

Numerically and parallel scalable TFETI based algorithms for quasistatic contact problems of mechanics

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Abstract

This paper deals with the solution of the discretized quasistatic 3D Signorini problems with local Coulomb friction. After a time discretization we obtain a system of static contact problems with Coulomb friction. Each of these problems is decomposed by the TFETI domain decomposition method used in auxiliary contact problems with Tresca friction. The algebraic formulation of these problems in 3D leads to the quadratic programming with equality constraints together with box and separable quadratic constraints. For the solution we used the scalable algorithm SMALSE developed at our department. The efficiency of the method is demonstrated by results of numerical experiments with parallel solution of 3D contact problems of elasticity.

1 Introduction

Contact problems represent a branch of mechanics of solids which analyzes the behavior of loaded, deformable bodies being in a mutual contact. If the system of bodies includes “floating” bodies, the resulting stiffness matrices from the discretization of such bodies are positive semi-definite. Moreover the occurrence of non-penetration and friction conditions implies the highly non-linear behavior of resulting problems. Both this phenomena depend on time so as the applied forces. If however applied forces vary only slowly in time, inertia of the system



can be neglected and one can use a quasistatic approximation (Eck *et al.* [1]). This typically arises in geomechanics (modelling of a movement of tectonic plates, prediction of earthquakes).

Using finite-difference approximation of the time derivative we arrive at a sequence of static contact problems with Coulomb friction, whose solutions are defined by using a fixed point approach. Each iterative step is represented by an auxiliary contact problem with a given slip bound (Tresca model of friction). Since its formulation is given by a variational inequality of the second kind, after discretization we obtain a convex non-smooth constrained minimization problem for a discrete total potential energy function. To increase the efficiency of the quadratic part of the minimized problem we used the TFETI method, a variant (see Dostál *et al.* [2] and [3]) of FETI domain decomposition methods framework (Farhat *et al.* [4]). Introducing the additional Lagrange multipliers one can release the non-penetration conditions and transform the frictional into a smooth one. Eliminating the displacement field we arrive at a quadratic minimization problem with box and separated quadratic inequality constraints.

For solving of such minimization problem we use the SMALSE algorithm (Dostál and Kozubek [5] and Dostál and Kučera [6]) with an error bound in terms of bounds on the spectrum of the Hessian to the quadratic cost function, independent of the conditioning of the constraints. Moreover if the problem has sparse enough representation of the Hessian, it simply follows that the cost of the solution is proportional to the number of unknowns.

This proceeding extends the results of Haslinger *et al.* [7] and [8] from unilateral 2D and 3D problems to multibody contact case.

2 Formulation of the quasistatic 3D contact problem with Coulomb friction

At first we briefly introduce the multibody quasistatic problem with Coulomb friction. The n_Ω isotropic elastic bodies are represented by the domains $\Omega^p \in \mathbb{R}^3$, with a sufficiently smooth boundary Γ^p , $p = 1, \dots, n_\Omega$. Suppose that each Γ^p consists of three disjoint parts Γ_U^p , Γ_F^p and Γ_C^p and consider zero displacements on Γ_U^p and given forces F^p . The part Γ_C^p denotes the part of Γ^p that may get into contact with some other body in a time interval $[0, T_0]$ and therefore the non-penetration and Coulomb friction conditions have to be prescribed. In particular, we denote by Γ_C^{pq} the part of Γ_C^p that can be, in the solution, in contact with the body Ω^q . For convenience, we introduce the index set of contact interfaces

$$\mathcal{I}_C = \{(\rho, q), \rho, q = 1, \dots, s : \rho < q \text{ and } \text{meas} \Gamma_C^{\rho q} > 0\}.$$

One can read more about the formulation for unilateral contact in [8, 9]. Our notation of the multibody case is similar to the prepared article [3] on transient problems.

By the *weak formulation* of the quasistatic contact problem with local Coulomb law of friction we mean the following problem formulated in terms of the



displacement vector u and the normal contact stress λ_N :

$$\left. \begin{aligned} & \text{Find } u \in W^{1,2}(0, T_0, \mathcal{V}), \lambda_N \in W^{1,2}(0, T_0, \mathcal{M}) \text{ such that} \\ & u \in \mathcal{K} \text{ for a.a. } t \in [0, T_0], \quad u(0) = u_0 \text{ in } \Omega \\ & a(u, v - \dot{u}) + j(\lambda_N, v) - j(\lambda_N, \dot{u}) \geq L(v - \dot{u}) + \\ & \quad + \sum_{(p,q) \in \mathcal{I}_C} \langle \lambda_N^{pq}, [v_N^{pq} - \dot{u}_N^{pq}] \rangle \quad \forall v \in \mathcal{V} \text{ and a.a. } t \in [0, T_0] \\ & \langle \lambda_N, z_N - u_N \rangle \geq 0 \quad \forall z \in \mathcal{K} \text{ and a.a. } t \in [0, T_0], \end{aligned} \right\} \quad (1)$$

where $[\]$ is the jump of the displacement across Γ_C^{pq} and $\langle \cdot \rangle$ is the appropriate duality pairing.

$$a(u, v) := \sum_{p=1}^{n_\Omega} \int_{\Omega^p} c_{ijkl}^p \varepsilon_{ij}(u^p) \varepsilon_{kl}(v^p) d\Omega, \quad j(\lambda_N, v) := - \sum_{(p,q) \in \mathcal{I}_C} \langle \mathcal{F} \lambda_N^{pq}, \llbracket [v_T^{pq}] \rrbracket \rangle$$

$$L(v) := \sum_{p=1}^{n_\Omega} \int_{\Omega^p} F_i^p v_i^p d\Omega + \int_{\Gamma_p} P_i^p v_i^p d\Gamma$$

where λ_N is equal to the normal contact stress on Γ_C .

After discretization of time (n_{ts} timesteps) and space and substitution of the difference instead of time derivation will be the quasistatic problem (1) approximated by the sequence of static contact problems with Coulomb friction. Only the small change will appear in the nonsmooth friction term formulation which is in each timestep corrected from the previous timestep solution. For details see Haslinger *et al.* [8]. The variational formulation of Coulomb friction is the *implicit* variational inequality due to the occurrence of the term λ_N in the nonsmooth part of it. To deal with this we used the alternative formulation based on the *fixed point* approach, where the sequence of auxiliary Tresca friction problems with previously mentioned correction is solved. It turned out to be efficient from the numerical point of view.

The discretized Tresca friction problem is described as a solution \mathbf{u} of the problem

$$\left. \begin{aligned} & \text{For given } \mathbf{g} \in \mathbb{R}^{n_N} \text{ find } \mathbf{u} \in \mathbb{K} \text{ such that} \\ & J(\mathbf{u}) \leq J(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{K}, \\ & \text{where } J(\mathbf{v}) = \frac{1}{2} \mathbf{v}^T \mathbf{K} \mathbf{v} - \mathbf{v}^T \mathbf{f} + \sum_{i=1}^{n_N} -\mathcal{F} \mathbf{g}_i \|\mathbf{T}_i(\mathbf{v} - \mathbf{z})\|_{\mathbb{R}^2}, \end{aligned} \right\} \quad (\mathcal{Q}_A^i)$$

where \mathbf{K} and \mathbf{f} is the stiffness matrix and right hand side, respectively. The matrices \mathbf{N} and \mathbf{T} describe mappings from \mathbf{u} to $[\mathbf{u}_N]$, $[\mathbf{u}_T]$, respectively. The vector \mathbf{g} is the given slip bound,

$$\mathbb{K} = \{ \mathbf{v} \in \mathbb{R}^{n_p} \mid \mathbf{v}_N := \mathbf{N} \mathbf{v} \leq \mathbf{c}_N \}$$

is the cone of admissible displacements and \mathbf{z} is the correction from the previous timestep solution. The gap between contact interfaces in the reference



configuration we denote by \mathbf{c}_N . The numbers n_p, n_N denote the displacement degrees of freedom count, the dual dof's (from the discretization of the duality pairing $\langle \cdot, \cdot \rangle$), respectively. Each of this matrices and vectors has a block structure according to the bodies of the multibody contact problem.

3 Domain decomposition and dual formulation of the auxiliary Tresca friction problem

Let mark all symbols from the previous section with g as “global”. At fig. 1 one can see the sketch of the domain decomposition of i th body Ω_i^g into subdomains Ω_{ij} . Let denote the space of the discretized displacements as V^g and the decomposed space (broken space) by V . Dealing with the decomposed displacement \mathbf{u} one have to enforce the continuity across the teared interface. This can be achieved by the “gluing” matrix \mathbf{B}_G and the equality constraint imposed on \mathbf{u} and in the same way can be handled the Dirichlet boundary condition

$$\begin{aligned} \mathbf{u}^g \in V^g &\Leftrightarrow (\mathbf{u} \in V) \wedge (\mathbf{B}_G \mathbf{u} = \mathbf{0}) \\ \mathbf{u}^g \in K^g &\Leftrightarrow (\mathbf{u} \in V) \wedge (\mathbf{B}_G \mathbf{u} = \mathbf{0}) \wedge (\mathbf{B}_D \mathbf{u} = \mathbf{0}) \Leftrightarrow (\mathbf{u} \in V) \wedge (\mathbf{B}_E \mathbf{u} = \mathbf{0}), \end{aligned}$$

where the matrix \mathbf{B}_E is formed from the blocks \mathbf{B}_G and \mathbf{B}_D . Substituting this into (Q_A^i) we obtain

$$\left. \begin{aligned} &\text{For given } \mathbf{g} \in \mathbb{R}^{n_N} \text{ find } \mathbf{u} \in V \text{ such that} \\ &J(\mathbf{u}) \leq J(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{K}, \text{ subject to } \mathbf{B}\mathbf{u} = \mathbf{B}\mathbf{v} = \mathbf{0} \\ &\text{where } J(\mathbf{v}) = \frac{1}{2} \mathbf{v}^T \mathbf{K} \mathbf{v} - \mathbf{v}^T \mathbf{f} + \sum_{i=1}^{n_N} -\mathcal{F} \mathbf{g}_i \| \mathbf{T}_i(\mathbf{v} - \mathbf{z}) \|_{\mathbb{R}^2}, \end{aligned} \right\} \quad (2)$$

Let us mention that all the vectors and matrices have the block structure according to subdomains and so the computation with them is much more convenient. All the blocks in the stiffness matrix are positive semidefinite with the kernel of the dimension 6 in the case of 3D problems. Using the standard

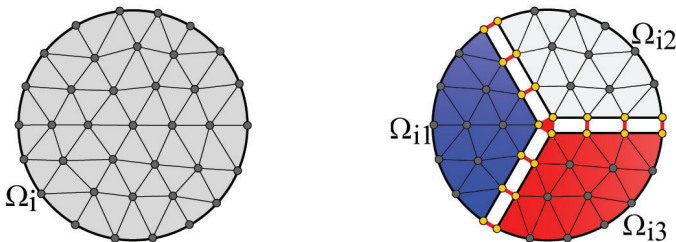


Figure 1: Original (left) and decomposed (right) domain.

procedure to modify the non-differentiable term (see Dostál *et al.* [10]), we get

$$\mathcal{F}g_i \|T_i(\mathbf{v} - \mathbf{z})\| = \max_{\|\lambda_{Ti}\|_{\mathbb{R}^2} \leq \mathcal{F}g_i} \lambda_{Ti}^\top T_i(\mathbf{v} - \mathbf{z}).$$

Now, we can employ the additional Lagrange multipliers; λ_E to enforce the equality condition, λ_N to enforce the non-penetration condition and λ_T that regularizes the non-differentiability. Finally we obtain the saddle point problem

$$\left. \begin{aligned} & \text{Find } (\mathbf{u}, \lambda_N, \lambda_T, \lambda_E) \in \mathbb{R}^{n_p} \times \Lambda(\mathbf{g}) \text{ such that} \\ & \mathcal{L}(\mathbf{u}, \lambda_N, \lambda_T, \lambda_E) = \sup_{\substack{\lambda_N \geq \mathbf{0} \\ \|\lambda_{Ti}\|_{\mathbb{R}^2} \leq \mathcal{F}g_i \\ \lambda_E \in \mathbb{R}^{n_E}}} \inf_{\mathbf{v} \in \mathbb{R}^{n_p}} \mathcal{L}(\mathbf{v}, \mu_N, \mu_T, \mu_E), \end{aligned} \right\} \quad (3)$$

where

$$\begin{aligned} \mathcal{L}(\mathbf{v}, \mu_N, \mu_T, \mu_E) &= \frac{1}{2} \mathbf{v}^\top \mathbf{K} \mathbf{v} - \mathbf{v}^\top \mathbf{f} + \\ &+ \mu_T^\top (\mathbf{T} \mathbf{v} - \mathbf{T} \mathbf{z}) + \mu_N^\top (\mathbf{N} \mathbf{v} - \mathbf{c}_N) + \mu_E^\top (\mathbf{B}_E \mathbf{v} - \mathbf{c}_E) \\ \Lambda(\mathbf{g}) &= \mathbb{R}_+^{n_N} \times \{ \mu_T \in \mathbb{R}^{2n_T} \mid \|\lambda_{Ti}\|_{\mathbb{R}^2} \leq \mathcal{F}g_i \} \times \mathbb{R}^{n_E}. \end{aligned}$$

After small change of notation to group the indices N, T, E together, swapping the supremum and infimum, eliminating of displacement using the pseudoinverse matrix \mathbb{K}^+ and the kernel matrix \mathbf{R} (see all this steps in Dostál *et al.* [2]) we obtain the *dual formulation*

$$\text{Find } \lambda \in \Lambda^{eq}(\mathbf{g}) \text{ such that } S(\lambda) \leq S(\mu) \quad \forall \mu \in \Lambda^{eq}(\mathbf{g}), \quad (4)$$

where

$$\begin{aligned} S(\mu) &= \frac{1}{2} \lambda^\top \overbrace{\mathbf{B} \mathbf{K}^\dagger \mathbf{B}^\top}^{\mathbf{F}} \lambda - \lambda^\top \overbrace{\mathbf{B} \mathbf{K}^\dagger \mathbf{f} - \mathbf{c}}^{\mathbf{d}} \\ \Lambda^{eq}(\mathbf{g}) &= \{ \lambda \in \Lambda(\mathbf{g}) \mid \underbrace{\mathbf{R}^\top \mathbf{B}^\top}_{\mathbf{G}} \lambda = \underbrace{\mathbf{R}^\top \mathbf{f}}_{\mathbf{e}} \} \end{aligned}$$

4 Optimality

After homogenization of the equality constraint in (4) and returning back to the same notation one can enhance the resulting problem by putting the projectors from the homogenized equality constraints, which results in the problem

$$\text{Find } \lambda \in \Lambda^*(\mathbf{g}) \text{ such that } S^*(\lambda) \leq S^*(\mu) \quad \forall \mu \in \Lambda^*(\mathbf{g}), \quad (5)$$

where

$$\begin{aligned} S^*(\mu) &= \frac{1}{2} \lambda^\top (\mathbf{P} \mathbf{F} \mathbf{P} + \rho \mathbf{Q}) \lambda - \lambda^\top \mathbf{P} \mathbf{d} \\ \Lambda^*(\mathbf{g}) &= \{ \lambda \in \Lambda(\mathbf{g}) \mid \mathbf{G} \lambda = \mathbf{0} \}, \quad \mathbf{Q} = \mathbf{G}^\top (\mathbf{G} \mathbf{G}^\top)^{-1} \mathbf{G}, \quad \mathbf{P} = \mathbf{I} - \mathbf{Q}. \end{aligned}$$



A good choice of the regularization parameter is given by $\bar{\rho} = \|\mathbf{PFP}\|$, as this is the largest value for which

$$\|\mathbf{PFP}\| \geq \|\mathbf{PFP} + \rho\mathbf{Q}\|.$$

The classical estimates by Farhat *et al.* [4] of the extreme eigenvalues says that if the decompositions and the discretizations are sufficiently regular, then there exists constants $C_1 > 0$ and $C_2 > 0$ independent of the discretization parameter h and the decomposition parameter H such that

$$C_1 \frac{h}{H} \leq \lambda_{\min}(\mathbf{PFP}|\mathbf{ImP}) \quad \text{and} \quad \lambda_{\max}(\mathbf{PFP}|\mathbf{ImP}) = \|\mathbf{PFP}\| \leq C_2, \quad (6)$$

where λ_{\min} and λ_{\max} denote the extremal eigenvalues of the corresponding matrices.

For the solution of 5 we propose to use SMALSE (semi-monotonic augmented Lagrangian method for separable and equality constraints) Dostál and Kozubek [5]. SMALSE enforces the equality constraints by the Lagrange multipliers generated in the outer loop, while the auxiliary inner quadratic programming problems with separable inequality constraints but without equality constraints are solved approximately in the inner loop by the MGP algorithm proposed by Dostál and Kozubek [5].

The unique feature of combination of SMALSE with MGP for solving the quadratic programming problems (5) that arise from the auxiliary Tresca friction problems is the bound on the number of iterations whose cost is proportional to the number of variables, so that it can return an approximate solution for the cost proportional to the number of variables.

5 Numerical experiments

In this section we want to test the previously mentioned algorithms to show their optimality. The algorithms were implemented into our MatSol software (Kozubek *et al.* [11]) and tested on academic and real world problems.

5.1 Structured grid: Two cantilever beams example

We shall consider two elastic bodies represented by the bricks Ω_i made of a homogenous and isotropic material which are characterized by Young's modulus E_i and Poisson's ratios σ_i , $i = 1, \dots, 2$:

$$\begin{aligned} \Omega_1 &= (0, 2) \times (-0.05, 1.05) \times (-1, 0), \quad E_1 = 30000 \text{ MPa}, \quad \sigma_1 = 0.27 \\ \Omega_2 &= \left(\frac{2}{3}, \frac{8}{3}\right) \times (0, 1) \times (0.1, 1.1), \quad E_2 = 30000 \text{ MPa}, \quad \sigma_2 = 0.35. \end{aligned}$$

The bodies are subjected by the surface tractions $P_{iU} \cdot \phi_a(\mathbf{t})$, $P_{iL} \cdot \phi_b(\mathbf{t})$ as is depicted in fig. 2. The functions ϕ_a , ϕ_b (fig. 3) characterizes the history of the



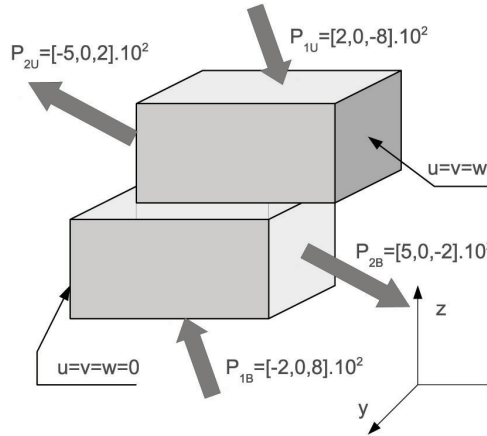


Figure 2: Geometry of cantilever beams.

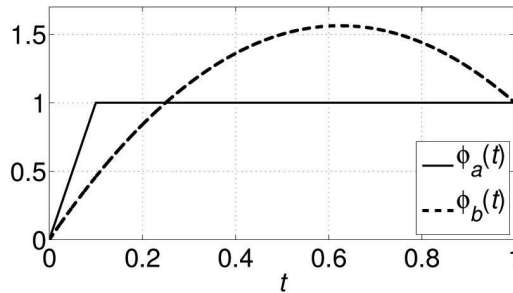


Figure 3: Geometry of cantilever beams.

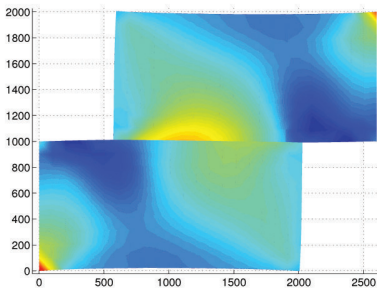
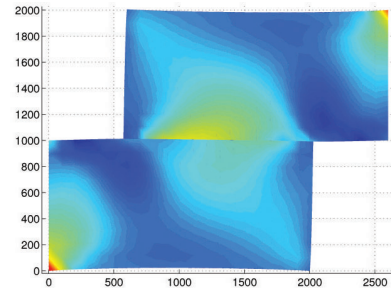
loading process. We used structured trilinear grid with varying discretizations. The discretization and decomposition parameters were properly chosen to keep the ratio H/h constant. The computation was done with the Coulomb friction coefficient $\mathcal{F} = 0.5$ and for 10 timesteps.

The resulting von Mises stress in timesteps 5 and 10 are depicted in figures 4 and 5. The characteristics of computation are summarized in table 1. We can see that the number of matrix–vector multiplications and SMALSE inner iterations increases with the increasing problem size. The number of inner iterations increases due to the use of Coulomb friction boundary condition. In addition, if we take into account the fact that the first approximately ten iterations of the SMALSE algorithm (in each timestep) are inexpensive to Hessian multiplications then one can see that the ratio of Hessian multiplication per inner iteration varies only a little, as was predicted by the theory.



Table 1: Numerical characteristics of 3D cantilever beams example.

Number of subdomains	32	108	256	500
Number of CPUs	20	20	20	20
Primal variables	127,775	431,244	1,022,208	1,996,500
Dual variables	23,763	89,283	222,462	446,916
Null space dimension	192	648	1536	3000
SMALSE-M iterations	123	125	136	147
Hessian multiplications	1834	2160	2522	3099
Solution time [s]	197	607	1813	4446
Total time [s]	302	1181	3586	9301

Figure 4: two beams, $t_s = 5$.Figure 5: two beams, $t_s = 10$.

5.2 Real world problem application: yielding clamp connection

To show that our approach is capable also to solve real world contact problems we consider the analysis of the yielding clamp connection of steel arched supports depicted in fig. 6. This type of construction is used to support the mining openings. It is a typical multibody contact, where the yielding connection plays the role of the mechanical protection against destruction, i.e., against the total deformation of the supporting arches. We consider again the quasistatic Coulomb friction problem contact with $\mathcal{F} = 0.5$ and 10 timesteps. The problem was decomposed by METIS into 150 subdomains and discretized by 520,833 and 102,724 primal and dual variables, respectively. The total displacements for timesteps 6 and 10 are depicted in fig. 6. The computation required 8278 matrix-vector multiplications. The solution time was 2315 s and the total time 4293 s.

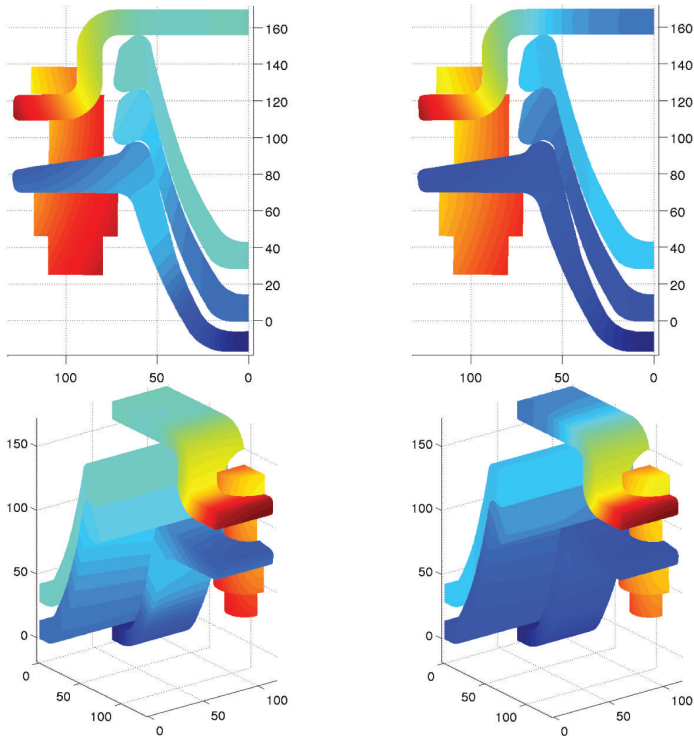


Figure 6: Steel support, displacement for $t_s = 6$ (left) and $t_s = 10$ (right).

Acknowledgements

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