Multidisciplinary Optimization*, **53**, pp. 1081–1098, 2016.

[25] Navarrina, F., Muiños, I., Colominas, I. & Casteleiro, M., Topology optimization of structures: A minimum weight approach with stress constraints. *Advanced Engineering Software*, **36**, pp. 599–606, 2005.

[26] Paris, J., Navarrina, F., Colominas, I. & Casteleiro, M., Topology optimization of continuum structures with local and global stress constraints. *Structural Multidisciplinary Optimization*, **39**(4), pp. 419–437, 2009.

[27] Paris, J., Restricciones en tension y minimización del peso: Una metodología general para la optimización topológica de estructuras, PhD thesis, University of A Coruña, 2007.