Parallel solution of contact shape optimization problems with Coulomb friction based on domain decomposition

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Abstract

We shall first briefly review the FETI based domain decomposition methodology adapted to the solution of multibody contact problems in 3D with Coulomb friction. These problems play a role of the state problem in contact shape optimization problems with Coulomb friction. We use a modification of FETI that we call Total FETI, which imposes not only the compatibility of a solution across the subdomain interfaces, but also the prescribed displacements. For solving a state problem we use the method of successive approximations. Each iterative step of the method requires us to solve the contact problem with Tresca friction.

The discretized problem with Coulomb friction has a unique solution for small coefficients of friction. The uniqueness of the equilibria for fixed controls enables us to apply the so-called implicit programming approach. Its main idea consists in minimization of a nonsmooth composite function generated by the objective and the control-state mapping. The implicit programming approach combined with the differential calculus of Clarke was used for a discretized problem of 2D shape optimization. There is no possibility to extend the same approach to the 3D case. The main problem is the nonpolyhedral character of the second-order cone, arising in the 3D model. To get subgradient information needed in the used numerical method we use the differential calculus of Mordukhovich. Application of the

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Total FETI method to the solution of the state problem and sensitivity analysis allows massively parallel solution of these problems and stable identification of rigid body modes which are a priori known. The effectiveness of our approach is demonstrated by numerical experiments.
Keywords: total FETI, contact problems, Coulomb friction, shape optimization, nonsmooth optimization.

1 Introduction

Contact shape optimization is a special branch of structural optimization whose goal is to find shapes of deformable bodies which are in a mutual contact. A specific feature of contact shape optimization is its nonsmooth character due to the fact that the respective state mapping is given by various types of variational inequalities. We must use some special minimization algorithm of nonsmooth optimization for this kind of problem.

The contact problems are in the heart of contact shape optimization applications in mechanical engineering. Solving large multibody contact problems of linear elastostatics is complicated by the inequality boundary conditions, which make them strongly non-linear, and, if the system of bodies includes "floating" bodies, by the positive semi-definite stiffness matrices resulting from the discretization of such bodies. Observing that the classical Dirichlet and Neumann boundary conditions are known only after the solution has been found, it is natural to assume the solution of contact problems to be more costly than the solution of a related linear problem with the classical boundary conditions.

2 State problem: 3D contact problem with Coulomb friction

2.1 Primal formulation

To apply the TFETI domain decomposition, we tear each body from the part of the boundary with the Dirichlet boundary condition, decompose each body into subdomains, assign each subdomain a unique number, and introduce new "gluing" conditions on the artificial intersubdomain boundaries and on the boundaries with imposed Dirichlet condition. The gluing conditions enforce continuity of the displacements and of their normal derivatives across the intersubdomain boundaries and prescribed displacement on the Dirichlet boundaries.

In the following, we use the symbols E and N to distinguish the parts corresponding to the equalities and inequalities, respectively. The finite element discretization of $\overline{\Omega} = \overline{\Omega}^1 \cup \ldots \cup \overline{\Omega}^s$ with a suitable numbering of nodes results in the problem

$$\min_{\mathbf{u}} J_h(\mathbf{u}) \text{ subject to } \mathbf{B}_N\mathbf{u} \leq \mathbf{c}_N \text{ and } \mathbf{B}_E\mathbf{u} = \mathbf{c}_E, \tag{1}$$

where

$$J_h(\mathbf{u}) = f(\mathbf{u}) + j_h(\mathbf{u}), \quad f(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{f}^T\mathbf{u},$$

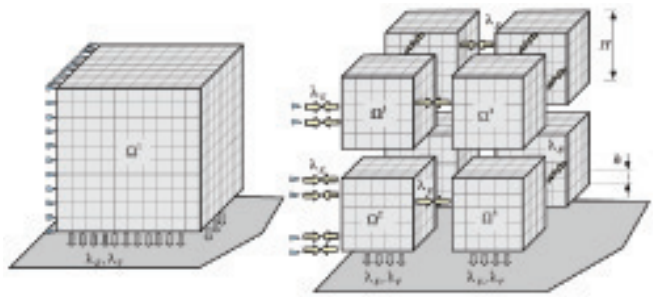

Figure 1: TFETI decomposition with subdomain renumbering.

j_h denotes the discretized dissipative term, $\mathbf{K} = \mathrm{diag}(\mathbf{K}_1, \ldots, \mathbf{K}_s)$ denotes a symmetric positive semidefinite block-diagonal matrix of the order n, $\mathbf{B}_N$ denotes an $m_C \times n$ full rank matrix, $\mathbf{B}_E$ denotes an $m_E \times n$ full rank matrix, $\mathbf{f} \in \mathbb{R}^n$, $\mathbf{c}_N \in \mathbb{R}^{m_C}$, and $\mathbf{c}_E \in \mathbb{R}^{m_E}$. After applying a numerical integration to a term with friction representing the Tresca friction law, we get

$$j_h(\mathbf{u}) = \sum_{i=1}^{m_c} \Psi_i \|\mathbf{T}_i \mathbf{u}\|, \tag{2}$$

where Ψ_i is the slip bound associated with $\mathbf{T}_i$. Using the standard procedure to modify the non-differentiable term j_h, we get

$$j_h(\mathbf{u}) = \sum_{i=1}^{m_c} \Psi_i \|\mathbf{T}_i \mathbf{u}\| = \sum_{i=1}^{m_c} \max_{\|\tau_i\| \le \Psi_i} \tau_i^T \mathbf{T}_i \mathbf{u}, \tag{3}$$

where $\tau_i \in \mathbb{R}^2$ can be considered as Lagrange multipliers. We assume that $\mathbf{B}_N$, $\mathbf{B}_E$, and $\mathbf{T}$ are the full rank matrices.

2.2 Dual formulation

The problem (1) is not suitable for numerical solution. The reasons are that the stiffness matrix $\mathbf{K}$ is typically ill-conditioned and singular, and the feasible set is in general so complex that projections into it can hardly be effectively computed. Under these circumstances, it would be very difficult to achieve fast identification of the active set at the solution and fast solution of the auxiliary problems.The complications mentioned above may be essentially reduced by applying the duality theory of convex programming . In the dual formulation of problem (1), we use three types of Lagrange multipliers, namely $\lambda_N \in \mathbb{R}^{m_C}$ associated with the non-interpenetration condition, $\lambda_E \in \mathbb{R}^{m_E}$ associated with the "gluing" and prescribed displacements, and

$$\tau = [\tau_1^T, \tau_2^T, \ldots, \tau_{m_C}^T]^T \in \mathbb{R}^{2m_C}$$

which regularizes the non-differentiability. The Lagrangian associated with problem (1) reads

$$L(\mathbf{u}, \lambda_N, \lambda_E, \tau) = f(\mathbf{u}) + \tau^T \mathbf{T}\mathbf{u} + \lambda_N^T(\mathbf{B}_N\mathbf{u} - \mathbf{c}_N) + \lambda_E^T(\mathbf{B}_E\mathbf{u} - \mathbf{c}_E). \quad (4)$$

Using the convexity of the cost function and constraints, we can use the classical duality theory to reformulate problem (1) to get

$$\min_{\mathbf{u}} \sup_{\substack{\lambda_E \in \mathbb{R}^{m_E},\ \lambda_N \geq \mathbf{o} \\ \|\tau_i\| \leq \Psi_i,\ i=1,\dots,m_C}} L(\mathbf{u}, \lambda_N, \lambda_E, \tau) = \max_{\substack{\lambda_E \in \mathbb{R}^{m_E},\ \lambda_N \geq \mathbf{o} \\ \|\tau_i\| \leq \Psi_i,\ i=1,\dots,m_C}} \min_{\mathbf{u}} L(\mathbf{u}, \lambda_N, \lambda_E, \tau).$$

To simplify the notation, we denote

$$\lambda = \begin{bmatrix} \lambda_E \\ \lambda_N \\ \tau \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_E \\ \mathbf{B}_N \\ \mathbf{T} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{c}_E \\ \mathbf{c}_N \\ \mathbf{o} \end{bmatrix},$$

and

$$\Lambda(\Psi) = \{(\lambda_E^T, \lambda_N^T, \tau^T)^T \in \mathbb{R}^{m_E+m_C+2m_C} :\ \lambda_N \geq \mathbf{o}, \|\tau_i\| \leq \Psi_i,\ i = 1, \dots, m_C\}$$

so that we can write the Lagrangian briefly as

$$L(\mathbf{u}, \lambda) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{f}^T\mathbf{u} + \lambda^T(\mathbf{B}\mathbf{u} - \mathbf{c})$$

and problem (1) is equivalent to the saddle point problem

$$L(\widehat{\mathbf{u}}, \widehat{\lambda}) = \max_{\lambda \in \Lambda(\Psi)} \min_{\mathbf{u}} L(\mathbf{u}, \lambda). \quad (5)$$

We assume that $\mathbf{B}$ is a full rank matrix. For fixed λ, the Lagrange function $L(\cdot, \lambda)$ is convex in the first variable and the minimizer $\mathbf{u}$ of $L(\cdot, \lambda)$ satisfies

$$\mathbf{K}\mathbf{u} - \mathbf{f} + \mathbf{B}^T\lambda = \mathbf{o}. \quad (6)$$

Equation (6) has a solution if and only if

$$\mathbf{f} - \mathbf{B}^T\lambda \in \mathrm{Im}\mathbf{K}, \quad (7)$$

which can be expressed more conveniently by means of a matrix $\mathbf{R} \in \mathbb{R}^{n \times 6s}$ whose columns span the null space of $\mathbf{K}$ as

$$\mathbf{R}^T(\mathbf{f} - \mathbf{B}^T\lambda) = \mathbf{o}. \quad (8)$$

Let us note that the action of a generalized inverse may be evaluated at the cost comparable with that of Cholesky's decomposition applied to the regularized $\mathbf{K}$. It may be verified directly that if $\mathbf{u}$ solves (6), then there is a vector $\alpha \in \mathbb{R}^{6s}$ such

that

$$\mathbf{u} = \mathbf{K}^{+}(\mathbf{f} - \mathbf{B}^{T}\lambda) + \mathbf{R}\alpha. \tag{9}$$

After substituting expression (9) into problem (5), changing the signs, and omitting the constant term, we get that λ solves the minimization problem

$$\min\ \Theta(\lambda) \quad \text{s.t.} \quad \lambda \in \Lambda(\Psi) \quad \text{and} \quad \mathbf{R}^{T}(\mathbf{f} - \mathbf{B}^{T}\lambda) = \mathbf{o}, \tag{10}$$

where

$$\Theta(\lambda) = \frac{1}{2}\lambda^{T}\mathbf{B}\mathbf{K}^{+}\mathbf{B}^{T}\lambda - \lambda^{T}(\mathbf{B}\mathbf{K}^{+}\mathbf{f} - \mathbf{c}). \tag{11}$$

Once the solution $\widehat{\lambda}$ of (10) is known, the solution $\widehat{\mathbf{u}}$ of (1) may be evaluated by (9) with

$$\alpha = (\mathbf{R}^{T}\widetilde{\mathbf{B}}^{T}\widetilde{\mathbf{B}}\mathbf{R})^{-1}\mathbf{R}^{T}\widetilde{\mathbf{B}}^{T}(\widetilde{\mathbf{c}} - \widetilde{\mathbf{B}}\mathbf{K}^{+}(\mathbf{f} - \mathbf{B}^{T}\widehat{\lambda})),$$

where the matrix $\widetilde{\mathbf{B}}$ and the vector $\widetilde{\mathbf{c}}$ are formed by the rows of $\mathbf{B}$ and $\mathbf{c}$ corresponding to all equality constraints and all free inequality constraints.

2.2.1 Coulomb's friction

Here we report approach for solving of the contact problem with Coulomb friction (with the coefficient of friction $\mathcal{F}$). The Tresca friction can be used to introduce a mapping

$$\Xi : \widetilde{\Lambda}^{N}(\boldsymbol{\Psi}) \mapsto \widetilde{\Lambda}^{N}(\boldsymbol{\Psi}), \quad \Xi(\boldsymbol{\Psi}) = \mathcal{F}\boldsymbol{\lambda}_{N}$$

with

$$\widetilde{\Lambda}^{N}(\boldsymbol{\Psi}) = \{\mathcal{F}\boldsymbol{\lambda}_{N} :\ \boldsymbol{\lambda} \in \widetilde{\Lambda}(\boldsymbol{\Psi})\},$$

whose fixed point is a solution to the problem with the Coulomb friction. Here $\boldsymbol{\Psi}$ is a vector with the entries Ψ_i introduced in (2). It is well-known that the sequence $\{\mathcal{F}\boldsymbol{\lambda}_{N}^{k}\}$, $\mathcal{F}\boldsymbol{\lambda}_{N}^{k+1} = \Xi(\mathcal{F}\boldsymbol{\lambda}_{N}^{k})$ with initial $\mathcal{F}\boldsymbol{\lambda}_{N}^{0} \in \widetilde{\Lambda}^{N}(\boldsymbol{\Psi})$, converges to the fixed-point $\mathcal{F}\boldsymbol{\lambda}_{N}$ if Ξ is contractive in $\widetilde{\Lambda}^{N}(\boldsymbol{\Psi})$. In addition, such a fixed-point is unique. This property holds provided that the coefficient of friction is small enough.

For more details see [4] and [8].

3 Shape optimization

So far, the shape of contact boundary has been fixed. Next we will look at $\boldsymbol{\alpha}$ as a control variable parameterizing our state problem. We denote by S the *control-to-state mapping* which assigns $\boldsymbol{\alpha} \in \mathbb{R}^{d}$ the solutions $(\boldsymbol{u}, \boldsymbol{\lambda})$ of the contact problem with Coulomb friction. We know that $S(\boldsymbol{\alpha})$ is nonempty for all $\boldsymbol{\alpha} \in \mathcal{U}$ ($\mathcal{U}$ is set of admissible design variables) and a singleton if coefficient of friction $\mathcal{F} \in (0, \mathcal{F}_0)$. For details see [2]. Let $J :\ \mathrm{Gr}\, S \longmapsto \mathbb{R}$ be an objective function.

The *discrete optimal shape design problem* reads as follows:

$$\left.\begin{array}{c}\text{Find } z^* := (\boldsymbol{\alpha}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*) \in \ \mathrm{Gr}\, S \text{ such that} \\ \mathcal{J}(z^*) \leq \mathcal{J}(z) \quad \forall z \in \ \mathrm{Gr}\, S.\end{array}\right\} \tag{$\mathbb{P}$}$$

If $\mathcal{F} \in (0, \mathcal{F}_0)$, then S is single-valued and ($\mathbb{P}$) takes the form

$$\left.\begin{array}{c}\text{Find } \boldsymbol{\alpha}^* \in \mathcal{U} \text{ such that} \\ \Theta(\boldsymbol{\alpha}^*) \leq \Theta(\boldsymbol{\alpha}) \quad \forall \boldsymbol{\alpha} \in \mathcal{U},\end{array}\right\} \tag{$\tilde{\mathbb{P}}$}$$

where $\Theta(\boldsymbol{\alpha}) := \mathcal{J}(\boldsymbol{\alpha}, S(\boldsymbol{\alpha}))$.

We use Matlab implementation of bundle trust method (see [7]) for the solution of shape optimization problem. This method is very robust and proper for the minimization of non-differentiable (nonsmooth) functions. This iterative method needs in every step value of the objective function and one arbitrary Clarke subgradient (for more details see [3]), i.e. we must be able for each admissible $\boldsymbol{\alpha}$ to find a solution of the state problem $(\boldsymbol{u}, \boldsymbol{\lambda}) = S(\boldsymbol{\alpha})$ and to compute one arbitrary Clarke subgradient. This is the goal of the following section.

4 Sensitivity analysis

The control-state mapping $\mathcal{S}(\boldsymbol{\alpha})$ can be transformed into the following generalized equation (GE):

$$\begin{aligned}
&\mathbf{0} = \boldsymbol{K}_f(\boldsymbol{\alpha})\boldsymbol{u} - \boldsymbol{f}_f(\boldsymbol{\alpha}) - \boldsymbol{B}(\boldsymbol{\alpha})_f^T \boldsymbol{\lambda} \ \ldots \text{ free nodes,}\\
&\mathbf{0} \in \boldsymbol{K}_{ct}(\boldsymbol{\alpha})\boldsymbol{u} - \boldsymbol{f}_{ct}(\boldsymbol{\alpha}) - \boldsymbol{B}(\boldsymbol{\alpha})_{ct}^T \boldsymbol{\lambda} + \tilde{\boldsymbol{Q}}(\boldsymbol{u}_{ct}, \boldsymbol{\lambda}_c) \ \ldots \text{ contact nodes,}\\
&\mathbf{0} = \boldsymbol{K}_{c\nu}(\boldsymbol{\alpha})\boldsymbol{u}_t - \boldsymbol{f}_{c\nu}(\boldsymbol{\alpha}) - \boldsymbol{B}(\boldsymbol{\alpha})_{c\nu}^T \boldsymbol{\lambda} \ \ldots \text{ contact nodes,}\\
&\mathbf{0} \in \boldsymbol{c}(\boldsymbol{\alpha}) - \boldsymbol{B}(\boldsymbol{\alpha})_c \boldsymbol{u} + N_{R_+^p}(\boldsymbol{\lambda}_c) \ \ldots \text{ contact nodes,}
\end{aligned}$$

where $_t$ denotes the tangential components, $_\nu$ denotes the normal components, $\boldsymbol{c}(\boldsymbol{\alpha})$ denotes the shape of contact boundary of the optimized body controlled by design variable $\boldsymbol{\alpha}$ and

$$\tilde{\boldsymbol{Q}}(\boldsymbol{u}_t, \boldsymbol{\lambda}_c) := \partial_{\boldsymbol{u}_t} j(\boldsymbol{u}_t, \boldsymbol{\lambda}_c), \quad j(\boldsymbol{u}_t, \boldsymbol{\lambda}_c) := \mathcal{F} \sum_{i=1}^{p} \lambda_c^i ||\boldsymbol{u}_t^i||, N_{R_+^p}$$

is standard normal cone.

This generalized equation can be written as:

$$\mathbf{0} \in F(\boldsymbol{\alpha})\boldsymbol{y} - l(\boldsymbol{\alpha}) + Q(\boldsymbol{y}),$$

where

$$\boldsymbol{y} := (\boldsymbol{u}, \boldsymbol{\lambda})^T,$$

$$F(\boldsymbol{\alpha}) := \begin{bmatrix} \boldsymbol{K} & -\boldsymbol{B} \\ -\boldsymbol{B} & \boldsymbol{0} \end{bmatrix},$$

$$l(\boldsymbol{\alpha}) := (\boldsymbol{f}(\boldsymbol{\alpha}), -\boldsymbol{c}(\boldsymbol{\alpha}))^T,$$

$$Q(\boldsymbol{y}) := \left(\boldsymbol{0}, \tilde{\boldsymbol{Q}}(\boldsymbol{u}_t, \boldsymbol{\lambda}_c), \boldsymbol{0}, N_{\mathbb{R}^p_+}(\boldsymbol{\lambda}_c)\right)^T.$$

We are searching for one (arbitrary) Clarke subgradient:

$$\partial\Theta(\boldsymbol{\alpha}) \ni \xi = \nabla_1 \mathcal{J}(\boldsymbol{\alpha}, \mathcal{S}(\boldsymbol{\alpha})) + \{\boldsymbol{C}^T \nabla_2 \mathcal{J}(\boldsymbol{\alpha}, \mathcal{S}(\boldsymbol{\alpha})) : \boldsymbol{C} \in \partial\mathcal{S}(\boldsymbol{\alpha})\}.$$

Because it holds

$$\forall \boldsymbol{y}^* : \quad D^*\mathcal{S}(\boldsymbol{\alpha})(\boldsymbol{y}^*) \neq \emptyset$$

and

$$\text{conv } (D^*\mathcal{S}(\boldsymbol{\alpha}))(\boldsymbol{y}^*) = \{\boldsymbol{C}^T \boldsymbol{y}^* : \boldsymbol{C} \in \partial\mathcal{S}(\boldsymbol{\alpha})\},$$

it suffices to compute just one element from the set $D^*\mathcal{S}(\boldsymbol{\alpha})(\nabla_2 \mathcal{J}(\boldsymbol{\alpha}, \mathcal{S}(\boldsymbol{\alpha})))$. We can find the elements from the limiting (Mordukhovich) coderivative

$$D^*\mathcal{S}(\boldsymbol{\alpha})(\boldsymbol{y}^*) := \{\boldsymbol{x}^* \in \mathbb{R}^d : (\boldsymbol{x}^*, -\boldsymbol{y}^*) \in N_{\text{Gr } \mathcal{S}}(\boldsymbol{\alpha})\}$$

on the basis of the following theorem.

Theorem Consider the reference pair $(\boldsymbol{\alpha}, \boldsymbol{y})$, with $\boldsymbol{\alpha} \in U_{ad}$, $\boldsymbol{y} = \mathcal{S}(\boldsymbol{\alpha})$. Then $\forall \boldsymbol{y}^* \in \mathbb{R}^m$

$$D^*\mathcal{S}(\boldsymbol{\alpha})(\boldsymbol{y}^*) \subset (\nabla_1(F(\boldsymbol{\alpha})\boldsymbol{y} - l(\boldsymbol{\alpha})))^T \mathcal{V},$$

provided $\mathcal{V}$ is the set of solutions $\boldsymbol{v}$ to the (limiting) adjoint generalized equation

$$\boldsymbol{0} \in \boldsymbol{y}^* + (F(\boldsymbol{\alpha}))^T \boldsymbol{v} + D^* Q(\boldsymbol{y}, -F(\boldsymbol{\alpha})\boldsymbol{y} + l(\boldsymbol{\alpha}))(\boldsymbol{v}).$$

For more details see [2].

We can use Total FETI method for the searching of solution to the limiting adjoint generalized equation. Our approach is based on [5].

5 Numerical results

Described methods were implemented and the following experiment was solved in MatSol library [6] developed in Matlab environment.

Two chain links (see Figure 2) were uniformly carved into $48 \times 8 \times 20 = 7680$ bricks. The finite element discretization was constructed by using hexahedron elements. The total number of nodal displacements was 66528. The total number of design variables (control points of surface of the chain link) was 28. Figure 3 presents meaning of the control points. These two chain links were decomposed in

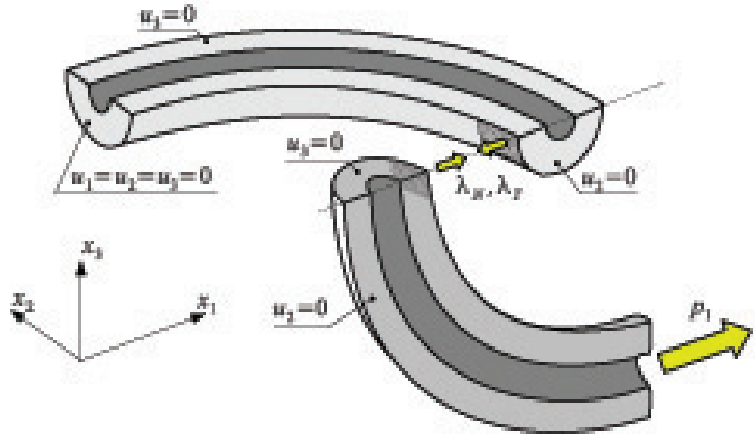

Figure 2: Two chain links.

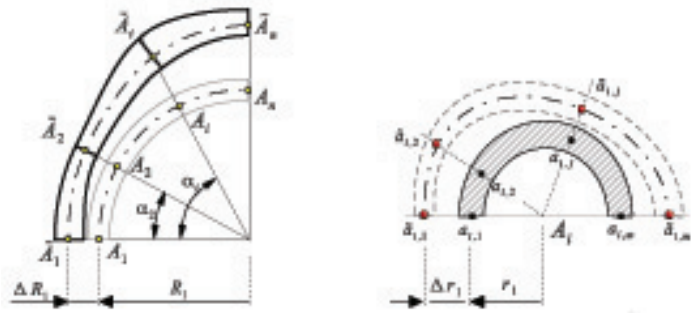

Figure 3: Design variables for controlling of shape of chain link.

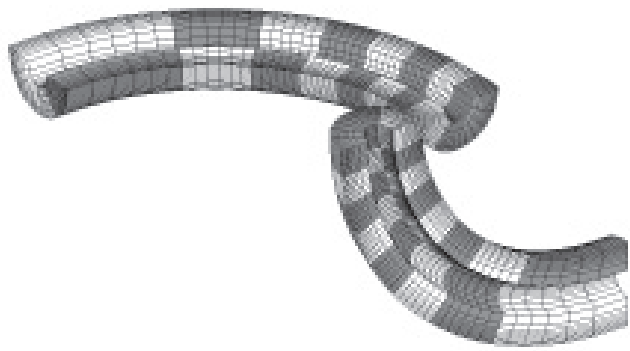

Figure 4: Domain decomposition of chain links.

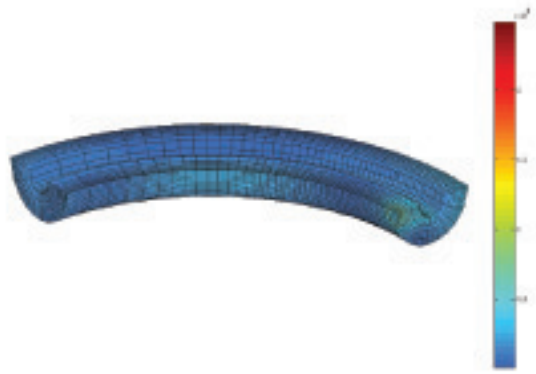

Figure 5: The distribution of the von Misses stresses for the initial shape.

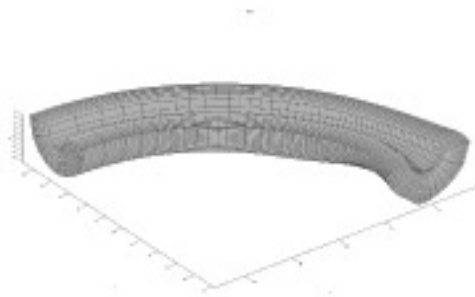

Figure 6: The optimal shape of the contact boundary.

our experiment into 32 bodies. See Figure 4. We try to find a shape of the contact surface of the chain link for which the total potential energy functional is minimal. The corresponding problem ($\tilde{\mathbb{P}}$) can be formulated as

$$\begin{aligned} &\text{minimize } \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{f} \\ &\text{subject to } \quad \boldsymbol{\alpha} \in \mathcal{U}. \end{aligned}$$

Figure 5 shows the distribution of the von Misses stresses for the initial shape, given by the constant vector $\boldsymbol{\alpha}_0 = [0, \ldots, 0]$. The objective value for the initial design was equal to $\Theta(\boldsymbol{\alpha}_0) = -3.387422 \cdot 10^5$. The relative stopping criterion for the code Bundle Trust was chosen as $\varepsilon = 1 \cdot 10^{-4}$. This required precision was reached after 2 iterations. Figure 6 presents the optimal solution, i.e., the optimal shape of the contact boundary, while Figure 7 shows the von Misses stresses. The optimal value of the objective function was $\Theta_{\text{opt}} = -3.807087 \cdot 10^5$.

Figure 7: The distribution of the von Misses stresses for the optimized shape.

Acknowledgements

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