Dynamic analysis of plates stiffened by parallel beams

E. J. Sapountzakis & V. G. Mokos
School of Civil Engineering, National Technical University, Athens, Greece

Abstract

In this paper a general solution for the dynamic analysis of plates stiffened by arbitrarily placed parallel beams of arbitrary cross section subjected to an arbitrary dynamic loading is presented. According to the proposed model, the stiffening beams are isolated from the plate by sections in the lower outer surface of the plate, taking into account the arising tractions in all directions at the fictitious interfaces. The aforementioned integrated tractions result in the loading of the beams as well as the additional loading of the plate. Their distribution is established by applying continuity conditions in all directions at the interfaces. The analysis of both the plate and the beams is accomplished on their deformed shape taking into account second-order effects. The method of analysis is based on the capability to establish a flexibility matrix with respect to a set of nodal mass points, while a lumped mass matrix is constructed from the tributary mass areas to these mass points. Both free and forced damped or undamped transverse vibrations are considered and numerical examples with great practical interest are presented. The discrepancy in the obtained eigenfrequencies using the presented analysis (which approximates better the actual response of the plate-beams system since it permits the evaluation of the shear forces at the interfaces in both directions) and the corresponding ones ignoring the inplane forces and deformations justify the analysis based on the proposed model.
Keywords: reinforced plate with beams, nonuniform torsion, warping, ribbed plate, slab-and-beam structure, vibrations, dynamic analysis.

WIT Transactions on The Built Environment, Vol 93, © 2007 WIT Press
www.witpress.com, ISSN 1743-3509 (on-line)
doi:10.2495/ERES070421

1 Introduction

Structural plate systems stiffened by beams are widely used in buildings, bridges, ships, aircrafts and machines. In this paper a general solution for the analysis of plates stiffened by arbitrarily placed parallel beams is presented. The adopted structural model is a refined one of that proposed by Sapountzakis and Katsikadelis in [1]. Six boundary value problems with respect to the plate transverse deflection, to the plate inplane displacement components, to the beam transverse deflections, to the beam axial deformation and to the beam nonuniform angle of twist are formulated and solved using the Analog Equation Method (AEM) [2], a BEM based method. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

i. The stiffened plate is subjected to an arbitrary dynamic loading, while both the number and the placement of the parallel stiffening beams are also arbitrary (eccentric beams are also included).
ii. The influence of the transverse traction component at plate-beams interfaces is taken into account. A nonuniform variation of the distribution of the transverse shear interface force is taken into account by applying compatibility equations on points in the transverse direction. Thus, the adopted model permits the evaluation of the shear connectors in both directions.
iii. Displacement continuity conditions at the interfaces are applied along all three axes of the coordinate system, leading to the formulation of a system of equations involving two nonlinear functions, namely the longitudinal and transverse inplane shear forces at the interfaces.
iv. The eccentricities of both the centroid and the shear center axes with respect to the midline of the plate – beam interface are also included.
v. The nonuniform torsion in which the stiffening beams are subjected is taken into account by solving the corresponding problem and by comprehending the arising twisting and warping in the corresponding displacement continuity conditions.
vi. Terms arising from the internal variable axial loading of both the plate and the beams coming from the longitudinal and transverse inplane shear forces at the interfaces are taken into account.
vii. Damping resistance is also included.

2 Statement of the problem

Consider a thin plate of homogeneous, isotropic and linearly elastic material with modulus of elasticity E and Poisson ratio ν, having constant thickness h_p and occupying the two dimensional multiply connected region Ω of the x,y plane bounded by the piecewise smooth $K+1$ curves $\Gamma_0,\Gamma_1,...,\Gamma_{K-1},\Gamma_K$, as shown in Fig.1. The plate is stiffened by a set of $i=1,2,...,I$ arbitrarily placed parallel beams of homogeneous, isotropic and linearly elastic material with modulus of elasticity E_b^i and Poisson ratio v_b^i, which may have either internal or boundary point supports. For the shake of convenience the x axis is taken parallel to the beams. The stiffened plate is subjected to the lateral load $g = g(\mathbf{x},t)$, $\mathbf{x} : \{x,y\}, t \geq 0$. For the analysis of the aforementioned problem a global coordinate system Oxy for the analysis of the plate and local coordinate ones $O_i x_i y_i$ and $O_i \tilde{x}_i \tilde{y}_i$ corresponding to the centroid and shear center axes of each beam are employed as shown in Fig.1.

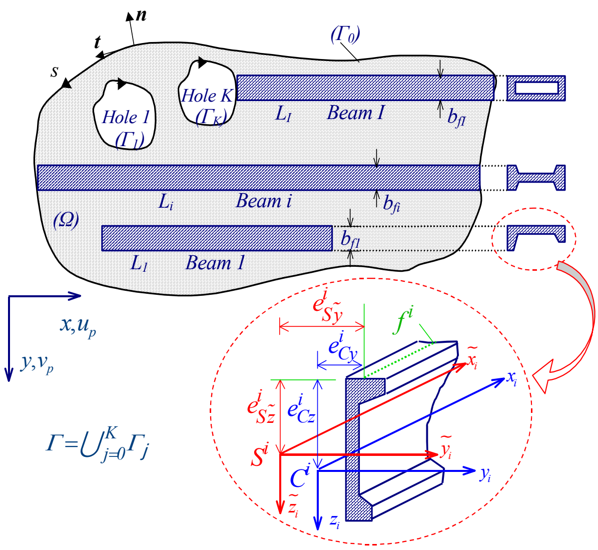

Figure 1: Two-dimensional region Ω occupied by the plate.

The solution of the problem at hand is approached by a refined model of that proposed by Sapountzakis and Katsikadelis in [1]. According to this model, the stiffening beams are isolated from the plate by sections in the lower outer surface of the plate, taking into account the arising tractions at the fictitious interfaces (Fig.2). Integration of these tractions along the width of the i-th beam results in line forces per unit length, which are denoted by q_x^i, q_y^i and q_z^i encountering in this way the influence of the transverse component q_y, which in the aforementioned model [1] was ignored. The aforementioned integrated tractions result in the loading of the i-th beam as well as the additional loading of the plate. Their distribution is unknown and can be established by imposing displacement continuity conditions at the interfaces along x_i, y_i and z_i local axes following the procedure developed in this investigation.

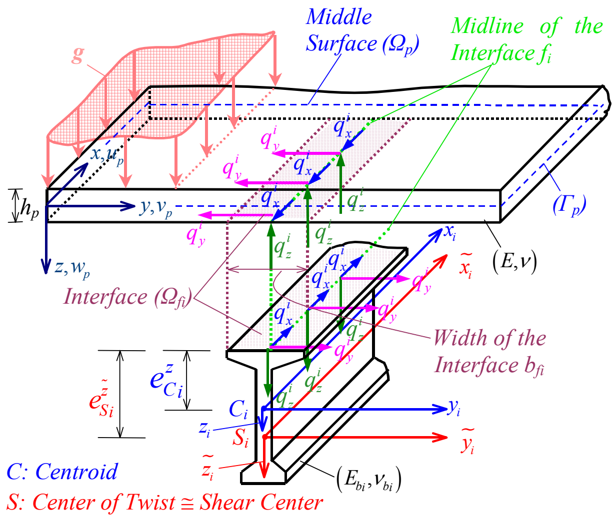

Figure 2: Isolation of the beams from the plate.

On the basis of the above considerations the response of the plate and of the beams may be described by the following initial boundary value problems.

(a) For the plate. The plate undergoes transverse deflection and inplane deformation. Thus, for the transverse deflection the equation of equilibrium employing the linearized second order theory can be written as

$$D\nabla^4 w_p + \rho_p \ddot{w}_p + c_p \dot{w}_p - \left(N_x \frac{\partial^2 w_p}{\partial x^2} + 2N_{xy} \frac{\partial^2 w_p}{\partial x \partial y} + N_y \frac{\partial^2 w_p}{\partial y^2} \right) = g - \sum_{i=1}^{I} \left(q_z^i + \frac{\partial m_{py}^i}{\partial x} + \frac{\partial m_{px}^i}{\partial y} - q_x^i \frac{\partial w_p}{\partial x} - q_y^i \frac{\partial w_p}{\partial y} \right) \delta(y - y_i) \quad \text{in } \Omega \qquad (1)$$

the corresponding boundary conditions as

$$\alpha_{p1} w_p + \alpha_{p2} R_{pn} = \alpha_{p3} \qquad \beta_{p1} \frac{\partial w_p}{\partial n} + \beta_{p2} M_{pn} = \beta_{p3} \qquad \text{on } \Gamma \qquad (2a,b)$$

and the initial conditions as

$$w_p(\mathbf{x}, 0) = w_{p0}(\mathbf{x}) \qquad \dot{w}_p(\mathbf{x}, 0) = \bar{w}_{p0}(\mathbf{x}) \qquad (3a,b)$$

where $w_p = w_p(x, y)$ is the time dependent transverse deflection of the plate; $D = Eh_p^3 / 12(1 - \nu^2)$ is its flexural rigidity; $N_x = N_x(\mathbf{x}, t)$, $N_y = N_y(\mathbf{x}, t)$, $N_{xy} = N_{xy}(\mathbf{x}, t)$ are the membrane forces per unit length of the plate cross section; $m_{py}^i = q_x^i h_p / 2$; $m_{px}^i = q_y^i h_p / 2$; $\rho_p = \rho h_p$ is the surface mass density of the plate with ρ being the volume mass density; c_p is the plate flexural damping constant; $w_{p0}(\mathbf{x})$, $\bar{w}_{p0}(\mathbf{x})$ are the initial deflection and the initial velocity of the points of the middle surface of the plate; $\delta(y-y_i)$ is the Dirac's delta function in the y direction; M_{pn} and R_{pn} are the bending moment normal to the boundary and the effective reaction along it, respectively. Finally, a_{pi}, β_{pi} $(i=1,2,3)$ are functions specified on the boundary Γ.

Since linearized plate bending theory is considered, the components of the membrane forces N_x, N_y, N_{xy} are given as

$$N_x = C\left(\frac{\partial u_p}{\partial x} + \nu\frac{\partial v_p}{\partial y}\right) \qquad N_y = C\left(\nu\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y}\right)$$

$$N_{xy} = C\frac{1-\nu}{2}\left(\frac{\partial u_p}{\partial y} + \frac{\partial v_p}{\partial x}\right) \tag{4a,b,c}$$

where $C = Eh_p/\left(1-\nu^2\right)$; $u_p = u_p(\mathbf{x},t)$ and $v_p = v_p(\mathbf{x},t)$ are the displacement components of the middle surface of the plate arising from the line body forces q_x^i, q_y^i $(i=1,2,...I)$. These displacement components are established by solving independently the plane stress problem, which is described by the following quasi-static (inplane inertia forces are ignored) boundary value problem (Navier's equations of equilibrium)

$$\nabla^2 u_p + \frac{1+\nu}{1-\nu}\frac{\partial}{\partial x}\left[\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y}\right] - \frac{1}{Gh_p}\sum_{i=1}^{I} q_x^i \delta\left(y-y_i\right) = 0 \tag{5a}$$

$$\nabla^2 v_p + \frac{1+\nu}{1-\nu}\frac{\partial}{\partial y}\left[\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y}\right] - \frac{1}{Gh_p}\sum_{i=1}^{I} q_y^i \delta\left(y-y_i\right) = 0 \quad \text{in } \Omega \tag{5b}$$

$$\gamma_{p1}u_{pn} + \gamma_{p2}N_n = \gamma_{p3} \qquad \delta_{p1}u_{pt} + \delta_{p2}N_t = \delta_{p3} \qquad \text{on } \Gamma \tag{6a,b}$$

in which $G = E/2(1+\nu)$ is the shear modulus of the plate; N_n, N_t and u_{pn}, u_{pt} are the boundary membrane forces and displacements in the normal and tangential directions to the boundary, respectively; γ_{pi}, δ_{pi} $(i=1,2,3)$ are functions specified on the boundary Γ.

(b) For each beam. Each beam undergoes transverse deflection with respect to z_i and y_i axes, axial deformation along x_i axis and nonuniform angle of twist along $\tilde{x}_i$ axis. Thus, for the transverse deflection with respect to z_i axis the equation of equilibrium employing the linearized second order theory can be written as

$$E_b^i I_y^i \frac{\partial^4 w_b^i}{\partial x_i^4} + \rho_b \ddot{w}_b^i + c_b^i \dot{w}_b^i - N_b^i \frac{\partial^2 w_b^i}{\partial x_i^2} = q_z^i - q_x^i \frac{\partial w_b^i}{\partial x_i} + \frac{\partial m_{by}^i}{\partial x_i} \quad \text{in } L_i \tag{7}$$

the corresponding boundary conditions as

$$a_{1i}^{z} w_{b}^{i} + a_{2i}^{z} R_{z}^{i} = a_{3i}^{z} \qquad \beta_{1i}^{z} \theta_{y}^{i} + \beta_{2i}^{z} M_{y}^{i} = \beta_{3i}^{z} \qquad \text{at the beam ends} \qquad (8a,b)$$

and the initial conditions as

$$w_{b}^{i}\left(x,0\right) = w_{b0}^{i}\left(x\right) \qquad \dot{w}_{b}^{i}\left(x,0\right) = \overline{w}_{b0}^{i}\left(x\right) \qquad (9a,b)$$

where $w_{b}^{i} = w_{b}^{i}\left(x_{i},t\right)$ is the time dependent transverse deflection of the i-th beam with respect to z_i axis; I_{y}^{i} is its moment of inertia with respect to y_i axis; $N_{b}^{i} = N_{b}^{i}\left(x_{i},t\right)$ is the axial force at the x_i centroid axis; ρ_b is the surface mass density of the beams; $m_{by}^{i} = q_{x}^{i} e_{Cz}^{i}$; c_{b}^{i} is the i-th beam flexural damping constant; $w_{b0}^{i}\left(x\right)$, $\overline{w}_{b0}^{i}\left(x\right)$ are the initial deflection and the initial velocity of the points of the neutral axis of the i-th beam with respect to z_i axis; a_{ji}^{z}, β_{ji}^{z} $(j = 1,2,3)$ are coefficients specified at the boundary of the i-th beam; θ_{y}^{i}, R_{z}^{i}, M_{y}^{i} are the slope, the reaction and the bending moment at the i-th beam ends, respectively.

The $v_{b}^{i} = v_{b}^{i}\left(x_{i}\right)$ transverse deflection with respect to y_i axis must satisfy the following quasi-static (transverse inertia forces with respect to y_i axis are ignored) boundary value problem

$$E_{b}^{i} I_{z}^{i} \frac{\partial^{4} v_{b}^{i}}{\partial x_{i}^{4}} - N_{b}^{i} \frac{\partial^{2} v_{b}^{i}}{\partial x_{i}^{2}} = q_{y}^{i} - q_{x}^{i} \frac{\partial v_{b}^{i}}{\partial x_{i}} - \frac{\partial m_{bz}^{i}}{\partial x_{i}} \qquad \text{in } L_{i},\ \mathrm{i} = 1,2,\dots,I \qquad (10)$$

$$a_{1i}^{y} v_{b}^{i} + a_{2i}^{y} R_{y}^{i} = a_{3i}^{y} \qquad \beta_{1i}^{y} \theta_{z}^{i} + \beta_{2i}^{y} M_{z}^{i} = \beta_{3i}^{y} \qquad \text{at the beam ends} \qquad (11a,b)$$

where I_{z}^{i} is the moment of inertia of the i-th beam with respect to y_i axis; $m_{bz}^{i} = -q_{x}^{i} e_{Cy}^{i}$; a_{ji}^{y}, β_{ji}^{y} $(j = 1,2,3)$ are coefficients specified at its boundary; θ_{z}^{i}, R_{y}^{i}, M_{z}^{i} are the slope, the reaction and the bending moment at the i-th beam ends. Since linearized beam bending theory is considered the axial deformation u_{b}^{i} of the beam arising from the arbitrarily distributed axial force q_{x}^{i} $(i=1,2,\dots I)$ is described by solving independently the following quasi-static (axial inertia forces are neglected) boundary value problem

$$E_{b}^{i} A_{b}^{i} \frac{\partial^{2} u_{b}^{i}}{\partial x_{i}^{2}} = -q_{x}^{i} \qquad \text{in } L_{i},\ \mathrm{i} = 1,2,\dots,I \qquad (12)$$

$$a_{1i}^{x} u_{b}^{i} + a_{2i}^{x} N_{b}^{i} = a_{3i}^{x} \qquad \text{at the beam ends} \qquad (13)$$

where N_{b}^{i} is the axial reaction at the i-th beam ends given as

$$N_b^i = E_b^i A_b^i \frac{\partial u_b^i}{\partial x_i} \tag{14}$$

Finally, the nonuniform angle of twist with respect to $\tilde{x}_i$ shear center axis has to satisfy the following quasi-static (torsional and warping inertia moments are ignored) boundary value problem

$$E_b^i I_w^i \frac{\partial^4 \theta_{\tilde{x}}^i}{\partial \tilde{x}_i^4} - G_b^i I_{\tilde{x}}^i \frac{\partial^2 \theta_{\tilde{x}}^i}{\partial \tilde{x}_i^2} = -q_y^i e_{S\tilde{z}}^i + q_z^i e_{S\tilde{y}}^i \quad \text{in } L_i,\ \text{i} = 1,2,...,I \tag{15}$$

$$a_{1i}^{\tilde{x}} \theta_{\tilde{x}}^i + a_{2i}^{\tilde{x}} M_{\tilde{x}}^i = a_{3i}^{\tilde{x}} \qquad \beta_{1i}^{\tilde{x}} \frac{\partial \theta_{\tilde{x}}^i}{\partial \tilde{x}_i} + \beta_{2i}^{\tilde{x}} M_w^i = \beta_{3i}^{\tilde{x}} \qquad \text{at the beam ends} \tag{16a,b}$$

where $\theta_{\tilde{x}}^i = \theta_{\tilde{x}}^i\left(\tilde{x}_i\right)$ is the variable angle of twist of the i-th beam along the $\tilde{x}_i$ shear center axis; $G_b^i = E_b^i / 2(1+\nu_b^i)$ is its shear modulus; I_w^i, $I_{\tilde{x}}^i$ are the warping and torsion constants of the i-th beam cross section, respectively $a_{ji}^{\tilde{x}}$, $\beta_{ji}^{\tilde{x}}$ $(j = 1,2,3)$ are coefficients specified at the boundary of the i-th beam; $M_{\tilde{x}}^i$ is the twisting moment and M_w^i is the warping moment due to the torsional curvature at the boundary of the i-th beam.

Eqns. (1), (5a), (5b), (7), (10), (12), (15) constitute a set of seven coupled partial differential equations including ten unknowns, namely w_p, u_p, v_p, w_b^i, v_b^i, u_b^i, $\theta_{\tilde{x}}^i$, q_x^i, q_y^i, q_z^i. Three additional equations are required, which result from the displacement continuity conditions in the direction of x_i, y_i and z_i local axes at the midline of each (i-th) plate – beam interface. These conditions can be expressed as

$$w_p - w_b^i = e_{S\tilde{y}}^i \theta_{\tilde{x}}^i \qquad \text{in the direction of } z_i \text{ local axis} \tag{17}$$

$$u_p - u_b^i = \frac{h_p}{2}\frac{\partial w_p}{\partial x} - e_{Cz}^i \frac{\partial w_b^i}{\partial x_i} - e_{Cy}^i \frac{\partial v_b^i}{\partial x_i} + \left(\phi_S^P\right)_{f_i} \frac{\partial \theta_{\tilde{x}}^i}{\partial \tilde{x}_i} \quad \text{in } x_i \text{ local axis} \tag{18}$$

$$v_p - v_b^i = -\frac{h_p}{2}\frac{\partial w_p}{\partial y} - e_{S\tilde{z}}^i \theta_{\tilde{x}}^i \qquad \text{in the direction of } y_i \text{ local axis} \tag{19}$$

where $\left(\phi_S^P\right)_{f_i}$ is the value of the primary warping function with respect to the shear center S of the beam cross section at the midline of the f_i (i-th) interface.

In all the aforementioned equations the values of all the eccentricities e^i_{Cz}, e^i_{Cy}, $e^i_{S\tilde{z}}$, $e^i_{S\tilde{y}}$ and of the primary warping function $\varphi^P_S(\tilde{y}_i, \tilde{z}_i)$ should be set having the appropriate algebraic sign corresponding to the local beam axes.

It is worth here noting that the coupling of the aforementioned equations is nonlinear due to the terms including the unknown q^i_x and q^i_y interface forces.

3 Solution procedure

The numerical solution of the aforementioned problem is achieved employing the method presented by Katsikadelis and Kandilas [3]. According to this method the domain Ω occupied by the plate is discretized by establishing a system of M nodal points on it, corresponding to M mass cells, to which masses are assigned according to the lumped mass assumption. Subsequently, the stiffness matrix, the damping matrix as well as the load vector with respect to these nodal points are established employing the Analog Equation Method [2], a BEM based method. This procedure leads to the typical equation of motion for the stiffened plate

$$[m]\{\ddot{w}\}+[c]\{\dot{w}\}+[k]\{w\}=\{g\} \tag{20}$$

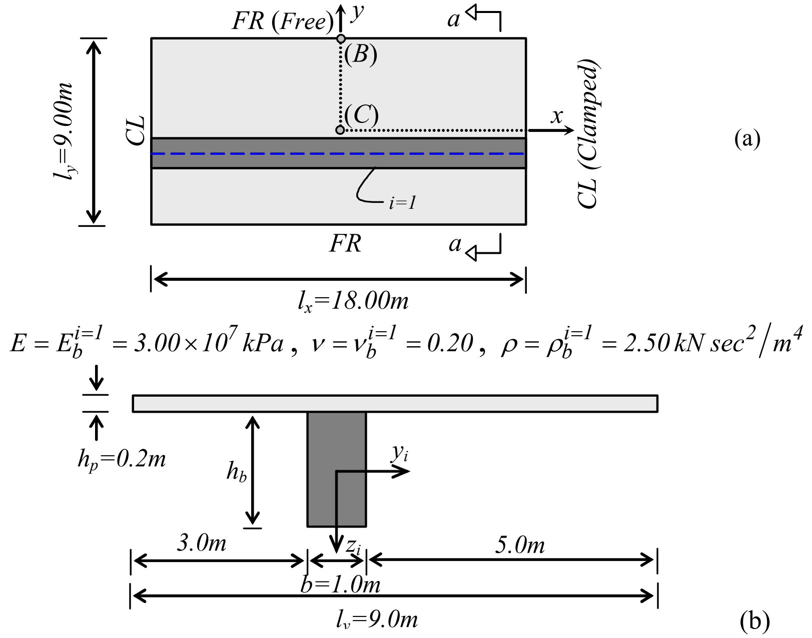

Figure 3: Plan view (a) and section a-a (b) of the stiffened plate.

4 Numerical examples

A rectangular plate stiffened by an eccentrically placed rectangular beam as shown in Fig.3 has been studied. In Table 1 the first four eigenfrequencies taking into account or ignoring the interface forces are presented as compared with those obtained from FEM solutions.

Table 1: $\Omega_n = \omega_n \sqrt{\rho}$ for various beam heights of the stiffened plate.

Ω_n	AEM with q_x, q_y (Present study)	AEM without q_x, q_y	Shell–Beam FE (SAP 2000)	Shell FE (NASTRAN)	Solid FE (NASTRAN)
			No beam		
1	–	22.1365	22.1434	22.1092	22.1159
2	–	37.6026	37.1742	37.0602	37.1034
3	–	61.419	61.0651	60.9110	60.9295
4	–	85.2215	84.1991	83.9974	84.0378
			Beam height 50cm		
1	36.42768	35.0639	40.8344	41.5054	44.4339
2	73.06466	49.0746	58.8534	59.3171	64.5881
3	82.96732	81.4040	85.9298	86.1819	90.4215
4	122.3293	108.9031	114.3645	113.3465	126.3263
			Beam height 100cm		
1	41.6788	37.8244	48.2654	48.3007	52.6324
2	84.6763	76.65077	86.4500	86.0006	94.2175
3	89.0430	85.9815	89.4149	89.2065	98.2054
4	135.9904	128.5372	135.5596	135.2461	151.5446
			Beam height 150cm		
1	47.7668	41.5472	50.3380	50.1915	55.0737
2	92.05131	84.1049	90.0815	89.8198	95.0838
3	93.50865	89.7506	95.9265	95.2206	112.6931
4	146.0206	138.0287	138.5487	138.0804	152.9531
			Beam height 200cm		
1	51.2582	45.5581	51.1244	50.9369	56.1035
2	93.2207	89.6741	90.3146	90.0342	95.4062
3	101.8806	91.7293	99.4490	98.6612	118.3639
4	151.0629	144.7945	139.5643	139.0592	153.1325

5 Concluding remarks

The proposed model permits the study of a stiffened plate subjected to an arbitrary loading, while both the number and the placement of the parallel stiffening beams are also arbitrary (eccentric beams are also included). The accuracy of the results compared with solid FE is remarkable.

References

[1] Sapountzakis, E.J. & Katsikadelis, J.T., Analysis of Plates Reinforced with Beams, *Computational Mechanics*, 26, pp. 66-74, 2000.

[2] Katsikadelis, J.T., The Analog Equation Method. A Boundary – only Integral Equation Method for Nonlinear Static and Dynamic Problems in General Bodies, *Theor. and Appl. Mech.*, 27, pp.13-38, 2002.

[3] Katsikadelis, J. T. & Kandilas, C.B., A Flexibility Matrix Solution of the Vibration Problem of Plates Based on the Boundary Element Method, *Acta Mechanica,* 83, pp.51-60, 1990.