TOPOLOGY OPTIMIZATION OF STRUCTURES WITH STRESS, BUCKLING AND DYNAMIC BEHAVIOUR CONSTRAINTS

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ABSTRACT

We can define topological optimization of structures as the branch of computational mechanics that seeks to obtain optimal material layouts in a given domain under specific boundary conditions. This discipline, pioneered by Bendsøe and Kikuchi in the late 1980s, is becoming increasingly relevant due to its direct application in various sections of applied physics such as civil, mechanical and aeronautical engineering. In recent years, the advancements in computational capacity have enabled a significant leap in the topological optimization of structures. This work aims, first, to systematize and compile the current knowledge surrounding the behaviour of structural models with intrinsic relative densities used in the SIMP method, in order to design solutions to the various physical and numerical challenges arising from the use of Galerkin interpolation methods (FEM, IGA, etc.) for solving elasticity equations with relative density. Specifically, we will attempt to address the problems arising from topological optimization processes in which dynamic structural analysis or second-order theory plays a fundamental role. Ultimately, the main objective of this work is to design topological optimization formulations for structures of isotropic materials using the SIMP method integrating advanced constraints and combining them in a way that accounts for an increasing number of physical phenomena, thereby reducing the structure's weight to optimal level. The reduction in structural weight introduces the potential for instabilities, dynamic effects and nonlinearities, which will be addressed throughout the optimization process using well-known structural analysis techniques such as the calculation of natural vibration frequencies, linear buckling or full geometric nonlinearity.
Keywords: structures, optimization, buckling, frequencies, stress.

1 INTRODUCTION

Topological optimization of structures is a discipline that began to take its first steps in the 1980s, with the development of computers and the evolution of the finite element method [1]. This discipline can be understood as the intersection of three distinct scientific branches: on one hand, optimization, a field that aims to minimize or maximize an objective function in a given problem, subject (or not) to certain constraints; on the other hand, topology, a branch of mathematics that studies the properties of continuous geometric bodies; and finally, structural analysis, a branch of engineering that seeks to determine the behaviour of a specific structure using physical principles.

Based on this foundation, this discipline can be defined as the combination of the three aforementioned branches, as it seeks to maximize or minimize a property of a structure without imposing a discontinuity or abrupt change in it, that is, while maintaining a global behaviour. The usefulness of topological optimization of structures has been widely demonstrated in all fields of engineering, from industrial engineering – where the automated execution of industrial components makes optimization play a fundamental role – to civil engineering, where reducing material costs is one of the primary objectives.

As of today, advancements in structural topology optimization have made it possible to integrate greater control over the dynamic and energy stability of structures into these processes. Significant progress has been made toward achieving a comprehensive

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optimization process that considers both stress and stability factors. However, there is still a need to develop a global optimization framework for steel structures that brings us closer to bridging the gap between mathematical optimization and the design of realistic structures that comply with established regulations.

2 STRUCTURAL RESPONSES

2.1 Introduction to structural responses

When we calculate and design structures, simplifications can be made by assuming that the state in the deformed position does not change or hardly changes compared to the initial geometric arrangement. The formulation of this hypothesis allows us to simplify structural calculations while obtaining highly accurate results for specific cases. However, when dealing with slender structures, neglecting nonlinear behaviour can lead to significant errors in the final results. Therefore, we must calculate the structure while considering that the deformed position will differ significantly from the initial position.

2.2 Geometrically non-linear formulation

First, it is important to highlight that the structural analysis method used in this work is linear. However, the explanation of the nonlinear formulation is necessary because the implementation of linear instability, which will be addressed in later sections, as well as the linear formulation itself and dynamic analysis, are based on the development of the updated Lagrangian formulation, which we will now present. So, we begin with the incremental expression of Principle of Virtual Work:

$$\int_{V_0} \delta\boldsymbol{\gamma}^T \boldsymbol{\tau}^{t+\Delta t} dV = \int_{S_0} \delta\boldsymbol{u}^T \boldsymbol{t_0}\ dS + \int_{V_0} \rho_0 \delta\boldsymbol{u}^T \boldsymbol{g}\ dV \tag{1}$$

where the right-hand side corresponds to the terms of the virtual external work, with $\boldsymbol{\gamma}$ as the Euler–Lagrange strain tensor, $\boldsymbol{\tau}$ as the second Piola–Kirchoff stress tensor, $\boldsymbol{t}$ being the nominal forces on the boundary of the continuous medium, $\boldsymbol{g}$ the gravitational forces, and $\boldsymbol{u}$ the displacements.

Since we will not discuss plasticity in this work, we can establish a linear relationship between the stress increment of the second Piola–Kirchhoff tensor and the Green–Lagrange strain tensor. In summary, we can introduce the material's constitutive equation, which for linear elasticity will be the well-known Hooke's Law:

$$\Delta\boldsymbol{\tau} = \boldsymbol{D}\Delta\boldsymbol{\gamma} \tag{2}$$

where $\boldsymbol{D}$ is the constitutive tensor.

Since we are working in a nonlinear formulation, the increment of the Green–Lagrange strain tensor will have a linear part and a nonlinear part. Discarding the nonlinear part of the tensor and rearranging the terms, we obtain the following:

$$\int_{V_0} \delta\Delta\boldsymbol{e}^T \boldsymbol{D}\, \Delta\boldsymbol{e}\, dV + \int_{V_0} \delta\boldsymbol{\eta}^T \boldsymbol{\tau}^t\ dV = \int_{S_0} \delta\boldsymbol{u}^T \boldsymbol{t_0}\ dS + \int_{V_0} \rho_0 \delta\boldsymbol{u}^T \boldsymbol{g}\ dV - \int_{V_0} \delta\boldsymbol{e}^T \boldsymbol{\tau}^t\ dV \tag{3}$$

where $\boldsymbol{e}$ and $\boldsymbol{\eta}$ are the linear and non-linear parts of the Green–Lagrange strain tensor respectively. Having obtained the present equation, it is important to note that we will use the finite element method for the discretization of the problem.

Thus, we will obtain an implicit iterative scheme where the resolution of each step can be carried out through a Newton–Raphson scheme, displacement control, or the most suitable method for the problem at hand. After applying the FEM, we obtain the following:

$$(\boldsymbol{K_L} + \boldsymbol{K_G})\,\Delta\boldsymbol{a} = \left(\boldsymbol{f}_{\mathbf{ext}}^{t+\Delta t} + \boldsymbol{f}_{\mathbf{int}}^{t}\right) \tag{4}$$

$$\boldsymbol{f}_{\mathbf{ext}}^{t+\Delta t} = \int_{S_0} \boldsymbol{H}^{\mathrm{T}}\boldsymbol{t_0}\ \mathrm{d}S + \int_{V_0} \rho_0 \boldsymbol{H}^{\mathrm{T}}\boldsymbol{g}\ dV\ ;\ \boldsymbol{f}_{\mathbf{int}}^{t} = \int_{V_o} \boldsymbol{B}_{\boldsymbol{L}}^{T}\boldsymbol{\tau}^{t}\ dV$$

$$\boldsymbol{K_L} = \int_{V_0} \boldsymbol{B}_{\boldsymbol{L}}^{T}\boldsymbol{D}\boldsymbol{B_L}dV\ ;\boldsymbol{K_G} = \int_{V_0}^{t} \boldsymbol{B}_{\boldsymbol{NL}}^{T}\boldsymbol{T}\boldsymbol{B_{NL}}dV$$

where $\boldsymbol{a}$ are the nodal displacements of the mesh, $\boldsymbol{K_L}$ the linear stiffness matrix, $\boldsymbol{K_{NL}}$ the geometric stiffness matrix (the sum of both is called the tangent matrix), $\boldsymbol{f}_{\mathbf{ext}}^{t+\Delta t}$ the external forces of the structure at the final of the load step, $\boldsymbol{f}_{\mathbf{int}}^{t}$ the internal forces of the structure and $\boldsymbol{H}, \boldsymbol{B_L}, \boldsymbol{B_{NL}}$ are the matrices that contain, respectively, the shape functions of the vector space and their derivatives. For a much more exhaustive explanation of this and the following formulations, it is recommended to consult [2].

Following this scheme, what we are doing is minimizing the residual of the structure. In equilibrium for predetermined loads, the following occurs:

$$\boldsymbol{f}_{\mathbf{ext}}^{end} = \boldsymbol{f}_{\mathbf{int}}^{end} \tag{5}$$

And taking this into account, if (for example) we solve the equation using the Newton–Raphson method, the tangent stiffness matrix is nothing more than the derivative of the residual we want to minimize and, therefore, the derivative of the internal force with respect to the displacements, since in most cases, the derivative of the external forces does not depend on the displacements.

Thus, if we wanted to introduce a variable such as relative density into the formulation, the proper approach would first be to implement it in the computation of the internal forces due to the volume integral performed to obtain the forces, which depends on the density of each element. Thus, developing the by-element formulation:

$$\boldsymbol{f}_{\boldsymbol{int,e}}(\rho_{\mathrm{e}}, \mathbf{u_e}) = \int \boldsymbol{\rho_e}\, \boldsymbol{B}_{\boldsymbol{L,e}}^{T}\boldsymbol{\tau}_{\boldsymbol{e}}^{t} dV \rightarrow \mathbf{K_{m,e}} = \frac{\partial \boldsymbol{f}_{\boldsymbol{int,e}}}{\partial \boldsymbol{u}} = \rho_{\mathrm{e}}\boldsymbol{K_m} \tag{6}$$

with $\boldsymbol{K_m}$ as the tangent stiffness matrix. Thus, considering the previous development and assembling the elemental matrices, we obtain the following formulation of the matrices for meshes where the elements have relative density.

$$\boldsymbol{K_m}\,\Delta\boldsymbol{a} = \left(\boldsymbol{f}_{\mathbf{ext}}^{t+\Delta t} + \boldsymbol{f}_{\mathbf{int}}^{t}\right) \tag{7}$$

$$\boldsymbol{f}_{\mathbf{ext}}^{t+\Delta t} = \int_{S_0} \boldsymbol{H}^{\mathrm{T}}\boldsymbol{t_0}\ \mathrm{d}S + \int_{V_0} \rho_0 \boldsymbol{H}^{\mathrm{T}}\boldsymbol{g}\ \boldsymbol{\delta_e}\boldsymbol{\rho_e}\ dV\ ;\ \boldsymbol{f}_{\mathbf{int}}^{t} = \int_{V_o} \boldsymbol{B}_{\boldsymbol{L}}^{T}\boldsymbol{\tau}^{t}\ \boldsymbol{\delta_e}\boldsymbol{\rho_e}\ dV$$

$$\boldsymbol{K_L} = \int_{V_0} \boldsymbol{B}_{\boldsymbol{L}}^{T}\boldsymbol{D}\boldsymbol{B_L}\ \boldsymbol{\delta_e}\boldsymbol{\rho_e}\ dV\ ;\boldsymbol{K_G} = \int_{V_0}^{t} \boldsymbol{B}_{\boldsymbol{NL}}^{T}\boldsymbol{T}\boldsymbol{B_{NL}}\ \boldsymbol{\delta_e}\boldsymbol{\rho_e}\ dV\ ;\boldsymbol{K_m} = \boldsymbol{K_L} + \boldsymbol{K_{NL}}$$

2.3 Linear formulation

The finite element method in the linear regime will be used to calculate the structures within this project, so we will define it up to the calculation of the system of linear equations to be

implemented. We will start from all the previously developed formulation, which will also be necessary in later sections. For the formulation of the finite element method in the linear regime, we can make a series of simplifications based on the statement of the principle of virtual work. The simplifications are as follows:

- Displacements are small.
- The displacement gradient is also small.
- The material is elastic and linear.

Under these simplifications, the formulation remain like it follows:

$$\boldsymbol{K}\,\boldsymbol{a} = \boldsymbol{f} \tag{8}$$

$$\boldsymbol{f} = \int_{S_0} \boldsymbol{H}^{\mathrm{T}}\boldsymbol{t_0}\ \mathrm{d}S + \int_{V_0} \rho_0 \boldsymbol{H}^{\mathrm{T}}\boldsymbol{g}\ \boldsymbol{\delta_e}\boldsymbol{\rho_e}\ dV$$

$$\boldsymbol{K_L} = \int_{V_0} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\ \ \boldsymbol{\delta_e}\boldsymbol{\rho_e}\ dV$$

where $\boldsymbol{B}$ is the matrix that contain the derivatives of the shape functions. The difference between this matrix and $\boldsymbol{B_L}$ is that $\boldsymbol{B}$ does not depend on the strain gradient.

2.4 Linear buckling analysis

The method of linear instability is based on the existence of the equilibrium bifurcation point. That is, along the force–displacement curve of the structure, there is a point at which the trajectory splits into a basic equilibrium branch and an instability branch. This method is used to obtain the bifurcation points of instability. By simplifying the non-linear formulation, we can obtain the bifurcation points by solving an eigenvalue problem like:

$$(\boldsymbol{K} + \lambda_i \boldsymbol{K_{NL}})\boldsymbol{\phi}_i = \boldsymbol{0} \tag{9}$$

where $\boldsymbol{K}$ and $\boldsymbol{K_{NL}}$ are the same matrix as in the previous sections.

In the case of applying the linear instability method to models with relative densities, we must be careful with the appearance of spurious buckling modes. These buckling modes appear due to an artificial increase in stress in areas with very low relative densities. This formulation does not pose any numerical problems when the constraints to be considered in the optimization problem are not intrinsic to the material's density and its effect on the different structural responses. However, in problems related to eigenvalues and eigenvectors, where the material's own density conditions the final calculations, this effect can lead to numerical instabilities [3].

2.5 Dynamics and natural frequencies of vibrations

The natural frequencies of a structure are those vibration frequencies at which a structure continues to vibrate naturally, that is, the ones that indicate the vibration modes of a structure. They are closely related to the number of degrees of freedom of the structure itself, as the number of natural frequencies of a structure is equal to its number of degrees of freedom. Each natural frequency is associated with a vibration mode, in such a way that the structure vibrates differently according to each frequency.

We can define the matrix equation that governs the motion problem of a body with multiple degrees of freedom (without considering damping forces) [4]:

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}(t) \tag{10}$$

If we assume that the movements of the structure will follow the following scheme $[u(t) = \boldsymbol{\phi} e^{i\omega t}; \ddot{u} = -\omega^2\, \boldsymbol{\phi} e^{i\omega t}]$, we obtain the eigenvalue formulation that allows us to solve the natural vibration modes of the structure, as we can see in the following development:

$$(\boldsymbol{K} + \omega_i^2\, \boldsymbol{M})\boldsymbol{\phi}_i = \boldsymbol{0} \tag{11}$$

where $\boldsymbol{M}$ is the mass matrix of the structure which, considering the implementation of relative densities, can be calculated as:

$$\boldsymbol{M} = \int_{V_0} \boldsymbol{H}^T \rho_0\, \boldsymbol{H}\ \boldsymbol{\delta_e}\boldsymbol{\rho_e} dV \tag{12}$$

3 OPTIMIZATION PROBLEM, OBJECTIVE FUNCTION AND CONSTRAINTS

As we mentioned earlier, the optimization problem we are going to solve focuses on obtaining the material distribution in a specific domain under a set of imposed loads, minimizing the weight of the structure, and potentially being subject to constraints on stress, buckling, or frequencies. The general mathematical formulation that encompasses all cases of this problem is the following:

$$\begin{aligned}
&\text{find:} && \boldsymbol{\rho} = \{\rho_i\} \\
&\text{that minimise:} && F(\boldsymbol{\rho}) \\
&\text{verifying:} && g_j(\boldsymbol{u}, \boldsymbol{\rho}) \ \leq\ 0 \\
& && h_1(\boldsymbol{u}, \boldsymbol{\rho}) \ \leq\ 0 \\
& && h_2(\boldsymbol{u}, \boldsymbol{\rho}) \ \leq\ 0 \\
&\text{subject to:} && \boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f} \\
& i = 1, \ldots, N_e \ \ ; \ \ && j = 1, \ldots, N_{\text{rest}}
\end{aligned} \tag{13}$$

where $\boldsymbol{\rho}$ is the vector of design variables, $F(\boldsymbol{\rho})$ is the objective function, $g_j(\boldsymbol{u}, \boldsymbol{\rho})$ are the local constraints (such as tensions, in our case), and $h_k(\boldsymbol{u}, \boldsymbol{\rho})$**)** are the global constraints, such as linear buckling constraints or eigenfrequencies constraints. Some studies have partially used this formulation (with different objective functions and modifications in the constraints), such as [5]–[7].

3.1 Objective function

As an objective function, as indicated in previous sections, we will use weight. A weight minimization formulation that does not penalize intermediate densities may lead to the appearance of areas in the domain with a high predominance of these densities. To correct this effect, a modified objective function is proposed, in which the cost is penalized. Thus,

the use of the objective function proposed by Chopra [4] is suggested, which penalizes intermediate densities through a factor that does not affect the final behaviour of the structure.

$$F = \sum_{e=1} \int_{V_e} (\rho_e)^{1/p} \, dV \tag{14}$$

where ρ_e is the relative density of a certain element and p is the penalization factor.

3.2 Stress constraints

The stress constraints at the centroid of each element must follow specific criteria during their definition. In the current structural calculation model, there are six stress values calculated at each Gaussian integration point: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$. These stresses form the stress tensor of the structure. The constraint will be implemented through an equivalent stress criterion. Since the material in the proposed formulation is steel, the stress used will be the Von Mises equivalent stress:

$$\sigma_{vm} = \sqrt{\frac{1}{2}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]} \tag{15}$$

In this way, we can define the stress constraints as:

$$g_e(\boldsymbol{\rho}) = \sigma_{vm,e}\left(\boldsymbol{\sigma}^h(\boldsymbol{u}_e^o, \rho)\right) - \sigma_{vm,\max} \leq 0 \tag{16}$$

where $g_e(\boldsymbol{\rho})$ is the constraint, $\sigma_{vm,e}$ is the Von Mises equivalent stress at the centroid of the element and $\sigma_{vm,\max}$ is the limit of the constraint.

3.3 Buckling load factor constraint

The linearized buckling constraint is a global constraint, meaning that it is obtained collectively for the entire structure through the model presented in this work. Thus, through the previously presented eigenvalue problem, it is possible to determine the linearized buckling factor λ, which is used in the constraint, resulting in the following form:

$$h_{\text{buck}}(\rho) = \lambda_{\text{lim}} - \lambda(\boldsymbol{u}, \boldsymbol{\rho}) \leq 0 \tag{17}$$

where λ_{lim} is the limit imposed on the first factor of linearized buckling.

3.4 Natural frequencies constraints

Similar to the case of linearized buckling, this constraint is also global in nature and is calculated through the model proposed and validated in this work. While, in the case of the buckling factor, there was an interest in restricting the value from below, this is not the case here, since the limit of the first natural frequency of vibration can depend on multiple factors, ranging from the structural objective to the dynamic loads it must withstand throughout its service life. In some cases, these limits on natural frequencies may be imposed by the relevant regulations in each structural domain. In this way, we obtain two natural frequency constraints that can be used separately or together, depending on the problem being posed.

$$h_{\text{inf,frec}}(\rho) = \lambda_{\text{lim}}^{\text{inf}} - \lambda(\boldsymbol{u}, \boldsymbol{\rho}) \leq 0 \tag{18}$$

$$h_{\text{sup,frec}}(\rho) = \lambda(\boldsymbol{u}, \boldsymbol{\rho}) - \lambda_{\text{lim}}^{\text{sup}} \leq 0 \tag{19}$$

where $h_{\text{inf,frec}}$ is the lower limit imposed on the first natural frequency, and $h_{\text{sup,frec}}$is the upper limit.

4 SENSITIVITY ANALYSIS

The optimization process presented in previous sections, in most cases and depending on the resolution algorithm used, requires the computation of the derivatives of the constraints and the objective function to be implemented. Thus, its proper functioning depends on the accuracy of these derivative calculations and the quality of the data provided in this regard. This process of computing derivatives with respect to the design variables in an optimization problem is known as sensitivity analysis and is one of the key components of the problem. It consumes a very high percentage of the computation time and, in turn, significantly influences our ability to reach the global optimum of the problem. In our case, we will present the process for obtaining the first-order derivatives of the functions involved in the problem so that they can be incorporated into the formulation.

4.1 Sensitivity of the objective function

Recalling the previously defined objective function, it is straightforward to obtain that its sensitivity with respect to the relative density of any element is:

$$\frac{dF}{d\rho_e} = \int_{V_e} \frac{1}{p} \rho_e^{\left(\frac{1}{p}-1\right)} dV \tag{20}$$

4.2 Stress sensitivity

Since we have defined the stress to be used in the optimization problem as the von Mises comparison stress, we need its derivative. We will express it using the chain rule as follows:

$$\frac{d\boldsymbol{t}}{d\boldsymbol{a}} = \frac{d\sigma_{vm}}{d\boldsymbol{\sigma}}\bigg|_{\boldsymbol{\sigma}(\boldsymbol{a})} \frac{d\boldsymbol{\sigma}}{d\boldsymbol{a}} \frac{d\boldsymbol{a}}{d\rho_i} + \frac{d\boldsymbol{t}(\rho_i)}{d\rho_i} \tag{21}$$

where $\boldsymbol{t}$ is the stress constraint. The last term must be zero in our case because there is not direct dependency between the constraint and the relative density. First, we can define the sensitivity of the Von Mises stress with respect to the stress tensor as:

$$\frac{d\sigma_{vm}}{d\boldsymbol{\sigma}} = \left[\frac{\partial\sigma_{vm}}{\partial\sigma_x} \quad \frac{\partial\sigma_{vm}}{\partial\sigma_y} \quad \frac{\partial\sigma_{vm}}{\partial\tau_{xy}}\right] \tag{22}$$

where:

$$\frac{\partial\sigma_{vm}}{\partial\sigma_x} = \frac{2\sigma_x - \sigma_y}{2\sigma_{vm}}; \frac{\partial\sigma_{vm}}{\partial\sigma_y} = \frac{2\sigma_y - \sigma_x}{2\sigma_{vm}}; \frac{\partial\sigma_{vm}}{\partial\tau_{xy}} = \frac{6\tau_{xy}}{2\sigma_{vm}} \tag{23}$$

In the formulation of the finite element method, the stress tensor of a continuous medium is calculated as:

$$\boldsymbol{\sigma}(\boldsymbol{r}) = \boldsymbol{DBu}(\boldsymbol{r}) \tag{24}$$

where $\boldsymbol{u}(\boldsymbol{r})$ are the displacements at point $\boldsymbol{r}$. Applying the discretization, we can also define the tensor as:

$$\boldsymbol{\sigma}(\boldsymbol{r}) = \sum_{i=1}^{n} \boldsymbol{DB}\, \boldsymbol{H}_i(\boldsymbol{r})\boldsymbol{a_i} \tag{25}$$

Thus, we can formulate the derivatives of the stress tensor with respect to the nodal displacements of the structure as:

$$\frac{d\boldsymbol{\sigma}}{da_i} = \boldsymbol{DBH}_i(\boldsymbol{r})\boldsymbol{\delta}_i \tag{26}$$

where $\boldsymbol{D}$ is the constitutive matrix, $\boldsymbol{B}$ is the matrix of shape function derivatives, and $\boldsymbol{H}$ is the shape function matrix.

In order to obtain the last term, we can start from eqn (8). Differentiating this expression, we obtain:

$$\boldsymbol{K}\frac{d\boldsymbol{a}}{d\rho_i} = \frac{d\boldsymbol{f}}{d\rho_i} - \frac{d\boldsymbol{K}}{d\rho_i}\boldsymbol{a} \tag{27}$$

from which, by solving, we can obtain the desired term:

$$\frac{da}{d\rho_i} = \boldsymbol{K}^{-1}\left[\frac{d\boldsymbol{f}}{d\rho_i} - \frac{d\boldsymbol{K}}{d\rho_i}\boldsymbol{a}\right] \tag{28}$$

The full development of this formulation can be found in París et al. [8].

4.3 Sensitivity of the natural vibrations

We will start from the development defined in the previous section to obtain the sensitivity of the first natural vibration frequency of the structure. The sensitivity formulation of the eigenvalue problems has been obtained following Hernández [9]. To obtain the natural frequencies, it was necessary to solve the eigenvalue problem defined previously:

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{\phi} = 0 \tag{29}$$

However, our intention was not to obtain all the natural vibration frequencies but rather the smallest one in absolute value. Therefore, we reformulated the problem by $\nu_i = \frac{1}{\omega_i{}^2}$, leading to:

$$(\boldsymbol{M} - \nu_i\boldsymbol{K})\boldsymbol{\phi_i} = 0 \tag{30}$$

For simplicity, we are going to omit the subscript i from the natural frequency and instead use it in reference to the design variables. By differentiating the previous expression, we obtain the following development:

$$\boldsymbol{\phi}^{\boldsymbol{t}}\left[\frac{\partial \boldsymbol{M}}{\partial \rho_i} - \frac{\partial \nu}{\partial \rho_i}\boldsymbol{K} - \nu\frac{\partial \boldsymbol{K}}{\partial \rho_i}\right]\boldsymbol{\phi} = 0 \tag{31}$$

Thus, we obtain the expression that gives us the sensitivity of the natural vibration frequency:

$$\frac{\partial \nu}{\partial \rho_i} = \boldsymbol{\phi}^{\boldsymbol{t}}\left[\frac{\partial \boldsymbol{M}}{\partial \rho_i} - \nu\frac{\partial \boldsymbol{K}}{\partial \rho_i}\right]\boldsymbol{\phi} \tag{32}$$

However, we cannot forget about the initial change made, nor that we are looking for the natural dimensional frequency, in Hz, so we need to make the following changes:

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{\nu}} \quad \rightarrow \quad \frac{\partial f}{\partial \rho_i} = -\frac{1}{4\pi}\nu^{-\frac{3}{2}}\frac{\partial \nu}{\partial \rho_i} \tag{33}$$

4.4 Sensitivity of the buckling load factor

The formulation of buckling load factor is very similar in its development to that of frequency calculation, with the exception of the subsequent treatment of the derivative of the matrices. Thus, we start from the development defined in previous sections:

$$[\boldsymbol{K} + \lambda_i \boldsymbol{K_G}]\boldsymbol{\phi}_i = 0 \tag{34}$$

However, just as in the case of frequencies, we need to obtain the minimum factor in absolute value, so the actual development will be through the inverse Mises method by defining the variable $\nu_i = \frac{1}{\lambda}$

$$[\boldsymbol{K_G} + \nu_i \boldsymbol{K}]\boldsymbol{\phi}_i = 0 \tag{35}$$

In this case, we will also omit the subscript referring to the buckling mode and will use it in reference to the design variables. Differentiating, we obtain:

$$\boldsymbol{\phi}^t \left[\frac{\partial \boldsymbol{K_G}}{\partial \rho_i} + \frac{\partial \nu}{\partial \rho_i}\boldsymbol{K} + \nu\frac{\partial \boldsymbol{K}}{\partial \rho_i}\right]\boldsymbol{\phi} = 0 \tag{36}$$

Thus, we obtain the expression that gives us the sensitivity of the first inverse linearized buckling factor:

$$\frac{\partial \nu}{\partial \rho_i} = -\boldsymbol{\phi}^t \left[\frac{\partial \boldsymbol{K_G}}{\partial \rho_i} + \nu\frac{\partial \boldsymbol{K}}{\partial \rho_i}\right]\boldsymbol{\phi} \tag{37}$$

However, we must not forget that we are seeking the derivative of the linearized buckling factor with the eigenvalue problem, therefore

$$\frac{\partial \lambda}{\partial \rho_i} = \frac{\partial \lambda}{\partial \nu}\frac{\partial \nu}{\partial \rho_i} = \frac{1}{\nu^2}\boldsymbol{\phi}^t \left[\frac{\partial \boldsymbol{K_G}}{\partial \rho_i} + \nu\frac{\partial \boldsymbol{K}}{\partial \rho_i}\right]\boldsymbol{\phi} \tag{38}$$

5 BENCHMARK CASES AND IMPLEMENTATION

Once the formulation has been presented, we will develop a series of benchmark cases to observe compliance with the constraints and the weight reduction of two structures. The ARPACK library [10] has been used to obtain the eigenvalues associated with each problem. For the process of obtaining the structural responses, proprietary software written in FORTRAN has been used, as well as for solving the optimization process, where sequential linear programming has been applied. To eliminate stress singularities and spurious buckling modes, the stress relaxation formulation [8], [11] has been implemented.

We will test two benchmark cases, both related to the reduction of the steel structure's weight but using a compression column to constrain the buckling factor and an MBB beam to impose frequency constraints. We have run the benchmark cases with frequency constraints of 140/150 Hz and buckling factor constraints of 40/50. In Figs 1–4, the evolution of the structure's weight and the relevant constraints can be observed. The computation time

is significantly higher (two orders of magnitude greater) in the case of the compression column, as the sensitivity of the buckling factor is computationally much more expensive than the sensitivity of the frequencies.

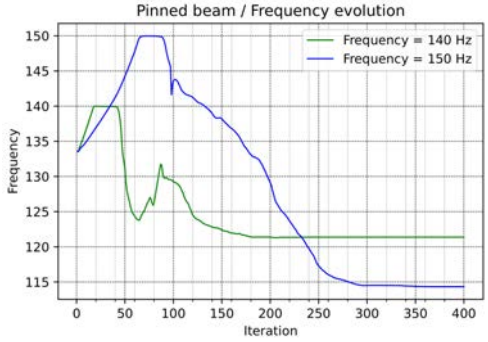

Figure 1: Beam frequency evolution.

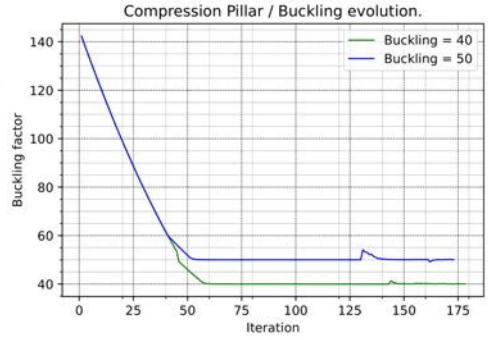

Figure 2: Pillar buckling evolution.

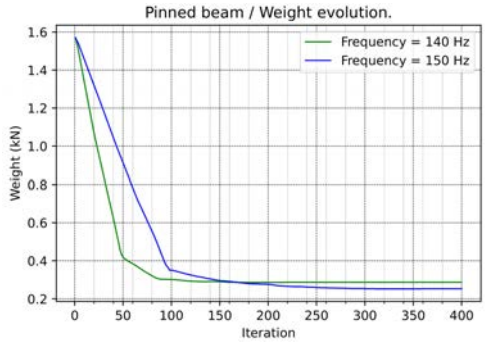

Figure 3: Beam weight evolution.

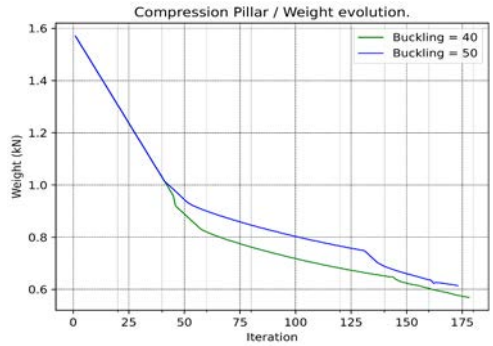

Figure 4: Pillar weight evolution.

6 CONCLUSIONS

This work presents the formulation required to carry out a topology optimization process using the SIMP method, starting from the linear and nonlinear structural response of the structure to the sensitivity of all functions involved in the problem. A general optimization formulation has been obtained, combining local and global constraints related to both stress and energy and dynamic instability. The work has focused solely on the physical problem, not on the numerical issues arising from the implementation of this type of formulation. The dynamic and buckling analysis is obtained through eigenvalue problems, which have also been studied to analytically determine their sensitivity.

Future research directions should consider the combined behaviour of buckling and natural vibration frequencies, as well as their mutual couplings, alongside improvements in computational efficiency. Additionally, a deeper study of existing works on numerical instabilities and spurious buckling modes is necessary. Furthermore, the integration of

material nonlinearities should be addressed to ensure the proper performance of the structure, especially if geometric nonlinearity is to be incorporated during the optimization process.

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