DISTANCE MEASURES FROM REPLICATOR SYSTEMS WITH NONLINEAR PAIRWISE INTERACTIONS FOR ENVIRONMENTAL RISK MANAGEMENT

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ABSTRACT

Replicator dynamics is an evolutionary strategy well established in different disciplines of environmental sciences. It describes the evolution of self-reproducing entities called replicators in various independent models of, for example, genetics, ecology, prebiotic evolution, and socio-biology. So, the replicator systems arise in an extraordinary variety of ecosystem modelling situations. In this report we combine the three universalisms in modelling: nonlinear pairwise interactions concept, dynamical systems theory and generalized relative entropy as a distance measure for environmental risk management. We introduce two hypotheses concerning the structure and types of properties of the system's entities' nonlinear interactions and derive generalized replicator dynamic equations. If there exists a nontrivial equilibrium point for a generalized replicator system then we construct an entropy-like Lyapunov-Meyer function (LMF) for this system and prove that it is a relative entropy function or the function of information divergence. We have proven that negative relative entropy is a convex function for a probability space and receive new distance measures (divergence) between two probability distributions. In conclusion, we show, as an example, that relative entropy as distance measures may be used as sustainability indicators for estimating the efficiency energy use of wastewater treatment. These results allow us to suggest that these new distance measures can be applicable to a wide set of real environmental situations.

Keywords: nonlinear pairwise interactions, replicator dynamics, Lyapunov functions, distance measure, divergence, sustainable development.

1 INTRODUCTION

Sustainable development is a complicated concept that has arisen to synthesize ideas that simultaneously address the risk of environmental impacts and the needs of people. Sustainable development commonly is defined as the process of meeting the needs of the present generation without compromising the ability of future generations to meet their needs. This concept arose in the late 1980s as an approach to balance economics and environment. Sustainable development embodies several advancements over other models that have characterized the development process. One advancement is drawing distinctions between the growth component and the efficiency component of development. It is evident that measures of divergence between two points play a key role in many environmental problems. One can find descriptions of energy, water and agricultural aspects of sustainable development in the books [1]–[3].

Mathematical models are useful tools for both understanding the dynamics of populations and impact assessment for several reasons. In the first place, models can lead to an increase of the general understanding of a system and help generate hypotheses and direct field experiments. In the second place, models can be used as a tool for evaluating alternative plans (scenarios) or by providing rules of thumb to policy makers.

Ecosystem stability and the response of ecosystems to disturbance are of crucial importance for conservation management. A variety of ecological interpretations have been given to the term "stability". The most generally used concept refers to the stability in the

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vicinity of an equilibrium point in a deterministic system. Pimm [4] distinguished two aspects of this stability: “resilience”, the speed at which a system returns to its equilibrium following a perturbation, and “resistance”, the ability of a system to withstand displacement by a disturbance in its environment. These two properties are usually held to be responsible for the persistence of ecosystems in an unpredictable environment. A “stable” ecosystem might be expected to exhibit both of these properties to some degree. The linkages between resilience and the stability of dynamical systems are discussed, along with its role in understanding of the evolution of such systems.

Many problems of investigating stability and calculating characteristics of dynamical systems (time and value of transient processes, speed of convergence, stability region, integral criteria, etc.) are solved using the direct Lyapunov method. The method is based on searching the function having certain properties along the system solutions. See, for example Common and Perrings [5].

The primary goal of this paper is to study the implications of Lyapunov functions theory for the analysis of nonlinear pairwise interactions replicator systems. We build Lyapunov-Meyer functions as entropy-like functions (distance measures). The existence of the Lyapunov-Meyer functions allows us to construct practically all known entropy and relative-entropy functions. Our investigations are based mainly on the stability postulate stated by Chetaev in 1936 [6].

Stability postulate. Stability, which is a fundamentally general phenomenon, apparently, must manifest itself in basic laws of nature in some way. If knowledge is constructed from the requirement of small deviations from nature, then scientific thinking must (or can) rely on some Lyapunov function. Certainly, this function always exists according to the stability postulate.

Many years later in 1967 stability postulate, in more narrow sense, was used by Coleman and Mizel in their famous paper “Existence of Entropy as a Consequence of Asymptotic stability” [7].

In 1968, this postulate was stated mathematically by Meyer [8], who proved that, for dynamical systems whose limit sets consist of only isolated rest points or cycles, i.e., for Morse–Smale systems, there always exists a Lyapunov function, which increases on the set of wandering points of the system. Meyer, followed by Smale [9], suggested the term energy functions for such functions; however, it is more natural and convenient to refer to them as Lyapunov–Meyer functions. One of the best-known examples of such function is entropy in Boltzmann’s H-theorem.

Prigogine [10] first pointed out the importance of the relationship between Lyapunov functions and entropy: “The positive time direction is associated with the increase of entropy. Let us emphasize the strong and very specific way in which the one-sidedness of time appears in the second law. According to its formulation it leads to the existence of a function having quite specific properties as expressed by the fact that for an isolated system it can only increase in time. Such function plays an important role in modern theory of stability as initiated by the classic work of Lyapunov. For this reason they are called Lyapunov functions (or functional).

The entropy S is a Lyapunov function for isolated systems. As shown in all textbooks, thermodynamic potentials such as the Helmholtz or Gibbs free energy are also Lyapunov functions for other “boundary conditions” (such as imposed values of temperature and volume). In all these cases the system evolves to an equilibrium state characterized by the existence of a thermodynamic potential. This equilibrium state is an “attractor” for non-equilibrium states.

For dynamical systems arising from physics the Lyapunov functions will typically have a thermodynamic interpretation (energy, entropy etc.) but its origin is not evident.

In an attempt to show how such an interpretation this idea might proceed, we analyze below a class of macrosystems with nonlinear pairwise interactions for with an assumption of structure and type of interactions has consequences that there exist entropy-like functions.

2 EQUATIONS OF MACROSYSTEMS WITH NONLINEAR PAIRWISE INTERACTIONS

One of the important assumptions in classic Lotka-Volterra and replicator systems is that the functional responses are linear functional responses. Usually additive linear models for predicting combined interactions effects cannot account for non-linearities in combined functional response introduced by non-trophic interactions Ayala et al. [11]. Exponential and power law responses were used to model social dynamics by Beechman and Farnsworth [12]. The PhD thesis by Merrifield [13] provides a review on the concept of a force including nonlinear in different interacting systems in Chapter 5.

Consider a macrosystem formed by a sufficiently large number *N* of interacting objects. The classical definition of macrosystems is as follows: these are systems in which a chaotic behavior at the microlevel transforms into a deterministic behavior at the macrolevel. Suppose that, at a moment *t,* the macrosystem under consideration contains *n* different types of objects and the number of objects of type i is $x_i(t)$, where $i=1,...,n$ and $\sum_{i=1}^{n} x_i(t) = N(t)$. Consider the relative numbers $p_i(t) = x_i(t)/N(t)$, $p_i(t)$ of various types' objects. Obviously, $\sum_{i=1}^{n} p_i(t) = 1$, i.e., vector $\mathrm{p}(t) \in \sigma_p^n = \{\mathrm{p} \in \mathbb{R}^n : p_i \geq 0, i = 1,2,...,n, \mathbf{e}^T \mathrm{p} = 1\}$, where σ_p^n is the standard simplex in Euclidean *n*-space $\mathbb{R}^n$ and, $\mathbf{e}$ is the vector of ones. Thus, the state of such a macrosystem at each moment t *is* determined by the vector $\mathrm{p}(t) = (p_i(t),..., p_n(t))$.

We consider the systems in which only pairwise interactions occur during a short time interval $(t, t + \vartriangle t)$; in other words, those systems in which simultaneous interactions of more than two particles are impossible.

We make the following two assumptions about interactions between objects in a macrosystem [14].

Hypothesis 1: The interaction between objects of types i and j is characterized by the so-called interaction strength, which we denote by w_{ij} and regard as a quantitative characteristic of the effect of the interaction between two objects of types i and j on the rate of change of the relative number $p_i(t)$ of objects of type i.

Remark 1: Apparently, the notion of an interaction strength first appeared in mathematical ecology. A fairly detailed study of this notion and a survey of related results are contained in Ayala et al. [11].

Remark 2: Note that the definition given above does not imply that $w_{ij} = w_{ji}$. We also emphasize that the asymmetry of results of interaction plays an essential role in many cases [15].

Hypothesis 2: For each macrosystem under consideration, there exists a set of probability distribution functions $f_i(p_i)$, where $i = 1,\ldots,n$, which determine the probability of the interaction of each object of type i with any other object in the macrosystems. Thus, the probability of the pairwise interaction between objects of types i and j is determined by the product $f_i(p_i)f_i(p_i)$.

Remark 3: The probability distribution functions f_i (nonlinear response functions) obeys the conditions $f_i(0)=0$, $\partial f_i/\partial p_i > 0$ for $p_i > 0$, $\partial f_i/\partial p_i \geq 0$ for $p_i = 0$ and $f_i(1)=1$ [16].

Note that the standardization condition for response functions was investigated in the article by Pykh and Malkina-Pykh [17].

It follows from the two hypotheses stated above that, for small $\vartriangle t$, the dynamics of the system is determined by the relation:

$$p_i\left(t+\vartriangle t\right)= p_i\left(t\right)+\vartriangle t\sum_{j=1}^{n} w_{ij} f_i\left(p_i\left(t\right)\right) f_j\left(p_j\left(t\right)\right)+ o\left(\vartriangle t\right) ,$$

subject to the constraints $\sum_{i=1}^{n} p_i(t)-1=0,$ and $f_i(0)=0,$ for $i=1,2,...,n$ which ensure the invariance of the simplex σ_p^n and all of its faces. Using an approach common in analytic mechanics, we treat the system under consideration as a nonfree system subject to the ideal holonomic constraints:

$$f_i\left(p_i\right)\left(\sum_{i=1}^{n} p_i(t)-1\right)=0, \quad i=1,...,n$$

and apply the principle of replacing constraints by reaction forces to this system [18]. Passing to the limit as $\vartriangle t \to 0$ and taking into account the obvious identity $\sum \dot{p}_i = 0$ for the sum of derivatives, we obtain the following equation for the constraint multipliers (the indeterminate Lagrange multipliers) λ_i :

$$\sum_{ij}^{n} w_{ij} f_i\left(p_i\right) f_j\left(p_j\right)-\sum_{i=1}^{n} \lambda_i f_i\left(p_i\right)=0$$

This equation has two obvious solutions, the trivial solution $\lambda_i = \sum_{j=1}^{n} w_{ij} f_j\left(p_j\right)$ and the nontrivial solution:

$$\lambda_i = \lambda = \frac{\sum_{ij}^{n} w_{ij} f_i\left(p_i\right) f_j\left(p_j\right)}{\sum_{i=1}^{n} f_i\left(p_i\right)}, \quad i=1,...,n$$

Using the nontrivial solution and denoting the sum $\sum_{i=1}^{n} f_i\left(p_i\right)$ by $\theta(\mathrm{p})$, we obtain the following system of differential equations determining the evolution of the probability distribution $\mathrm{p}(t)$:

$$\dot{p}_i = f_i\left(p_i\right)\left(\sum_{j=1}^{n} w_{ij} f_j\left(p_j\right)-\theta^{-1}(\mathrm{p})\sum_{ij}^{n} w_{ij} f_i\left(p_i\right) f_j\left(p_j\right)\right), \quad i=1,...,n . \qquad (1)$$

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This system of equations was first obtained from balance considerations in Pykh [19] without the use of the Lagrange method. In what follows, it is convenient to pass to the matrix form. System (1) is written in this form as:

$$\dot{\mathrm{p}} = D(\mathrm{f})\left(\mathrm{Wf} - \mathbf{e}\theta^{-1}(\mathrm{p})\langle \mathrm{f}, \mathrm{Wf}\rangle\right). \tag{2}$$

Here, $\mathrm{f}(\mathrm{p})$ is the vector $\left(f_1(p_1),...,f_n(p_n)\right)$; $D(\mathrm{f}) = diag\left(f_1, f_2..., f_n\right)$; $\mathrm{W} = \left(w_{ij}\right)$ is the matrix of interactions; and $\theta(\mathrm{p}) = \langle \mathbf{e}, \mathrm{f}(\mathrm{p})\rangle$, where $\langle \cdot, \cdot \rangle$ denotes inner product. Obviously, since $\langle \dot{\mathrm{p}}(t), \mathbf{e}\rangle \equiv 0$ and $f_i(0) = 0$ it follows that the simplex σ_p^n and each of its faces are invariant sets for system (2). Note that system (2) determines the dynamics of objects with nonlinear pairwise interactions; the matrix W determines the structure of interactions, and the response functions determine their types. The elements of the matrix W are generalized interaction strengths, i.e., by analogy with the thermodynamics of irreversible processes, are "reasons" causing changes in the speed of flows.

3 ESCORT DISTRIBUTIONS

Let us rewrite eqn (2) as:

$$\dot{\mathrm{p}} = \theta D(\mathrm{f})\left(\mathrm{Wf}\theta^{-1} - \mathbf{e}\mathrm{E}(\mathrm{p})\right), \tag{3}$$

where $\mathrm{E}(\mathrm{p}) = \theta^{-2}(\mathrm{p})\langle \mathrm{f}, \mathrm{Wf}\rangle$. Using the terminology of the theory of neural networks, we refer to $\mathrm{E}(\mathrm{p})$ as the energy function of the macrosystem under consideration. Note that in the population genetic theory this function named as population fitness [20], and in the evolutionary game theory as utility function [21]. System (3) and the energy function $\mathrm{E}(\mathrm{p})$ naturally determine the introduction of new additional variables:

$$x_i(\mathrm{p}) = f_i(p_i)\theta^{-1}(\mathrm{p}) \qquad i = 1,...,n. \tag{4}$$

Obviously, $\mathrm{x} = (x_1, x_2..., x_n) \in \sigma_x^n = \left\{\mathrm{x} \in \mathbb{R}^n : x_i \geq 0, \mathbf{e}^{\mathrm{T}}\mathrm{x} = 1\right\}$ for $\mathrm{p} \in \sigma_p^n$. The indices x and p are used in the notation of simplexes in order to avoid confusion. Consider change (4) in more detail. If this is a diffeo-morphism, then it can be regarded not only as a simplifying change of variables customary in the theory of differential equations but also as the definition of a set of quantities with particular physical meaning.

Theorem 1. [22]: Under the conditions (remark 3) used for the response functions f_i, for $\mathrm{p} \in \sigma_p^n$, there exists a one-to-one inverse mapping to (4), which is defined by:

$$p_i = f_i^{-1}(x_i) \Big/ \sum_{j=1}^n f_j^{-1}(x_j), \quad i = 1,...,n, \tag{5}$$

where $f_i^{-1}(\cdot)$ denotes the function inverse to $f_i(\cdot)$.

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Remark 4: Recall that the notation $f_i^{-1}(\cdot)$ is used for both functions inverse in the sense of function theory and functions inverse in the algebraic sense. It is always clear from the context what is meant.

4 EQUILIBRIUMS

To go further, we need the following assertion [22]:

Statement 1: System (1) is invariant with respect to the replacement of the interaction matrix W by a perturbed matrix $\mathrm{W}_\zeta = \left(\mathrm{W} + \mathrm{e}\zeta^T(\mathrm{p})\right)$, where the components of the vector function $\zeta(\mathrm{p}) = (\zeta_1(\mathrm{p}),...,\zeta_n(\mathrm{p})) : \sigma_p^n \to \mathbb{R}^n$ are bounded on σ_p^n.

We proceed to consider questions related to the existence of a nontrivial equilibrium point $\hat{\mathrm{p}} \in \mathrm{Int}\,\sigma_p^n$ for system (1). In Pykh [23], it was proved that $\hat{\mathrm{p}} > 0$ exists if and only if all components of the vector $\mathrm{W}^{-1}\mathrm{e}$ are of the same sign. It is clear already from this result that properties of the inverse interaction matrix play a substantial role in the evolution of macrosystems.

Statement 1 implies that the coordinates of a nontrivial equilibrium point remain invariable $\forall\zeta \in \mathbb{R}^n$. So we have:

Statement 2. [22]: If system (4) has a nontrivial equilibrium point $\hat{\mathrm{p}} \in \mathrm{Int}\,\sigma_p^n$, then:

$$\frac{\hat{\mathrm{f}}}{\left\langle \hat{\mathrm{f}}, \mathrm{e} \right\rangle} = \frac{\mathrm{W}^{-1}\mathrm{e}}{\left\langle \mathrm{e}, \mathrm{W}^{-1}\mathrm{e} \right\rangle} = \frac{\mathrm{W}_\zeta^{-1}\mathrm{e}}{\left\langle \mathrm{e}, \mathrm{W}_\zeta^{-1}\mathrm{e} \right\rangle} = \hat{\mathrm{x}} \qquad \forall\zeta(\mathrm{p}) \in \mathbb{R}^n. \tag{6}$$

5 ENTROPY-LIKE LYAPUNOV-MEYER FUNCTIONS

Theorem 2. [22]: If system (1) has a nontrivial equilibrium point $\hat{\mathrm{p}} \in \mathrm{Int}\,\sigma_p^n$ and the matrix $\left(\mathrm{W}_\zeta^{\mathrm{T}} + \mathrm{W}_\zeta\right)$, $\forall\zeta \in \mathbb{R}^n$ has $n-1$ negative characteristic numbers, then the function:

$$H(\mathrm{p}) = \sum_{i=1}^{n} \int_{\hat{p}_i}^{p_i} \frac{\hat{f}_i dx}{f_i(x)} + C, \tag{7}$$

where C is a constant, is a Lyapunov-Meyer function for system (1) in $\mathrm{Int}\,\sigma_p^n$, and the energy function $E(\mathrm{p})$ of the system attains its maximum value $E(\hat{\mathrm{p}})$ as $\mathrm{t} \to \infty$.

It is obvious that function $H(\mathrm{p})$ depends from interaction matrix W only through $\hat{\mathrm{p}}$ and its characteristic numbers. It means that for each $H(\mathrm{p})$ we have a set of different systems (1) and can use this function for any two points $\mathrm{p}(t)$, $\hat{\mathrm{p}} \in \mathrm{Int}\,\sigma_p^n$.

In the next section we use Theorem 2 to construct entropy characteristics and "distances" between probability characteristics. First, we note that, if $G : \mathbb{R}^1 \to \mathbb{R}^1$ is a monotonically increasing smooth function and $C_1 > 0$, $C_2 > 0$ and C_3 are the numbers, then the function:

$$H_G = C_1 G\left(C_2 H(\mathrm{p})\right) + C_3$$

is also a Lyapunov-Meyer function for system (1).

6 DISTANCE MEASURES

Following Shannon information measure let us introduce $S(\mathrm{p})$ as negative entropy defined by:

$$S(\mathrm{p}) = -H(\mathrm{p}) = -\sum_{i=1}^{n}\int_{\hat{p}_i}^{p_i} \frac{\hat{f}_i dx}{f_i(x)}, \qquad p \in \sigma_p^n .$$

It is easy to see, that:

$$\lambda \sum_{i=1}^{n}\int_{\hat{p}_i}^{p_i} dx = \lambda \sum_{i=1}^{n}(p_i - \hat{p}_i) \equiv 0 \quad \forall p, \hat{p} \in \sigma_p^n$$

and we clearly have the following relationship:

$$S(\mathrm{p}) = -\sum_{i=1}^{n}\int_{\hat{p}_i}^{p_i} \left(\frac{\hat{f}_i dx}{f_i(x)} + \lambda \right) dx \quad \forall \lambda \in \mathbb{R}^1 .$$

Since, by the evident equality $\partial S(\mathrm{p})/\partial p_i = 0$, with $p = \hat{p}$, we have that $\lambda = -1$. It is clear, that λ is a type of Lagrange multiplier. Therefore:

$$S(\mathrm{p}) = \sum_{i=1}^{n}\int_{\hat{p}_i}^{p_i} \left(1 - \frac{\hat{f}_i}{f_i(x)} \right) dx \qquad p \in \sigma_p^n . \tag{8}$$

The gradient of $S(p)$ with $p \in \mathrm{Int}\sigma_p^n$ is given by partial derivatives:

$$\nabla S(\mathrm{p}) = \left(1 - \frac{\hat{f}_i}{f_i(p_i)} \right)$$

that is strictly monotonically increasing for each p_i and Hessian of $S(p)$ on $\mathrm{Int}\sigma_p^n$ is $\mathcal{H}(\mathrm{p}) = diag\left(\hat{f}_i \frac{\partial f_i}{\partial p_i} f_i^{-2}(p_i) \right)$. Since $\partial f_i/\partial p_i > 0$ it follows that the Hessian is positive definite, and the function $S(p)$ according to definition is convex on $\mathrm{Int}\sigma_p^n$.

Another very well-known definition of convexity Jensen [24], Hardy et al. [25] is the next inequality with $p, \mathrm{q} = \mathrm{Int}\sigma_p^n$:

$$S(\mathrm{p}) - S(\mathrm{q}) - \langle \nabla_q S(\mathrm{q}), \mathrm{p} - \mathrm{q} \rangle \geq 0 . \tag{9}$$

Evident that the expression from left-side inequality is the Bregman divergences denote by $B_s(\mathrm{p,q})$. This name was first given the by Censor and Lent [26]. Bregman divergence or Bregman distance [27] is similar to a metric but does not satisfy the triangle inequality or symmetry. Using inequality (9) we can receive new weighted distance measure between two probability distributions:

$$B_s(\mathrm{p,q}) = \sum_{i=1}^{n}\left(\int_{p_i}^{q_i}\frac{\hat{f}_i dx}{f_i(x)} + \frac{\hat{f}_i}{f_i(q_i)}(p_i - q_i)\right) \geq 0$$

It is obvious that $B_s(\mathrm{p,q})$ for p=q , and $B_s(\mathrm{p,\hat{p}}) = S(p)$.

7 LEGENDRE-DONKIN-FENCHEL TRANSFORMATIONS

The Legendre-Donkin-Fenchel (LDF) transformation is a mathematical concept of great significance to thermodynamics, mechanics and probability theory is named after Legendre [28], Donkin [29] and Fenchel [30].

Let us denote $y_i = \partial S(\mathrm{p})/\partial p_i$. Since y_i is strictly monotonically increasing for p_i , this makes it clear that y_i and p_i are mutually reciprocal. The transformation from p to y is called LDF-transformation. We can find a convex function Υ of y defined by:

$$\Upsilon(y) = \sum p_i y_i - S(p). \tag{10}$$

when y and p are respective coordinates of the same point, and the inverse transformation from y to p is given by the gradient:

$$p = \nabla \Upsilon(y)$$

Concerning the (8) we clearly have:

$$y_i = \frac{\partial S(p)}{\partial p_i} = 1 - \frac{\hat{f}_i}{f_i(p_i)}$$

and:

$$p_i = f_i^{-1}\left(\frac{\hat{f}_i}{1 - y_i}\right) = \frac{\partial Y}{\partial y_i}$$

for $y_i \in \left(-\infty, 1 - \hat{f}_i\right)$. We also denote $\partial Y/\partial y_i = \varphi(y_i)$ and after simple counting we have the following relationship:

$$\Upsilon(y) = \sum_{i=1}^{n}\left(y_i\varphi(y_i) + \int_{0}^{\varphi(y_i)}\left(\frac{\hat{f}_i}{f_i} - 1\right)dx\right)$$

After simple computation one can check that $\partial Y/\partial y_i = p_i$. Another form of LDF-transformation, together with (10) is:

$$\Upsilon(y) = \max_p\left(\langle p, y\rangle - S(p)\right)$$

Follow Amari [31] and definition (10) one can define a divergence function (distance measure) between two points p and q on σ_p^n in the dual coordinates. This divergence function will be similar to the squared Riemannian distance. Additional new information on LDF-transformation is presented in Nielsen [32] and Zia et al. [33] papers.

8 ENTROPY CHARACTERISTICS AND DISTANCE FUNCTIONS

Note that Theorem 1 can be used in two ways:

1. The response functions can be found for previously known entropy characteristics. Relevant examples for the Shannon and Tsallis entropy were given in Pykh [34]. It is easy to show that this approach yields response functions for all the entropy characteristics proposed in Esteban and Morales [35].
2. New entropy characteristic can be derived from some functions obeying the conditions stated for response functions. As an example, we consider the logistic function, which is widely applied in mathematical ecology and economics. This function is defined as follows:

$$f(x)=\frac{c}{(b+c)}\frac{(1-e^{-\alpha x})}{(b+ce^{-\alpha x})},$$

where $b>0$, $c>0$, $\alpha>0$.

It is obvious that $f(0)=0, f(+\infty)=c/b(b+c)$. Evaluating the corresponding integral and taking into account that arbitrary parameters can be picked in (8) (to get rid of the factors and summands), we derive the following expression for the logistic entropy $H_l(p)$:

$$H_l(p)=\sum_{i=1}^{n}\ln(1-e^{-\alpha p_i}).$$

Recall that $H_l(p)$ is a Lyapunov energy function for system (1) if and only if the conditions of Theorem 2 are satisfied and the equilibrium $\hat{p}$ is a uniform distribution; i.e. $\hat{p}=en^{-1}$. Since all the response functions are identical in the case under study, it is necessary and sufficient that W be quasi-stochastic. Note that the interest in objects described by noncanonical distributions has increased noticeably over resent years [35]. It is this type of distribution that can be derived from formula (8) suggested for entropy characteristics. For further use of these characteristics, it is natural to apply Jaynes' principle of maximum information entropy [36].

Now, we underline that distance functions are defined in the case where the equilibrium distribution in system (1) is not uniform. As in the previous case, we have that new distance functions between distributions can be derived from given response functions. As before, we consider the logistic function as an example. Let the conditions of Theorem 2 be satisfied. Evaluating the integral in (8) gives the function

$$H_l(p,\hat{p})=\sum_{i=1}^{n}\hat{f}_i\left(\frac{1-e^{-\alpha p_i}}{1-e^{-\alpha\hat{p}_i}}\right), \tag{11}$$

that, by analogy, is naturally called the relative logistic entropy.

9 ESTIMATIING OF WASTEWATE TREATMENT (EXAMPLE)

Sustainable development is one of the great hopes for conservation and proper use of resources including biodiversity. Instead of a focus on biodiversity itself, ecosystems sustainability represents a focus on a set of ecosystems properties, especially on ecosystems resilience, persistence and integrity [37]. One of the most important characteristics among them is the measures of divergence between two points that plays a key role in many ecosystems' problems. As it was shown in the article of Sobczuk and Lagod on the base of the results from [37]–[38] for Wastewater Treatment Plant (WWTP) [39]: "the results of bioindication, apart from the other physical and chemical parameters, may be used in wastewater plant treatment process management". Note that authors mean there "sustainable management".

In Sobczuk and Lagod [39], the comparison of different distance measures presented in Fig. 1 demonstrates that function (11) is the best distance measure of all applied methods for estimating the efficiency of wastewater treatment. Also in Sobczuk and Lagod [39], the authors pointed out that entropy and distance measures should be appropriately chosen for complex data sets comparison. Eqn (1) with accompanying theory allows to generate distance measures according to data set character. It shows up that the distance measure (11) is best fitted to recognize differences in microbial distribution functions described there.

10 CONCLUSIONS

We note that the approach suggested for constructing entropy characteristics and divergences between probability distributions based on the direct Lyapunov method applied to kinetic replicator equations (i.e., equations determining the dynamics of systems with nonlinear pair interactions) provides new strategies in the macroscopic analysis of complex open objects in various areas of the natural sciences. It should also be emphasized that this approach makes it possible to introduce mixed entropy, i.e., an entropy characteristic arising when the interacting subsystems have different response functions. Due to the possibility to generate measures according to the functional form of the response function in (1) one can find the best distance measure for the certain purpose. More information about the systems with pairwise interactions can be obtained in the preprint [40].

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