OPTIMIZING INVERSE KINEMATICS IN THE SCREW THEORY FRAMEWORK WITH QUANTUM HYBRID GENETIC ALGORITHM

SORAYA ZENHARI
Institute for Machine Tools, Stuttgart, Germany

ABSTRACT
On five axis machines, it is necessary to compensate for position independent geometric errors (and position dependent geometric errors) to enhance machining precision. The purpose of this paper is to present an inverse kinematic model which provides clear solutions when executing motion commands according to an inverse kinematic model. A quantum hybrid genetic algorithm is used to determine the parameters of the kinematic model to achieve the most optimal performance possible. A different simulation result validated the proposed approach. This results in an 80% improvement in accuracy compared to conventional genetic algorithms with similar parameters, along with faster convergence. The model is versatile and can be used in a wide range of machine tools, especially those with rotary axes that are not orthogonal. The proposed algorithm can easily be adapted for use in computer-aided design and computer-aided manufacturing systems, depending on the machine's maximum range of motion.
Keywords: screw theory, five axis machine tool, hybrid genetic algorithm, error compensation.

1 INTRODUCTION

The increasing demand for precision manufacturing necessitates advanced machining technologies, especially in industries like aviation, aerospace, and automotive. At the same time, the machine tool industry is focused on improving performance, reducing costs, and minimizing energy consumption to meet the needs of today's economy [1]. Machine tools are prone to geometric errors, which can adversely affect the quality of manufactured components [2]. Among these, position-independent geometric errors (PIGEs) and position-dependent geometric errors (PDGEs) present significant challenges. These errors can arise from imperfections in machine components, thermal deformations, or inaccuracies in the machine's kinematic structure. Compensating for these errors is paramount to ensuring the reliability, quality, and consistency of machined parts. Advanced methods to detect, model, and correct these errors not only improve the precision of the machine tool but also extend its operational efficiency and versatility. To address this challenge, this paper presents an approach that combines screw theory with a quantum hybrid genetic algorithm. The approach is evaluated on real machine data and achieves favourable results. The paper is structured as follows: first, the current state of the art is presented; then, screw theory and the quantum hybrid genetic algorithm are described. Section 5 describes the experiment carried out; Section 6 shows the application of the approach in the form of a simulation. Section 7 then shows the postprocessor. The paper ends with a summary and future perspectives.

2 BACKGROUND AND STATE OF THE ART

In recent years, computer numerical control (CNC) machine tools have become more accurate due to the rapid development of manufacturing engineering. These machines are capable of working quickly and accurately, reducing the need for manual labour and resulting in faster production times. Error compensation allows CNC machines to produce parts with higher precision, reducing waste and improving product quality. It also minimizes the need

WIT Transactions on The Built Environment, Vol 216, © 2025 WIT Press
www.witpress.com, ISSN 1743-3509 (on-line)
doi:10.2495/HPSM250111

for costly rework and increases overall efficiency in the manufacturing process. By enhancing the accuracy of machine operations, companies can achieve more consistent and reliable outputs.

Many researchers have explored various approaches for compensating geometric errors. Using genetic algorithm, Creamer et al. [3] optimized machine tools for effective error compensation. According to Kvrgic et al. [4], geometric errors can be determined in rotational shafts and inverse kinematics in nonorthogonal five axis machines can be analysed. Using a particle swarm algorithm, Chen et al. [5] created a model for compensating geometric errors. A chicken swarm optimization method has been introduced by Fu et al. [6] for optimizing tool directions. The accurate machining of workpieces and cutters requires the use of NC codes generated from cutter location (CL) data. Additionally, each CNC machine requires a specific post-processor.

Based on force-induced, position-dependent errors, Zhang et al. [7] proposed a kinematic chain optimization method for five axis machine tools. By using the homogeneous transformation matrix (HTM) method, they develop a mathematical model to analyse how local deformations affect machining errors. In this process, frequent motion positions are planned, and sensitivity analyses are conducted using static finite element analysis. An alternative kinematic chain's sum and variance indices are used to perform optimization. In comparison with conventional approaches, the method is validated with a five axis vertical machining centre demonstrating a 71% reduction in variance.

By using a postprocessing program in MATLAB, Maletić et al. [8] developed a method for testing a machine's kinematic model. In this study, they presented an inverse kinematic model consisting of five equations and then translated the CL data into a G-code program for five axis machining. Further, they developed software for developing kinematic equations for a multi-axis post-processor and implemented a mathematical model using elementary transformational matrices. Post-processing software successfully converted CL data into G-code for the machining of concave workpiece [9].

Using an iterative control vector parametrization method, Ma et al. [10] recommend optimal federate planning for five axis CNC machines with geometric and kinematic constraints like velocity, acceleration, and jerk. An analysis based on CAD-generated virtual models of the main components of an eight-axis machine tool was conducted by Ullah et al. [11]. This study focused on optimizing the design for better machine performance and structural integrity. The lattice optimization resulted in a 64% reduction in energy consumption. A solution algorithm based on least squares and Particle Swarm Optimization PSO was proposed by Yan et al. [12], considering the error quantity function of each tool position. Zhang et al. [13] focused on determining and compensating for geometric errors in five axis machine tools to improve the quality of the products they produced. Inverse kinematics is used to calculate errors, and volumetric. In another study, Gao et al. [14] proposed an optimization method for the setup of workpieces, which took into account the need to avoid singularities and to ensure that the five axis system was stable kinematically. The solutions to ill-conditioning in the Jacobian matrix of five axis inverse kinematics were discussed by Lu et al. [15]. As part of another study, Dening et al. [16] used the Jacobian condition number to limit tool orientation to prevent singularities.

The combination of geometric error compensation and optimization processes has been studied relatively little. By applying improved genetic algorithms, Wei et al. [17] developed a novel optimization method for obtaining the NC code for geometric error compensation, which used the complete position deviation as the fitness function.

A quantum hybrid genetic algorithm (HQGA) can provide a promising solution for CNC machine tool error compensation and optimization due to its ability to combine quantum computing with classical genetic algorithms, thereby increasing the efficiency and capability of optimization. Hybrid methods excel at solving complex, NP-hard combinatorial problems typically encountered in manufacturing, resulting in improved accuracy and adaptability in dynamic machining environments. With HQGA, unlike traditional genetic algorithms which may become trapped in local optima, and pure quantum methods that are limited by current hardware constraints, HQGA provides an effective framework for addressing the high-dimensional, nonlinear challenges associated with CNC machining, making it the optimal choice for modern manufacturing operations.

3 SCREW THEORY

Screw theory models the relative position and orientation of each axis in a five axis machine tool, offering a clear understanding of its three-dimensional motion. By incorporating both rotational and translational movements, it provides a mathematical framework to describe the relative motion between the tool and the workpiece, enabling a more intuitive approach to kinematic analysis. To enhance the accuracy and reliability of the machine tool, error models must be applied to calculate discrepancies between the tool tip and the workpiece. An accurate representation of a five axis machine tool requires consideration of both the workpiece and the toolchain. Assuming that the machine tool moves by a certain amount (x, y, z, θ_A, θ_C), then the updated coordinates of the workpiece and the tool tip in the machine coordinate system (MCS) can be assessed using screw motions $e^{\hat{\xi}\cdot\theta}$ with twists ξ, described as [18]:

$$g_{bw} = e^{\hat{\xi}_X\cdot(-x)}.\, e^{\hat{\xi}_A\cdot(-\theta_A)}.\, e^{\hat{\xi}_C\cdot(-\theta_C)}.\, g_{bw}(0) \tag{1}$$

$$g_{bt} = e^{\hat{\xi}_Y\cdot(y)}.\, e^{\hat{\xi}_Z\cdot(z)}.\, g_{bt}(0) \tag{2}$$

where $g_{bw}(0)$ and $g_{bt}(0)$ are initial 4×4 transformation matrices of workpiece and tool tip relative to measuring coordinate system. Regarding the forward kinematics model, five axis machine tool was utilized for the experiment, its transformation matrix can be expressed as follows:

$$g_{bw}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -z0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$g_{wt} = g_{wb}.g_{bt} = (g_{bw})^{-1}.\, g_{bt} = g_{bw}^{-1}(0).\, e^{\hat{\xi}_C\cdot(\theta_C)}.\, e^{\hat{\xi}_A\cdot(\theta_A)}.\, e^{\hat{\xi}_X\cdot(x)} e^{\hat{\xi}_Y\cdot(y)}.\, e^{\hat{\xi}_Z\cdot(z)}.\, g_{bt}(0) \tag{4}$$

These relations can be used to drive mathematic equations for inverse kinematics. In the rotary axes, screw motion is defined as follows [19]:

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}, v = q \times \omega \tag{5}$$

where ω is the unit vector in the positive direction of the rotary axis, and q is any point on the axis expressed in the machine coordinate system (MCS). In the rotary axes, screw motion is defined as follows

$$e^{\hat{\xi}\cdot\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I_{3\times3} - e^{\hat{\omega}\theta}\right)(\omega \times v) + \omega\omega^{T} v\theta \\ 0_{1\times3} & 1 \end{bmatrix} \tag{6}$$

where θ is the angular rotation of the axis and $e^{\widehat{\omega}\theta}$ can be expanded as:

$$e^{\widehat{\omega}\theta} = I_{3\times3} + \widehat{\omega}.\sin\theta + \widehat{\omega}^2.(1-\cos\theta) \tag{7}$$

Error twists are used to model the geometric errors between the tool tip and workpiece and will be applied as q_d and ω_d into the equations. The ideal path that the machine should trace will differ slightly from the actual path, as shown in Fig. 1.

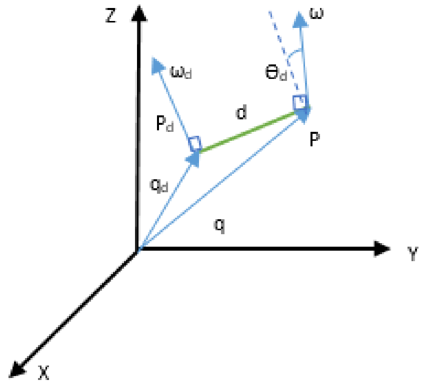


Figure 1: Error twist.

These deviations can lead to inefficiencies that could compromise the quality of the final product or process. As mentioned before the geometric errors of a machine tool can be categorized PDGEs and PIGEs. The five axis machine tool under study suffers from 41 errors. These errors can be detected using metrology techniques and devices such as laser interferometers and double ball bar devices [19]. To address these errors, several measurements were performed on the machine's structure to accurately identify and quantify them. Considering eqns (5)–(7), incorporating the error into the equations for a revolute joint, we can rewrite the equations as follows [19]:

$$\omega_e = \frac{\omega_d \times \omega}{\sin\theta_e}, h_e = \frac{d}{A_e} = \frac{|q-q_d|}{\theta_e} \tag{8}$$

$$v_e = \frac{q\times(\omega_d \times \omega)}{\sin\theta_e} + h_e \frac{\omega_d \times \omega}{\sin\theta_e} = \frac{q_d \times q}{d} + \frac{q-q_d}{\theta_e} \tag{9}$$

and for prismatic joints, where ω = 0:

$$\xi_e = \begin{bmatrix} \frac{q-q_d}{d} \\ 0 \end{bmatrix} \tag{10}$$

To correct geometric errors, the axis errors are measured and modelled as functions of position in the machine coordinate system. These errors are then translated to the tool's orientation and tip position using the forward kinematic model of the machine in the part coordinate system. This step considers the impact of axis errors on the tool's final position and orientation during machining. To address these errors, the tool position and orientation errors are converted into corrections for the machine's drive components using the inverse kinematic model in the part coordinate system. This ensures that the tool's movement is adjusted to counteract the measured errors, thereby enhancing machining accuracy.

4 QUANTUM HYBRID GENETIC ALGORITHM

The Genetic Algorithm (GA) is a metaheuristic inspired by natural selection, first introduced by John Holland and later expanded by Goldberg [20]. It employs evolutionary principles such as selection, mutation, and crossover to solve optimization and search problems. GAs are widely used in machine optimization to reduce geometric errors by optimizing machine tool movements, thereby increasing precision and product quality [21]. The algorithm begins with a population of potential solutions, referred to as chromosomes, which are evaluated based on a fitness function reflecting their performance relative to the optimization objective. The process iteratively selects the best solutions to form new generations, gradually improving the population until an optimal solution is found [22].

In the basic GA, the process begins by generating an initial population with diverse solutions. Each chromosome is evaluated using a fitness function, and the fittest individuals are selected as parents. Crossover combines genetic information from two parents to create offspring, while mutation introduces random changes to maintain diversity and prevent premature convergence. These steps population generation, fitness evaluation, selection, crossover, and mutation are repeated until the termination criteria are met, such as achieving a target fitness or reaching a set number of generations. While the basic GA is effective for many optimization tasks, it can struggle with premature convergence or fail to find globally optimal solutions in complex problems. To address these challenges, the Quantum Hybrid Genetic Algorithm (QHGA) incorporates two additional steps: The Newton–Raphson optimization and a quantum-inspired crossover phase, providing enhanced performance and robustness. The QHGA process begins like the basic GA, with the generation of an initial population, fitness evaluation, and parent selection. However, before performing the crossover phase, the algorithm applies the Newton–Raphson optimization method to the top-performing chromosomes. This gradient-based method refines the parameters of selected chromosomes by leveraging the cost function's gradient, enabling more accurate solutions and faster convergence. After this refinement, the crossover phase proceeds as in the basic GA, creating offspring by combining the genetic material of two selected parents. An additional innovation in the QHGA is the introduction of the quantum-inspired crossover phase, which addresses the issue of stagnation. If no improvement is observed in the fitness value according to threshold number, the algorithm detects stagnation and triggers the quantum-inspired crossover step. This phase introduces novel genetic combinations by leveraging probabilistic techniques inspired by quantum mechanics, helping the population escape local optima and continue improving. This mechanism significantly enhances the algorithm's ability to explore diverse regions of the solution space. In quantum-inspired crossover, a matrix called R is used to generate a child from two parents. This matrix will be added into the calculation as follows:

$$x_{\text{Child}} = R\,\frac{x_1+x_2}{2} \tag{11}$$

where $x_1$and x_2 were considered as selected parents and R is the proposed matrix. It is worth mentioning that R plays an essential role in introducing diversity and effectively exploring the solution space. Resulting in improved performance in trajectory optimization.

Finally, mutation is applied to introduce random alterations in the offspring, ensuring the population maintains genetic diversity and adaptability. The process continues iteratively, repeating the steps of Newton–Raphson optimization, crossover, quantum-inspired crossover (if necessary), and mutation until the termination criteria are met. By combining the Genetic Algorithm (GA) with the Newton–Raphson method and quantum-inspired crossover, the QHGA achieves convergence, prevents premature stagnation, and enhances solution diversity. This hybrid approach makes the QHGA particularly effective for solving complex optimization problems that require both precision and adaptability. The proposed algorithm is illustrated in Fig. 2.

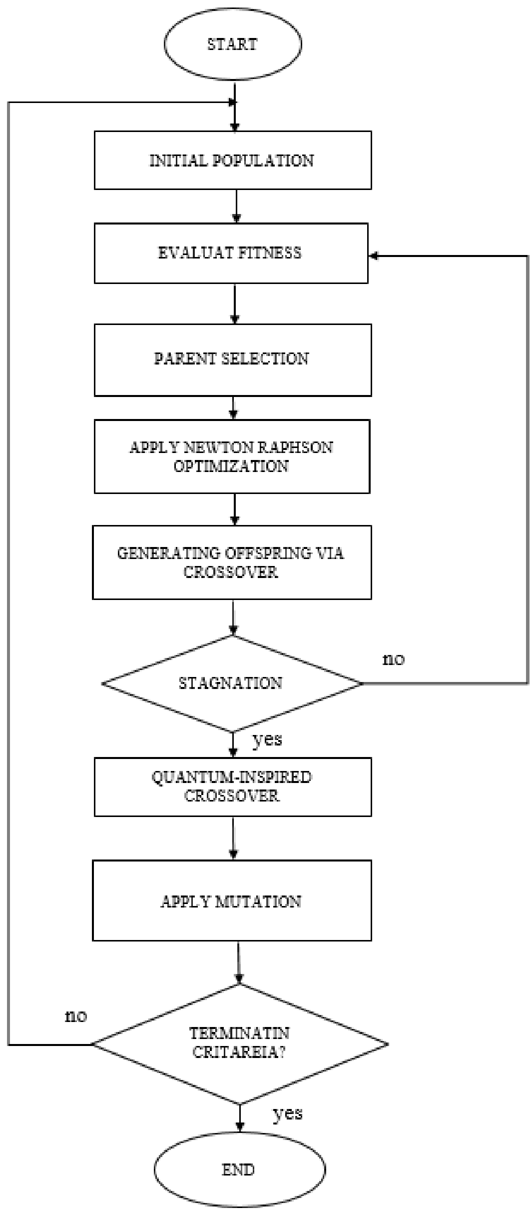


Figure 2: Proposed algorithm flow chart.

5 EXPERIMENT

The kinematic model of a five-axis machine tool consists of a combination of prismatic and revolute joints, enabling precise and flexible machining operations. Typically, such machines feature separate kinematic chains for the tool and workpiece, allowing for multi-axis motion. The tool's movement is often governed by a set of prismatic joints, while the workpiece is manipulated using a combination of prismatic and rotational joints. This structure provides extensive motion capabilities necessary for complex machining tasks. To analyse the inherent accuracy of multi-axis machines and quantify geometric errors, high-precision measurement devices such as laser trackers and double ball bar systems are commonly used. These tools facilitate the evaluation of linear and circular positioning accuracy by performing systematic tests across multiple planes. Circular tests typically assess motion precision in various planes, while linear measurements capture positional deviations along individual axes at different feed rates. By integrating these measurement techniques, it is possible to quantify axis-specific errors and develop compensation strategies to enhance machine accuracy. Geometric errors in multi-axis machines can be systematically analysed and modelled using screw theory. This framework represents both translational and rotational deviations as unified error twists, which integrate linear and angular errors into a single mathematical representation (see eqns (8)–(10)). By expressing these errors in a global coordinate system, screw theory eliminates the need for multiple local reference frames, simplifying error modelling and compensation. Each axis error can be expressed in terms of an exponential screw displacement, allowing for efficient computation of position and orientation deviations (see eqn (8)). The total error in tool positioning can be determined by summing the contributions of individual error components, including positional deviations along linear axes and angular misalignments in rotary joints. Using the product of exponentials (PoE) method, the overall transformation incorporating these errors is obtained, providing a structured approach to error compensation. Additionally, screw theory facilitates the efficient mapping of individual error components, such as those listed in Table 1 (e.g., δ_{xx}, δ_{yy} for translational axes and δ_{aa}, δ_{cc} for rotary axes), to the overall volumetric error. This enables a clear understanding of how specific errors impact the machine's accuracy. By systematically quantifying these deviations, error compensation strategies can be implemented to refine machine performance.

Table 1: Geometric errors (PDGEs and PIGEs) [19].

X-axis	Y-axis	Z-axis	Squareness error	A-axis	C-axis	PIGES of R rotary axes
δ_{xx}	δ_{yy}	δ_{zz}	S_{xy}	δ_{xa}	δ_{xc}	δ_{xoc} S_{boa}
δ_{yx}	δ_{xy}	δ_{xz}	S_{yz}	δ_{ya}	δ_{yc}	δ_{yoc} S_{coa}
δ_{zx}	δ_{zy}	δ_{yz}	S_{xz}	δ_{za}	δ_{zc}	δ_{zoa} S_{aoc}
ε_{xx}	ε_{xy}	ε_{xz}		ε_{xa}	ε_{xc}	δ_{zoa} S_{boc} (ISO 230-7)
ε_{yx}	ε_{yy}	ε_{yz}		ε_{ya}	ε_{yc}	
ε_{zx}	ε_{zy}	ε_{zz}		ε_{za}	ε_{zc}	

Once errors are measured and modelled as functions of position within the machine's coordinate system, a correction algorithm can be implemented to adjust machine movements accordingly. By continuously monitoring and compensating for deviations, this approach improves machining accuracy and consistency, ultimately enhancing the precision and reliability of multi-axis manufacturing processes.

6 SIMULATION

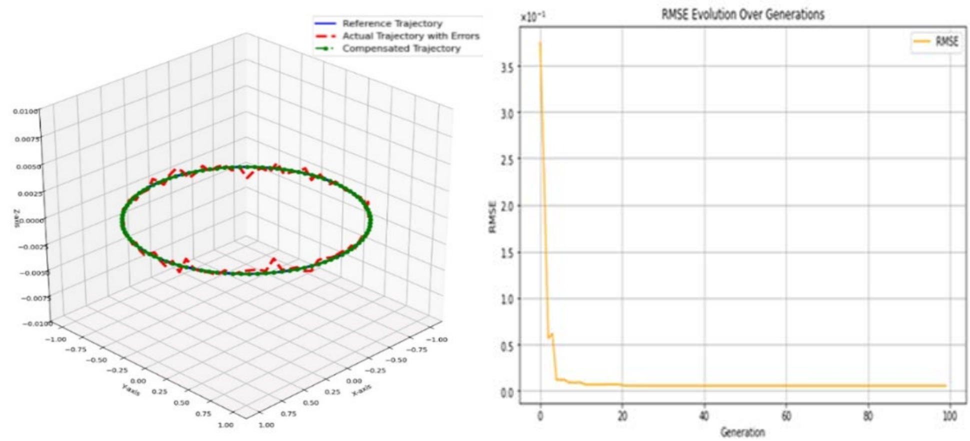
Further investigation is required to evaluate the efficiency of the proposed algorithm. The trajectory of the tool in the work piece's local frame is considered as follows:

$$x_{ref} = \sin(t) \tag{12}$$

$$y_{ref} = \cos(t) \tag{13}$$

$$z_{ref} = 0 \tag{14}$$

The objective is to find an optimal parameter set $[x_1, x_2, x_3, \theta_4, \theta_5]$ that minimize the cost function. According to proposed algorithm (see Fig. 3) after iteration 20 the results convergence to optimal values, considering home position of the machine tool optimized parameters for a machine tool's positioning can be found as: $x_1 = 4.07$ μm, $x_2 = -1.32$ μm, $x_3 = 5.62$ μm.



(a) (b)

Figure 3: (a) Reference and actual paths; and (b) RMSE error of the proposed algorithm.

In CNC machining, even small misalignments between the tool and the workpiece can lead to poor surface finishes, dimensional errors, or incorrect part geometries. To maintain high part quality, precise adjustments to the tool's position are necessary. The transformation matrix plays a crucial role in modelling and correcting these displacements by incorporating both translation and rotational components for accurate machining. Operators adjust the matrix parameters to compensate for machine errors. These transformations, typically expressed through rotation matrices and translation vectors, account for both linear and angular movements. The matrix is derived by identifying optimal parameters through empirical measurement, calibration, and simulation, using methods like matrix exponentiation and Euler angles. These adjustments ensure the tool's position is precise during operations. The three key parameters, expressed in micrometres, guide the adjustments required for optimal machine performance. Based on these, the transformation matrix for the tool relative to the workpiece is computed as follows [23], [24]:

$$g_{bt} = e^{\xi_z x1} e^{\xi_y x2} g_{bt0} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 0 & -1 & -x_2 \\ 0 & 1 & 0 & x_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$g_{bw} = e^{\xi_1 x3} e^{\xi_{\theta_y} \theta_4} e^{\xi_{\theta_z} \theta_5} g_{bw0}$$

$$= \begin{bmatrix} \cos(\theta_4)\cos(\theta_5) & -\cos(\theta_4)\sin(\theta_5) & -\sin(\theta_4) & 0 \\ \sin(\theta_5) & \cos(\theta_5) & 0 & -y_0 \\ \sin(\theta_4)\cos(\theta_5) & -\sin(\theta_4)\sin(\theta_5) & \cos(\theta_4) & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

Fig. 3(a) shows the ideal path, the actual path, and the path with compensation and (b) illustrating the algorithm's efficiency per iteration. The algorithm's efficiency per iteration demonstrates a significant improvement, indicating that the compensation method effectively enhances the overall performance. Consequently, the proposed approach not only refines the path but also optimizes the computational process, ensuring a more efficient and accurate outcome.

7 POST-PROCESSOR

A post-processor was designed based on the concepts to convert data from screw theory (see Fig. 4). The interface is divided into three steps:

1. General: Machine information, such as CAD data provided in the library, is selected in this step.
2. Process and environment: In this step, the volumetric error of the machine tool is uploaded. Based on the forward kinematics framework derived from screw theory, deviations in three dimensions are calculated.
3. Detail design: Users can evaluate and optimize parameters to reduce the corresponding errors. Various optimization methods can be employed in this step. In this paper, we focus on quantum hybrid genetic algorithms, which can provide accurate results.

Based on the designed postprocessor, engineers can easily evaluate the results without delving into complex calculations, allowing them to quickly assess simulation outcomes.

8 CONCLUSION AND OUTLOOK

This research introduces a novel hybrid algorithm for solving inverse kinematics problems, achieving an 80% improvement in proficiency over traditional genetic algorithms with similar parameters. The algorithm effectively addresses stagnation caused by singularities and demonstrates rapid convergence, finding solutions within just a few generations. Moreover, it can determine intermediate configurations for the end effector, enhancing its adaptability in complex scenarios. To improve usability, a user-friendly post-processing step was implemented, making the algorithm more accessible for practical applications. Future work will focus on expanding its adaptability to various machine tools and robotic systems across a broader range of scenarios. Additionally, incorporating Kalman filters to update feedback signals will enable real-time error compensation, enhancing precision during operation. Further enhancements include developing a digital twin architecture, which is expected to improve real-time accuracy and facilitate predictive maintenance, thereby increasing system reliability in dynamic environments. Another key advancement involves

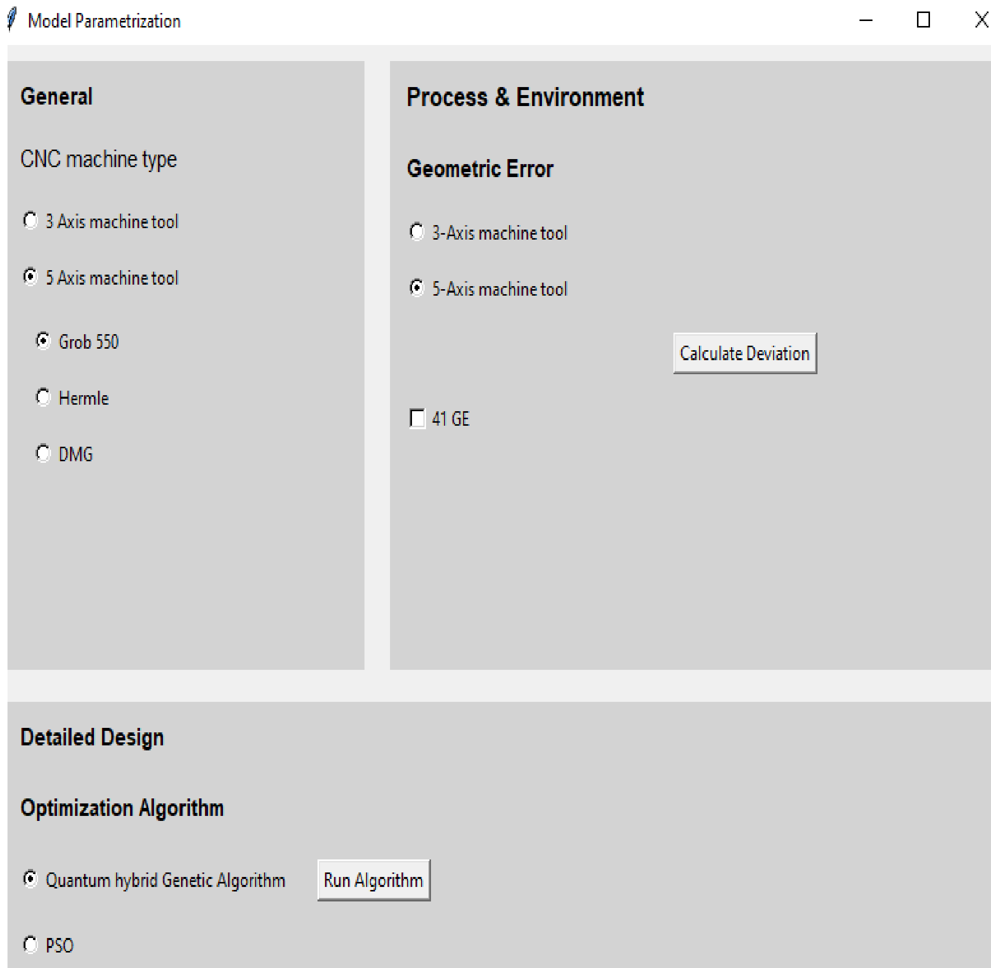


Figure 4: Dialog of the proposed five axis postprocessor.

refining the machine interface's G-code to incorporate error compensation, allowing for more precise control, reduced errors, and increased overall efficiency. Finally, integrating machine learning techniques for continuous optimization could lead to more intelligent and autonomous robotic systems, expanding their applications across industries. These advancements will contribute to greater reliability, adaptability, and efficiency in robotic and CNC-based machining environments.

REFERENCES

[1] Wang, H., Ran, Y., Zhang, S. & Li, Y., Coupling and decoupling measurement method of complete geometric errors for multi-axis machine tools. *Appl. Sci.*, **10**(6), 2020. https://doi.org/10.3390/app10062164.

[2] Zhang, Z., Jiang, F., Luo, M., Wu, B., Zhang, D. & Tang, K., Geometric error measuring, modeling, and compensation for CNC machine tools: A review. *Chinese J. Aeronaut.*, **March**, 2023. https://doi.org/10.1016/j.cja.2023.02.035.

[3] Creamer, J., Bristow, D.A. & Landers, R.G., Selection of limited and constrained compensation tables for five axis machine tools. *Int. J. Adv. Manuf. Technol.*, **92**, pp. 1315–1327, 2017.
[4] Kvrgic, V.M., Ribic, A.I., Dimic, Z., Zivanovic, S.T., Dodevska, Z.A., Equivalent geometric errors of rotary axes and novel algorithm for geometric errors compensation in a nonorthogonal five axis machine tool. *CIRP J. Manuf. Sci. Technol.*, **37**, pp. 477–488, 2022.
[5] Chen, P., Wang, S., Li, B. & Li, F., A novel geometric error compensation method for improving machining accuracy of spiral bevel gear based on inverse kinematic model. *Int. J. Adv. Manuf. Technol.*, **127**(9), pp. 4339–4355, 2023.
[6] Fu, W., Wang, B., Li, X., Liu, L. & Wang, Y., Ascent trajectory optimization for hypersonic vehicle based on improved chicken swarm optimization. *IEEE Access*, **7**, 151836–151850, 2019.
[7] Zhang, S., He, C., Liu, X., Xu, J. & Sun, Y., Kinematic chain optimization design based on deformation sensitivity analysis of a five axis machine tool. *Int. J. Precis. Eng. Manuf.*, **21**(12), pp. 2375–2389, 2020. https://doi.org/10.1007/s12541-020-00421-w.
[8] Maletić, J. & Živanović, S., Verification of inverse kinematic equations for a five axis machine tool with a spindle tilting configuration. *Adv. Technol. Mater.*, **47**(1), pp. 33–38, 2022. https://doi.org/10.24867/atm-2022-1-006.
[9] Maletić, J. & Živanovi, S., A post-processor for the five-axis machine MultiProDesk based on inverse kinematic transformation. *16th Int. Conf. on Accomplishments Mech. Ind. Eng.*, pp. 28–34, 2023.
[10] Ma, H.Y., Yuan, C.M., Shen, L.Y. & Gao, X.S., Optimal feedrate planning on a five axis parametric tool path with global geometric and kinematic constraints. *J. Comput. Des. Eng.*, **9**(6), pp. 2355–2374, 2022. https://doi.org/10.1093/jcde/qwac116.
[11] Ullah, A., Chan, T.-C. & Chang, S.-L., Enhancing five axis machine tool performance through esg-based design optimization. *Int. J. Precis. Eng. Manuf. Technol.*, pp. 1–18, 2024.
[12] Yan, Y., Zhang, L. & Gao, J., Tool path planning for flank milling of non-developable ruled surface based on immune particle swarm optimization algorithm. *Int. J. Adv. Manuf. Technol.*, **115**(4), pp. 1063–1074, 2021.
[13] Zhang, H., Xiang, S., Zhou, T. & Yang, J., Joint sensitivity analysis method of impeller aerodynamic performance to key geometric errors of five axis machine tool. *Eng. Sci. Technol. an Int. J.*, **57**, 101814, 2024. https://doi.org/10.1016/j.jestch.2024.101814.
[14] Gao, S., Zhou, H., Hu, P., Chen, J., Yang, J. & Li, N., A general framework of workpiece setup optimization for the five axis machining. *Int. J. Mach. Tools Manuf.*, **149**, 103508, 2020.
[15] Lu, Z., Huo, G. & Jiang, X., A novel method to minimize the five axis CNC machining error around singular points based on closed-loop inverse kinematics. *Int. J. Adv. Manuf. Technol.*, **128**(5–6), pp. 2237–2249, 2023.
[16] Dening, S., Chao, Z., Jianwei, M., Jinghua, L. & Yuguang, Z., An adaptive singularity avoidance method for five axis toolpath generation with invariant cutter-contact points and bounded tool orientation variation based on the reduction of Jacobian condition number. *J. Manuf. Process.*, **101**, pp. 1521–1537, 2023.
[17] Wei, J., Jiang, L. & Liang, B., Geometric error compensation method of five axis CNC machine tool based on improved genetic algorithm. *Journal of Physics: Conference Series*, 12029, 2024. https://doi.org/10.1088/1742-6596/2825/1/012029.

[18] Murray, R.M., Li, Z. & Sastry, S.S., *A Mathematical Introduction to Robotic Manipulation*, CRC Press, 2017.
[19] Xiang, S. & Altintas, Y., Modeling and compensation of volumetric errors for five axis machine tools. *International Journal of Machine Tools and Manufacture*, **101**, pp. 65–78, 2016.
[20] Goldberg, D.E. & Richardson, J., Genetic algorithms with sharing for multimodal function optimization. *Genetic Algorithms and Their Applications: Proceedings of the Second International Conference on Genetic Algorithms*, **4149**, pp. 414–425, 1987.
[21] Holland, J.H., Genetic algorithms. *Scientific American*, **267**(1), pp. 66–73, 1992.
[22] Ahmad, M., Kumar, N. & Kumari, R., A hybrid genetic algorithm approach to solve inverse kinematics of a mechanical manipulator. *International Journal of Scientific and Technology Research*, **8**(9), pp. 1777–1782, 2019.
[23] Spaney, P. et al., A model-driven digital twin for manufacturing process adaptation. *Proceedings of the 2023 ACM/IEEE International Conference on Model Driven Engineering Languages and Systems Companion (MODELS-C)*, pp. 465–469, 2023.
[24] Zenhari, S., Möhring, H.C. & Torbati, A.V., Comprehensive analysis of kinematic models based on the DH method and screw theory for a five axis machine tool. *Procedia CIRP*, **130**, pp. 1745–1751, 2024.