Optimal pre-stress and lay-ups in a thick-walled hollow cylinder for minimum stresses

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Abstract

This paper is concerned with the analysis of a laminated composite cylinder, considering both the pre-stress and lay-ups of unilateral filaments, i.e. filament winding or fiber placement, which involve fiber pre-stress and the best lay-up in laminas for waviness reduction. The fiber pre-stress applied in individual plies is shown to cause an eigenstress in the respective plies, and relaxation stresses in the already completed plies. Influence functions (influence tensors) that relate the ply stresses to the applied pre-stress forces are derived. The goal is to determine fiber pre-stress distributions through the wall thickness such that the total stresses due to external hydrostatic pressure and the fiber pre-stress in individual plies are as uniform as possible through the wall thickness and confined by the ply strength magnitudes. Generalized plain strain is the starting model for the formulation of the problem, which is to be solved. Optimal pre-stress of the pseudo 3D problem based on the generalized plain strain is solved in some previous papers by the author, which leads to uniformly distributed stresses in a selected direction through the thickness of the cylinder. It is of great interest to engineers to consider the optimization of stresses that are dependent also on the lay-ups of reinforcing filaments, for example in classical composites. In order to attain the optimal result, which depends also on the arrangement of filaments, an extended cost functional has to be suggested. For the practical production and creation of such composites, see the description of the build-up process in the previous publication by Dvorak and Procházka (Thick-walled composite cylinders with optimal fiber prestress. *Composites Part B* **27B**: 643-649 1996). The great advantage of the theoretical pseudo 3D model is in the fact that the solution of the basic problem of laminated cylinders, which describe the behavior of layered composite cylinders, is quite simple and leads to

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a system of simultaneous linear algebraic equations. While before only the isotropic structure of the composites envisaged was taken into account in practical examples, now, inevitably, anisotropic layers have to be considered in each step of the solution. Hence, the basic formulation is slightly more complicated, but still leads to a system of linear equations. Since the cost functional is quadratic, the optimum condition is formed in terms of a system of linear equations. A typical example will follow the theory.

Keywords: optimal fiber pre-stress, optimal lay-ups, laminated cylinder, hydrostatic pressure.

1 Introduction

It is well known that relatively thick unidirectional fiber composites that are carefully manufactured for reduced fiber waviness (and consequently possible kinking) can support axial and hoop compressive stresses of significant magnitudes. In the structures that are produced by fiber placement or filament winding, fiber waviness can be reduced by fiber pre-stress applied appropriately during the curing process. Apart from the potentially beneficial effect on ply compressive strength, the consequences of fiber pre-stress applied in a large laminated structure are not well understood. Throughout this paper, the same material properties are supposed for the laminas.

Considering the hydrostatic pressure of hollow laminated cylindrical structures (submersibles, tunnels, aircrafts, etc.), a conventional solution starting with generalized plain strain state is fully described in [1], which involves even eigenstrains and eigenstresses in the formulation. The fabrication process is also suggested in [1] and developed in [2] and [3]. In [4] it is proved that the optimization for minimum stress leads to the uniform distribution of prevailing stresses in hoop and axial directions. Moreover, it has been be shown in [1] that the number of eigenstresses in a laminated cylinder (or plate, shell, arch, etc.) can serve as a design parameter for the optimization problem; reducing stresses in the laminate structure is equal to the number of layers minus one. From this one can always arrange the pre-stress in such a way that all filaments are in tension due to superposition of the external load and optimal pre-stress. Constraint conditions are stated in [3], which are very reasonable: The stress cannot exceed a prescribed value of strength i.e., generally the solution could not exist at all. Then the structure has to be redesigned. In order to fulfill one constraint condition, that the pre-stress forces are as low as possible, while at the same time the pre-stress is optimal, the orientation of fibers (lay-up) can be taken into consideration. Recall that in all laminas the same material properties are used; only the set-up of plies influences the differences in stiffness or compliance with the generalized Hooke's law.

Using the theoretical framework developed in [1] and [2] we establish a set of influence functions that evaluate the ply eigenstresses in terms of the pre-stress forces applied to the individual layers of the laminate. Constant pre-stress applied uniformly to all plies is shown to reduce possibly high stress gradients, with compressive stresses at the inner surface that may impair the load bearing capacity of the structure. Finally, we establish a nonlinear optimization procedure for solving the problem of finding fiber pre-stress distribution that generates minimized residual stresses that do not exceed certain prescribed magnitudes. In superposition with the stresses due to the applied hydrostatic pressure, the eigenstresses produce total stresses that lie within given ply strength limits.

2 Generation of eigenstresses

The eigenstress caused by fiber pre-stress will be denoted by $\boldsymbol{\lambda} \equiv \{\boldsymbol{\lambda}^{\theta}, \boldsymbol{\lambda}^{z}\} = \{\lambda_1^{\theta}, \lambda_2^{\theta}, \lambda_3^{\theta}, ..., \lambda_n^{z}, \lambda_1^{z}, \lambda_2^{z}, \lambda_3^{z}, ..., \lambda_n^{z}\}$, in which n is the number of laminas, and only two out of three directions $\{ x, \theta, z \}$ are reflected. This is why it can be shown that the r-direction does not have any significant effect on overall stresses, either in individual layers or in the entire structure. Consider a cylindrical layer (i) of inner radius a_i, outer radius b_i, and thickness $t_i = b_i - a_i$. Let φ_i denote the angle that all fibers in the layer (i) contain with the longitudinal z-axis of the cylinder. The eigenstress $\boldsymbol{\lambda}_i \equiv \{\lambda_i^{\theta}, \lambda_i^{z}\}$ in layer (i) is caused by the application of a certain pre-stress force P_i to each fiber in its winding direction φ_i, both after curing; therefore, this stress is preserved after curing as part of the total fiber stress. The fiber pre-stress force components in the hoop and axial directions are, see Fig. 1,

$$P_i^{\theta} = P_i \sin \varphi_i \quad P_i^{z} = P_i \cos \varphi_i \tag{1}$$

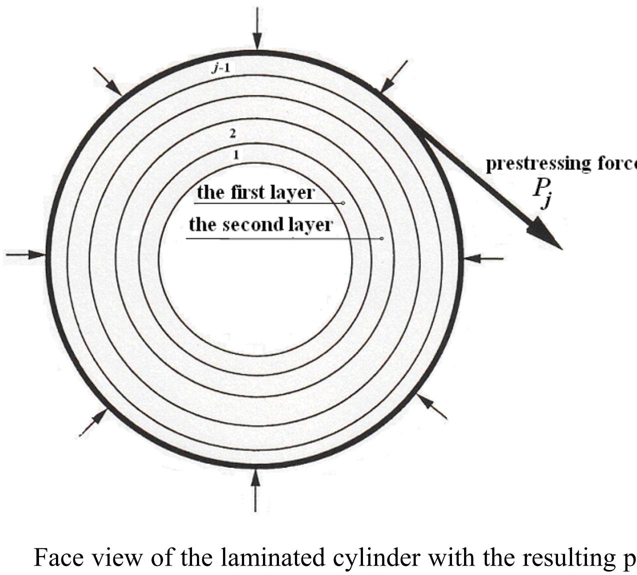

Figure 1: Face view of the laminated cylinder with the resulting pre-stressing force in layer j.

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The axial and hoop components of the eigenstress in the cylinder coordinate system can be found as follows. Consider a small square element of the *i*th layer in the θz -plane, where the θ -axis is in the hoop direction and the *z*-axis is in the longitudinal direction of the composite cylinder. For simplicity, let the reinforcement be represented by a monolayer of fibers of diameter d_i, and spacing s_i, evaluated in terms of fiber volume fraction c_i, as,

$$s_i = \frac{\pi d_i^2}{4 t_i c_i} \tag{2}$$

which straightforwardly follows from the definition of fiber volume ratio in a unit cell given by a rectangle $s_i \times t_i$.

This represents the average distance between the fibers' axes measured in the direction perpendicular to the fibers in each ply. However, in the planes perpendicular to the θ- and z-axes of the cylinder, the average distances between the fiber axes will be,

$$s_i^z = \frac{s_i}{\sin \varphi_i} \text{ in plane } \theta = \text{const.}, \qquad s_i^\theta = \frac{s_i}{\cos \varphi_i} \text{ in plane } z = \text{const.}, \tag{3}$$

Since the force P_i is applied to the individual fibers and the hoop and axial components are defined in (1), the eigenstresses are related to the pre-stressing force components in layer *i* as,

$$\lambda_i^\theta = \frac{P_i^\theta}{s_i^z t_i} = \frac{4 c_i P_i \sin^2 \varphi_i}{\pi d_i^2}, \qquad \lambda_i^z = \frac{P_i^z}{s_i^\theta t_i} = \frac{4 c_i P_i \cos^2 \varphi_i}{\pi d_i^2} \tag{4}$$

where the expression $\frac{4P_i}{\pi d_i^2}$ represents the magnitude of the actual pre-stress applied to the fibers. It obviously holds that $\lambda_i^\theta = 0$ for 0^0 plies, $\lambda_i^z = 0$ for 90^0 plies. Moreover, by dividing both formulas in (4) a relation between eigenstresses evidently follows,

$$\frac{\lambda_i^\theta}{\lambda_i^z} = \tan^2 \varphi_i \tag{5}$$

and $\lambda_i^\theta = 3\lambda_i^z$ for 60^0 plies.

Using the above notation we can eventually write in matrix form:

$$\boldsymbol{\lambda}^\theta = \boldsymbol{S}^\theta \boldsymbol{P}^\theta, \ \boldsymbol{\lambda}^z = \boldsymbol{S}^z \boldsymbol{P}^z, \text{ where } \boldsymbol{P}^\theta = \{P_1^\theta, P_2^\theta, ..., P_n^\theta\}, \boldsymbol{P}^z = \{P_1^z, P_2^z, ..., P_n^z\} \tag{6}$$

The ($n \times n$) matrices $\boldsymbol{S}^{\theta}, \boldsymbol{S}^{z}$ are of the form:

$$\boldsymbol{S}^{\theta} = \begin{bmatrix} \frac{4c_1P_1\sin^2\varphi_1}{\pi d_1^2} & 0 & 0 & . \; . \; . & 0 \\ 0 & \frac{4c_2P_2\sin^2\varphi_2}{\pi d_2^2} & 0 & . \; . \; . & 0 \\ 0 & 0 & \frac{4c_3P_3\sin^2\varphi_3}{\pi d_3^2} & . \; . \; . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & . \; . \; . & \frac{4c_nP_n\sin^2\varphi_n}{\pi d_n^2} \end{bmatrix}$$

$$\boldsymbol{S}^{z} = \begin{bmatrix} \frac{4c_1P_1\cos^2\varphi_1}{\pi d_1^2} & 0 & 0 & . \; . \; . & 0 \\ 0 & \frac{4c_2P_2\cos^2\varphi_2}{\pi d_2^2} & 0 & . \; . \; . & 0 \\ 0 & 0 & \frac{4c_3P_3\cos^2\varphi_3}{\pi d_3^2} & . \; . \; . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & . \; . \; . & \frac{4c_nP_n\cos^2\varphi_n}{\pi d_n^2} \end{bmatrix}$$

3 Stiffness matrix in one lamina

Since parallel winding of filaments in each lamina is assumed, the generalized Hooke's law holds valid,

$$\boldsymbol{\sigma} = \boldsymbol{L}\boldsymbol{\varepsilon} + \boldsymbol{\lambda} \tag{7}$$

where $\boldsymbol{\sigma} = \{\sigma^r, \sigma^\theta, \sigma^z\}^{\mathrm{T}}$ is the stress tensor with the only components being different from zero and similarly $\boldsymbol{\varepsilon} = \{\varepsilon^r, \varepsilon^\theta, \varepsilon^z\}^{\mathrm{T}}$ is the strain tensor, both written in vector notation, as usual. Tensor $\boldsymbol{\lambda}$ is defined in compliance with the previous section as: $\boldsymbol{\lambda} = \{0, \lambda^\theta, \lambda^z\}^{\mathrm{T}}$ because there is no pre-stress in the radial direction. Note that symmetric lay-up is supposed and the thickness of the layers is very small, so that no shear stress is induced in the composite cylinder. The stiffness matrix becomes:

$$\boldsymbol{L} = \begin{bmatrix} L_{rr} & L_{r\theta} & L_{rz} \\ & L_{\theta\theta} & L_{\theta z} \\ \text{symm.} & & L_{zz} \end{bmatrix} \tag{8}$$

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where $L_{rr} = \dfrac{E_r(1-\nu_{\theta z}\nu_{z\theta})}{D}$, $L_{\theta\theta} = \dfrac{E_\theta(1-\nu_{rz}\nu_{zr})}{D}$, $L_{zz} = \dfrac{E_z(1-\nu_{r\theta}\nu_{\theta r})}{D}$,

$$L_{r\theta} = \frac{E_r(\nu_{r\theta}-\nu_{rz}\nu_{z\theta})}{D} = \frac{E_\theta(\nu_{\theta r}-\nu_{\theta z}\nu_{zr})}{D},$$

$$L_{rz} = \frac{E_r(\nu_{rz}-\nu_{r\theta}\nu_{z\theta})}{D} = \frac{E_z(\nu_{zr}-\nu_{\theta r}\nu_{\theta z})}{D},$$

$$L_{\theta z} = \frac{E_\theta(\nu_{\theta z}-\nu_{\theta r}\nu_{rz})}{D} = \frac{E_\theta(\nu_{z\theta}-\nu_{zr}\nu_{r\theta})}{D},$$

$$D = [1 - (\nu_{r\theta}\nu_{\theta r}+\nu_{rz}\nu_{zr}+\nu_{\theta z}\nu_{z\theta}) - (\nu_{r\theta}\nu_{\theta z}\nu_{zr}+\nu_{\theta r}\nu_{z\theta}\nu_{rz})]$$

On the other hand the stresses and strains transformed by the angle φ from the coordinate system generated by the fiber direction $\begin{bmatrix} \sigma'_r & 0 & 0 \\ 0 & \sigma'_\theta & 0 \\ 0 & 0 & \theta'_z \end{bmatrix}$ to the system given by the original radial (r), hoop (θ) and axial (z) coordinates provide:

$$\sigma_r = \sigma'_r, \quad \sigma_\theta = \sigma'_\theta \cos^2\varphi + \sigma'_z \sin^2\varphi, \quad \sigma_z = \sigma'_z \cos^2\varphi + \sigma'_\theta \sin^2\varphi$$

$$\tau_{\theta z} = (\sigma'_\theta - \sigma'_z)\sin\varphi\cos\varphi \tag{9}$$

As said above, symmetric lay-up of $\pm\varphi$ declination from the z-axis is assumed, and the shear stress (similarly strain) disappears, i.e.

$$\tau_{\theta z} = (\sigma'_\theta - \sigma'_z)\sin\varphi\cos\varphi + (\sigma'_\theta - \sigma'_z)\sin(-\varphi)\cos(-\varphi) = 0 \tag{10}$$

Then a simple transformation rule holds valid:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\varphi & \sin^2\varphi \\ 0 & \sin^2\varphi & \cos^2\varphi \end{bmatrix} \begin{Bmatrix} \sigma'_r \\ \sigma'_\theta \\ \sigma'_z \end{Bmatrix} = \boldsymbol{T\sigma}', \quad \boldsymbol{\varepsilon} = \boldsymbol{T\varepsilon}' \tag{11}$$

Moreover, $\boldsymbol{\sigma}' = \boldsymbol{L}'\boldsymbol{\varepsilon}'$, $\boldsymbol{\sigma} = \boldsymbol{L\varepsilon}$ and

$$\boldsymbol{\sigma} = \boldsymbol{T\sigma}' = \boldsymbol{TL}'\boldsymbol{\varepsilon}' = \boldsymbol{TL}'\boldsymbol{T}^{-1}\boldsymbol{\varepsilon} = \boldsymbol{L\varepsilon} \Rightarrow \boldsymbol{L} = \boldsymbol{TL}'\boldsymbol{T}^{-1} \tag{12}$$

where

$$\boldsymbol{T}^{-1} = \frac{1}{\cos^2\varphi - \sin^2\varphi} \begin{bmatrix} \cos^2\varphi - \sin^2\varphi & 0 & 0 \\ 0 & \cos^2\varphi & -\sin^2\varphi \\ 0 & -\sin^2\varphi & \cos^2\varphi \end{bmatrix} \tag{13}$$

After substituting (13) into (12) it is seen that the material properties are dependent on the direction of the fiber on $\cos^2\varphi$ and $\sin^2\varphi$, see also (6). Similarly to the approach suggested in [4], where the decisive parameter was the volume fraction of fibers, here straightforwardly the assessment of material in an arbitrary direction φ being in a pair is

$$\boldsymbol{L}^{\varphi} = \boldsymbol{L}^{0}\cos^{2}\varphi + \boldsymbol{L}^{90}\sin^{2}\varphi \qquad (14)$$

where $\boldsymbol{L}^{\varphi}$ is the stiffness matrix in the φ direction, $\boldsymbol{L}^{0}$ is the stiffness matrix of $\varphi = 0$ and $\boldsymbol{L}^{90}$ is that of $\varphi = 90$. Recall that the angle φ is measured from the axial direction.

4 Algorithm of optimization

In the first step, optimization of laminated composite is carried out based on the algorithm described in [1], where no fabrication is considered, i.e. no effect of mandrel is employed. The solution is fully described in this paper and leads to the uniform stresses in each lamina in both the axial and hoop directions. Recall that the optimal solution is unique and exists under the condition that at least one pre-stress is given. This condition enables us to arrange that even only tensile stresses in fibers can be ensured as a result of the optimization; for details see [4] (compression in a fiber is not realistic, as the fibers work in a similar manner to ropes). In order to get uniform distribution of both axial and hoop stresses, we introduce the pre-stress in each lamina but one.

Suppose that we have available the coefficients of the stiffness matrix of the anisotropic composite involving fibers in the axial direction

$$\boldsymbol{L}^{0} = \begin{bmatrix} L_{rr}^{0} & L_{r\theta}^{0} & L_{rz}^{0} \\ L_{r\theta}^{0} & L_{\theta\theta}^{0} & L_{\theta z}^{0} \\ L_{rz}^{0} & L_{\theta z}^{0} & L_{zz}^{0} \end{bmatrix} \qquad (15)$$

and that in the hoop direction

$$\boldsymbol{L}^{90} = \begin{bmatrix} L_{rr}^{90} & L_{r\theta}^{90} & L_{rz}^{90} \\ L_{r\theta}^{90} & L_{\theta\theta}^{90} & L_{\theta z}^{90} \\ L_{rz}^{90} & L_{\theta z}^{90} & L_{zz}^{90} \end{bmatrix} \qquad 16)$$

Note that the value of uniformly distributed stresses along the thickness of the laminated cylinder is attained independently on the stiffness, as the result follows from statical equilibrium for a given geometry and hydrostatic loading. Let the hoop optimal stress be denoted as $\tilde{\sigma}_{\theta\theta}$ and in axial direction $\tilde{\sigma}_{zz}$, both being the same in each layer. The distribution of prestresses (eigenstresses) is strongly dependent on the stiffness matrices in the layers and the stiffnesses are again dependent on the orientation of the filaments (lay-ups) in each individual lamina. Consequently, we have to iterate the orientation of the fibers and accordingly also the current stiffness matrix. Recall that the goal of this optimization is to get such lay-ups, which deliver the correct pre-stressing force.

Procedure:

1. Carry out the optimization in the sense of [1] to get $\tilde{\sigma}_{\theta\theta}$ and $\tilde{\sigma}_{zz}$.
2. Select φ, say equal to 90°, and accomplish the optimization with the stiffness matrices according to (14), which are considered in each lamina. We get eigenstresses λ_i^θ and λ_i^z (in the first step $\lambda_i^z = 0$).
3. The angles φ corresponding to the optimal solution are calculated from (5) for each lamina.
4. Substituting the sines and cosines of the angles calculated in the previous step and the given matrices (15) and (16) into (14) yields a new matrix $\boldsymbol{L}^\varphi$ in each lamina.
5. Realize optimization with the new material coefficients in the sense of [1] to get $\tilde{\sigma}_{\theta\theta}$ and $\tilde{\sigma}_{zz}$ (they do not change) and also eigenstresses λ_i^θ and λ_i^z. New angles φ are calculated from (5) for each lamina.
6. If it holds that $|\varphi - \varphi^{\text{old}}| < \varepsilon$ everywhere, where φ^{old} is the angle obtained from the previous iteration step (the starting value is, according to our assumption, equal to 90°, but this choice is not mandatory, only $0 \le \varphi \le 90°$, and ε is the a priori selected error of iteration, then calculate the prestressing force P from (4). Otherwise go to point 4.

Note that the convergence is ensured for every choice of starting angle. In creating the algorithm for a computer program, the fact that the lay-ups do not differ in adjacent laminas too much can be utilized. It can speed up the consumption of computer time.

5 Example

In what follows, a laminate structure is considered with special realistic values of a classical composite based on an epoxy matrix. The material coefficients are taken from [6]. The stiffness coefficients of the AS4/3501-6 $(0_{12}/90_{38})$s laminate (in MPa) are listed below:

$$L_{rr}^0 = 14240, L_{r\theta}^0 = 6506, L_{rz}^0 = 5730, L_{\theta\theta}^0 = 41200.8, L_{\theta z}^0 = 11699.6, L_{zz}^0 = 94100$$

$$L_{rr}^{90} = 14240, L_{r\theta}^{90} = 5730, L_{tz}^{90} = 6506, L_{\theta\theta}^{90} = 94100, L_{\theta z}^{90} = 11699.6, L_{zz}^{90} = 41200.8$$

The loading is applied on the outer boundary of the hollow laminated composite cylinder by the hydrostatic pressure 800 MPa. The pre-stress is introduced in all layers but the first one, where implicitly zero eigenstress is assumed in both axial and hoop directions. The overall optimal stresses are $\tilde{\sigma}_{\theta\theta} = 1066.66$ MPa and $\tilde{\sigma}_{zz} = 853.33$ MPa. For $\varphi = 90°$ the axial and hoop pre-stresses are illustrated in Fig. 2. The following pictures are drawn in such a way that the horizontal axis describes the local radii along the thickness of the cylinder, i.e. the inner radius is 9.6 m and the outer radius is 10.0 m. The thickness is always measured in meters. In Fig. 2, the vertical axes describe the pre-stress in MPa. The triggering value in the iteration above described changes during the iteration and attains the values of the optimal degrees of filaments depicted in Fig. 3 and optimal forces in the fibers directed according to the orientation of angles. Here we considered the unit cell with the fiber volume ratio equal to 0.3, the radius of fibers is 1 mm, and the values of pre-stressing force are measured in the scale 10^5.

From Figs. 3 and 4 one can deduce that the improvement of the calculation is necessary to get more accurate results. This is obvious in case the only reinforcement and, consequently, the source of anisotropy are the filaments being winded according to the above calculation.

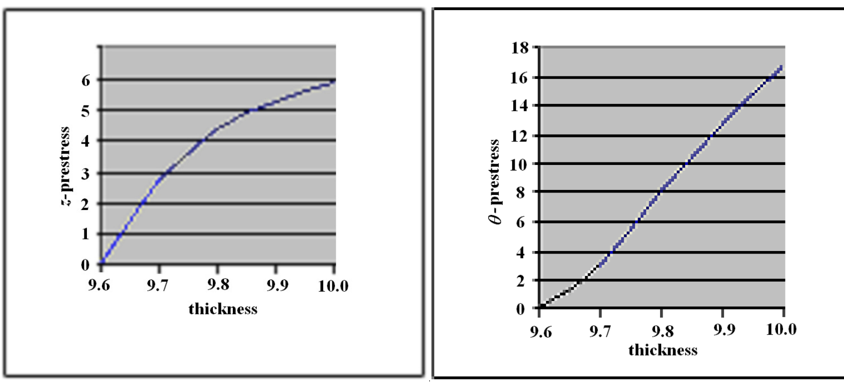

Figure 2: Pre-stress at φ = 90°.

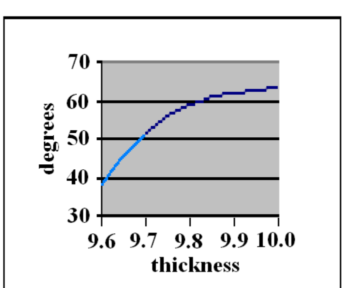

Figure 3: Optimal lay-ups.

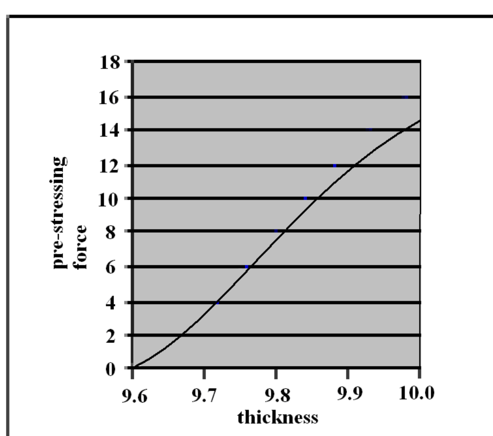

Figure 4: Optimal pre-stressing force.

6 Conclusion

Optimal pre-stress of a laminated anisotropic laminated cylinder under hydrostatic load is studied in this paper. More objective supposition is applied here: the fiber orientation is respected in the optimization for minimum stresses in both the axial and hoop directions. The radial direction is negligible; the stresses are relatively too small. Since the problem appears to be nonlinear, a natural iteration process is prescribed, starting from the orientation of fibers in the hoop direction. As a partial result, lay-ups in each lamina are derived, which are principally different from $\varphi = 90°$ or $\varphi = 0°$. They have to be taken into account when optimizing the pre-stress in one particular iteration step. The lay-up then generates the components of the stiffness matrices in laminas.

Acknowledgements

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