A dynamic model of mobile source air pollution and its properties

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Abstract

Air pollution has been recognised as a serious concern in modern society. Various adverse effects have been analysed with exposure to air contamination. In this paper, a (discrete-time) dynamic model of mobile source air pollution (DMSAP) to predict the emission levels from a vehicle population within a monitoring system over time is developed. An essential feature of the model is the dynamics of vehicle population. The vehicle population is structured into different interconnected "age" groups (cohorts). It is assumed that vehicles of the same group have similar physical characteristics and behaviour such as engine types, engine conditions, emissions amounts, travelling habits and depreciation factors. Therefore, the model is suitable for large cities or metropolitan areas, where small group characteristics do not affect the aggregate behaviour of the vehicle population. Because of the nature of mobile source air pollution process, the DMSAP model exhibits positive linear systems (PLS) behaviour. Two important properties of DMSAP model with direct impact on the decision making process are discussed in this paper, namely maintainability and controllability. Maintainability is a property of the system to maintain (preserve) desired emission levels of pollution. Controllability is a property that shows the ability of the system to achieve desired emission levels from given ones. Maintainability and controllability are strong intrinsic properties of the system. For example, if the system is controllable it can be "moved" from any initial to any other (terminal) desired emission levels. In this paper some very recent advances of control theory for positive systems are used to study maintainability and controllability properties of DMSAP model.
Keywords: environmental modelling, air pollution, mobile sources, dynamic models, positive linear systems, maintainability, null-controllability, reachability, controllability.

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1 Introduction

Mobile source air pollution as a percentage of total air emissions has dramatically increased over the past few decades. Motor vehicles are primary sources of air pollution today. For example, in the USA in 1995, according to Deaton and Winebrake [1], only highway vehicles contributed 64% of national CO emissions, 35% of national NO_x emissions and 27% of national volatile organic compound (VOC) emissions. Millions of vehicles (automobiles, trucks, buses etc.) are on the road and most of them operating four-stroke, spark ignition internal combustion engine (ICE) technology. The components of the exhaust gas such as CO and NO_x create air quality problems in urban areas throughout the world. These components, as well as some others are a major cause of lung cancer and a variety of respiratory illnesses.

In order to better understand the role of mobile source emission in urban areas models, which help to predict the emissions from such sources over time, are needed. The purpose of this paper is to develop a simple dynamic model of mobile source air pollution (DMSAP), which can be used to predict the emissions of the total vehicle population. The vehicle population is structured into different interconnected "age" groups (cohorts). It is assumed that vehicles of the same group have similar physical characteristics and behaviour such as engine types, engine conditions, emissions amounts, travelling habits and depreciation factors. Therefore, the model is suitable for large cities or metropolitan areas, where small group characteristics do not affect the aggregate behaviour of the vehicle population. Two important properties of DMSAP are also studied in the paper namely maintainability and controllability. Maintainability is a property of the vehicle population to maintain (preserve) desired emission levels of pollution. Controllability is a property that shows the ability of the system to achieve desired emission levels from given ones.

The paper is organised as follows. A discrete-time dynamic model of mobile source air pollution is developed in section 2. Section 3 is devoted to its maintainability property. Controllability of the system described by DMSAP is studied in section 4. The paper is concluded with some final remarks on how to achieve and maintain a desired emission levels.

2 The model

2.1 Evolution of vehicle population

Consider a vehicle population on the road in a given area (large city, metropolitan area, etc.). The total vehicle population can be divided into n different cohorts (categories, groups) C_i, $i = 1, 2,\ldots, n$, on the basis of the attribute "age", the cohort C_1 being the youngest and C_n the oldest.

Let $x_i(t)$, $t = 0, 1, 2,\ldots$, denote the number of vehicles of cohort C_i at the beginning of time period t, where a single time period corresponds to the time between the samplings of the system (normally a year). Clearly, $x_i(0)$ is the initial number of vehicles in the ith cohort. It is assumed in this paper that the cohort span is one year. During year t, $t = 0, 1, 2, \ldots$, a fraction α_i of the vehicles constituting the ith cohort will progress to $(i + 1)$-st cohort, whilst some of the vehicles will be scrapped due to accidents or breakdowns, or because of long lifetime. All vehicles $u_i(t)$ (new and used) purchased from outside the system (that is the total vehicle population) during year t are related to the beginning of the period. They can be used as decision (control) variables, which are, obviously, non-negative, $u_i(t) \geq 0$ for $t = 0, 1, 2, \ldots$, and $i = 1, 2, \ldots, p$, where $p < n$, assuming that vehicles are delivered into the first p cohort only.

The evolution of the vehicle population can, then, be modelled by the simple system of difference balance equations

$$x_1(t+1) = u_1(t), \tag{1}$$

$$x_{i+1}(t+1) = \alpha_i x_i(t) + u_{i+1}(t), \quad i = 1, 2, \ldots, n-1; t = 0, 1, 2, \ldots, \tag{2}$$

where $x_i(t)$ is the number of vehicles (state variables) in cohort C_i at the beginning of year t, $u_i(t)$ is the number of vehicles (decision variables) purchased from outside the system into C_i during year t, and $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, n - 1$, is the progression fraction of the ith cohort, which is strictly positive and less than unity under the realistic assumptions that (i) some, but not all, vehicles are disposed because of long lifetime or (ii) due to accidents or breakdowns.

The difference balance equations (1) and (2) together constitutes a discrete-time dynamic model of the vehicle population. It is a simple cohort-type model, see Rumchev, Caccetta and Kostova [2].

The system (1)-(2) can be rewritten in the following vector form

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,\boldsymbol{u}(t), t = 0, 1, 2, \ldots, \tag{3}$$

with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \alpha_{n-1} & 0 \end{bmatrix} \geq 0, \; 0 < \alpha_i < 1, i = 1, 2, \ldots, n-1, \tag{3a}$$

$$\boldsymbol{B} = [\boldsymbol{e}_1\, \boldsymbol{e}_2 \ldots \boldsymbol{e}_p] \geq 0, \tag{3b}$$

where $\boldsymbol{e}_i$, $i = 1, \ldots, p$ $(p < n)$, are the unit basis vectors, $\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))'$ is named a state vector and the decision (control) vector is non-negative, i.e.

$$\boldsymbol{u}(t) = (u_1(t), \ldots, u_p(t))' \geq 0, \tag{3c}$$

assuming that the decision (control) variables $u_j(t)$ affect directly the first p cohort only and '′' denotes the transpose.

Throughout this paper nonnegative real matrices, respectively vectors, are denoted as $\boldsymbol{A} \geq 0$, respectively $\boldsymbol{x} \geq 0$, where the inequality is component-wise; that is $a_{ij} \geq 0$, respectively, $x_j \geq 0$, for all i and j. The notation $\boldsymbol{A} > 0$ ($\boldsymbol{x} > 0$) is used for matrices (vectors) with all its entries positive, i.e. $a_{ij} > 0$ ($x_j > 0$) for all i and j.

An $n \times n$ matrix $\boldsymbol{A}$ is said to be nil-potent with index of nil-potency $k \leq n$ if $\boldsymbol{A}^s \neq 0$ for $s = 1,2,\ldots, k$-1 but $\boldsymbol{A}^s = 0$ for $s = k, k+1,\ldots, n$. An immediate consequence of Cayley-Hamilton theorem is that all eigenvalues of a nilpotent matrix are equal to zero, see, for example, Goldberg [3] and for more information about the properties of nil-potent matrices Bru and Johnson [4]. It is not difficult to find out that the cohort matrix (3a) is a nilpotent matrix and, therefore, all its eigenvalues are zeros.

The state vector $\boldsymbol{x}(t)$ represents the distribution of vehicles (population distribution) among the cohorts whilst the size of the vehicle population at the beginning of year t is given by $X(t) = \sum_{i=1}^{n} x_i(t)$.

The system (3) with (3a), (3b) and (3c) is a discrete-time positive linear dynamic system - see Kaczorek [5] or Farina and Rinaldi [6]. The state variables $x_i(t)$ of such systems are always positive (or at least non-negative in value) whenever the initial state and the controls are non-negative. This is a common property of positive systems.

2.2 Emission levels

Let m_i be the average annual distances (in *km* or *miles*) travelled by one vehicle of cohort C_i during one time period and e_{ji} (called emission level or factor) be the average emission (in *g/km* or *g/miles*) of pollutant type j, $j = 1, 2, \ldots, k$, from one vehicle of the ith cohort. Note that $e_{ji} > 0$ for all i, j. Then the total emission level $\varepsilon_j(t)$ for the jth type of pollutant during time period t is given by the following expression

$$\varepsilon_j(t) = \sum_{i=1}^{n} x_i(t) m_i e_{ji} \, . \tag{4}$$

Let, now, the total emission vector be denoted as $\boldsymbol{\varepsilon}(t) = (\varepsilon_1(t),\ldots, \varepsilon_k(t))$, the matrix of emission levels as $\boldsymbol{E} = \lfloor \varepsilon_{ji} \rfloor_{k \times n}$ and the average distances matrix as $\boldsymbol{M} = \mathrm{diag}\{ m_1,\ldots, m_n\}$. The total emissions vector then can be represented as

$$\boldsymbol{\varepsilon}(t) = \boldsymbol{E}\boldsymbol{M}\,\boldsymbol{x}(t) \tag{5}$$

The emission levels and the average annual distances as well as the progression fractions are usually known from demographic and engineering studies, and there are (at least in the developed countries) accepted standards for the emissions from mobile sources. These standards, indeed, are upper limits $\bar{\varepsilon}_j$ on total pollution of type j, $j = 1, 2,\ldots, k$, i.e. the total emissions are subject to the following restrictions:

$$\varepsilon_j(t) \le \bar{\varepsilon}_j, \quad j = 1, 2,\ldots, k, \tag{6}$$

or, in vector form,

$$\boldsymbol{EM}\,\boldsymbol{x}(t) \le \bar{\boldsymbol{\varepsilon}}, \tag{7}$$

where $\bar{\boldsymbol{\varepsilon}} = (\bar{\varepsilon}_1,\ldots,\bar{\varepsilon}_k)'$ is the vector of standards (allowed limits) for the total emissions.

Formulas (4)-(5) express the relationship between the vehicle population and the emissions of various pollutants while (6)-(7) introduce restrictions on the population (state variables) and, consequently, on the numbers of new purchases (decision variables, controls) from outside the vehicle population.

As a matter of fact the dynamic model of mobile air pollution includes (3)-(3c) (or, equivalently, (1)-(2)) and (5) (or, equivalently (4)) because any implementation of the decisions $u_j(t)$ affects the states $x_j(t)$ and, consequently, the emission levels $\varepsilon_j(t)$. At the same time the dynamics of the model is concentrated in the vehicle population and that is why in the following sections the main attention is on the equation (3) with (3a)-(3c).

3 Maintainabilty

Definition 1. The state $\boldsymbol{x}_e \ge 0$ of the system (3) (or (1) - (2)) is said to be maintainable at time t if and only if there exists a control $\boldsymbol{u} \ge 0$ such that $\boldsymbol{x}(t + 1) = \boldsymbol{x}(t) = \boldsymbol{x}_e$.

It readily follows from (3) that a state $\boldsymbol{x}_e \ge 0$ is maintainable at time t if and only if there exists a control vector $\boldsymbol{u} \ge 0$ such that

$$\boldsymbol{x}_e = \boldsymbol{A}\,\boldsymbol{x}_e + \boldsymbol{Bu} \qquad \text{with} \quad \boldsymbol{u} \ge 0 \text{ and } \boldsymbol{A}, \boldsymbol{B} \ge 0. \tag{8}$$

Equation (8) tells us that if the state $\boldsymbol{x}_e$ of the time-invariant system (3) is maintainable at time t it is maintainable for all t, i.e. it is maintainable at all times. Therefore, it is called simply a maintainable state.

Equation (8) is referred to as maintainability equation (equilibrium equation) and its solutions as maintainable or equilibrium states (points). In the case of time-invariant systems the maintainable states are also equilibrium states of the system but we prefer to use the term maintainability since equilibrium states are associated with the stability of the motion, which is an asymptotic notion (James and Rumchev [7]), while maintainability is a short term concept.

Equation (8) can be rewritten in the form

$$(\boldsymbol{I}-\boldsymbol{A})\,\boldsymbol{x_e}=\boldsymbol{Bu} \qquad \text{with} \quad \boldsymbol{u} \geq 0 \text{ and } \boldsymbol{A}, \boldsymbol{B} \geq 0. \tag{9}$$

Definition 2. The set X_e of all maintainable (equilibrium) states $\boldsymbol{x_e}$

$$X_e = \{\boldsymbol{x_e} \mid (\boldsymbol{I}-\boldsymbol{A})\,\boldsymbol{x_e}=\boldsymbol{Bu} \text{ with } \boldsymbol{u}, \boldsymbol{A}, \boldsymbol{B} \geq 0\} \tag{10}$$

is called a maintainable (equilibrium) set or a set of maintainable states.

A square matrix $\boldsymbol{C}$ is an M-matrix (see, for example, Berman and Plemmons [8]) if there exists a non-negative matrix $\boldsymbol{D} \geq 0$ with dominant (maximal) eigenvalue $\rho(\boldsymbol{D}) \geq 0$ such that $\boldsymbol{C} = c\boldsymbol{I} - \boldsymbol{D},$ where $c \geq \rho(\boldsymbol{D})$ and $\boldsymbol{I}$ is the $n \times n$ identity matrix. The M-matrix is non-singular if $c > \rho(\boldsymbol{D})$ and singular if $c = \rho(\boldsymbol{D})$. Moreover, when $\boldsymbol{C}$ is non-singular the inverse exists and is nonnegative, i.e. $\boldsymbol{C}^{-1} = (c\boldsymbol{I} - \boldsymbol{D})^{-1} \geq 0$. Thus, $\boldsymbol{C}^{-1}$ exists and is nonnegative if and only if $c > \rho(\boldsymbol{D})$.

All eigenvalues of the cohort matrix $\boldsymbol{A}$ given by (3a) are zeros. Therefore the matrix $(\boldsymbol{I} - \boldsymbol{A})$ is a non-singular M – matrix and its inverse exists and is non-negative, $(\boldsymbol{I}-\boldsymbol{A})^{-1} \geq 0$, so that the maintainable set (10) can be represented as

$$\begin{aligned} X_e &= \{\boldsymbol{x_e} \mid \boldsymbol{x_e} = (\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{Bu} \text{ with } \boldsymbol{u}, \boldsymbol{B} \geq 0\} = \\ &= \{\boldsymbol{x_e} \mid \boldsymbol{x_e} = \sum_{j=1}^{p} u_j \, col_j(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}, \, u_j \geq 0\} \end{aligned} \tag{11}$$

The maintainable set (11) is a polyhedral cone called cone of maintainable states or maintainable cone. The vectors $col_j\ (\boldsymbol{I}-\boldsymbol{A})^{-1}\,\boldsymbol{B} \geq 0$ are named edges (minimal generators, extreme rays) of the polyhedral cone. Let us specify the cone of maintainable states for the dynamic model of mobile source air pollution (3) – (3c). It is not difficult to obtain given (3a) that

$$(\boldsymbol{I}-\boldsymbol{A})^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_1 & 1 & 0 & \cdots & 0 & 0 \\ \alpha_1\alpha_2 & \alpha_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \alpha_1 \ldots \alpha_{n-2} & \alpha_2 \ldots \alpha_{n-1} & \cdots & \alpha_{n-2} & 1 & 0 \\ \alpha_1 \ldots \alpha_{n-2}\alpha_{n-1} & \alpha_2 \ldots \alpha_{n-1} & \cdots & \alpha_{n-2}\alpha_{n-1} & \alpha_{n-1} & 1 \end{bmatrix} \tag{12}$$

So, in this case, the first $p \leq n$ columns of (12) are the edges of the cone of maintainable states (11). Notice also that $\boldsymbol{B}'\boldsymbol{B} = \boldsymbol{I}$ for $\boldsymbol{B}$ given by (3b). Now, if a given state (distribution of vehicles among the different cohorts) $\boldsymbol{x}^*$ is in the maintainable cone (11) – (12) then there exists a decision vector

$$\boldsymbol{u}^* = (\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}^* = \boldsymbol{B}'(\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{x}^* \qquad (13)$$

such that this state can be maintained. If the state $\boldsymbol{x}^*$ does not belong to the maintainable cone then there is no admissible decisions that can keep that state and the desired distribution of vehicles among the vehicle population cannot be preserved to the next time period.

To see better that the set of maintainable states is not that large consider a vehicle population for which only the purchase of new vehicles is allowed, so $u_1 \geq 0$ but $u_j = 0$ for $j = 2,\ldots, p$. In this case the maintainable cone is a ray in the nonnegative orthant with the origin as its initial point (vertex).

$$X_e = \{\boldsymbol{x_e} \mid \boldsymbol{x_e} = u_1(1, \alpha_1, \alpha_1\alpha_2, \ldots, \alpha_1\ldots\alpha_{n-2}, \alpha_{n-1})',\ u_1 \geq 0\} \qquad (14)$$

Only states that are on the ray (14) are maintainable; all the other nonnegative states cannot be maintained by admissible decision $u_1 \geq 0$ and, consequently, the corresponding desired emission levels cannot be maintained.

4 Controllability

Definition 3. The positive system (3) (and the nonnegative pair ($\boldsymbol{A}$, $\boldsymbol{B}$)) is said be (Rumchev and James [9])

- **(a)** *reachable* (or controllable-from-the-origin) if for any state $\boldsymbol{x_f} \geq 0$ but $\boldsymbol{x_f} \neq 0$, and some finite t there exists a non-negative decision sequence $\{\boldsymbol{u}(s),\ s = 0, 1, 2, \ldots, t-1\}$ that transfers the system from the origin $\boldsymbol{x}(0) = 0$ to the state $\boldsymbol{x}(t) = \boldsymbol{x_f}$;
- **(b)** *null-controllable* (or controllable-to-the-origin) if for any state $\boldsymbol{x}_0 \geq 0$ and some finite t there exists a non-negative decision sequence $\{\boldsymbol{u}(s),\ s = 0, 1, \ldots, t-1\}$ that transfers the system from the state $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ to the origin $\boldsymbol{x}(t) = 0$;
- **(c)** *controllable* if for any non-negative pair $\boldsymbol{x}_0, \boldsymbol{x_f} \geq 0$ and some finite t there exists a non-negative decision sequence $\{\boldsymbol{u}(s),\ s = 0, 1, \ldots, t-1\}$ that transfers the system from the state to $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ the state $\boldsymbol{x_f} = \boldsymbol{x}(t)$.

Controllability is a fundamental property of dynamical systems that shows their ability to move in space. It has direct implications in many control problems such as optimal control, feedback stabilization, nonnegative realizations and system minimality among others.

Definition 4. The matrix

$$\mathcal{R}_t(\boldsymbol{A}, \boldsymbol{B}) = [\ \boldsymbol{B} \mid \boldsymbol{AB} \mid \boldsymbol{A}^2\boldsymbol{B} \mid \ldots \mid \boldsymbol{A}^{t-1}\boldsymbol{B}\ ] \geq \boldsymbol{0} \qquad (15)$$

is called t - step *reachability matrix*.

A column vector with exactly one non-zero entry is called *monomial*. The product of a non-singular diagonal matrix and a permutation matrix is called *monomial matrix*, see Berman and Plemmons [8]. A monomial matrix consists of linearly independent monomial columns. A monomial vector is called *i*-monomial if the non-zero entry is in the *i*th position. A selection of n non-zero entries, one from each row and column of a square matrix is called a *diagonal*. The non-zero entries of a monomial matrix always form a diagonal.

Proposition 1. The discrete-time positive linear system (3) (and the nonnegative pair $(\boldsymbol{A}, \boldsymbol{B}) \geq 0$) is

- **(a)** *reachable* if and only if the n – step reachability matrix $\mathcal{R}_n$ $(\boldsymbol{A}, \boldsymbol{B})$ contains an $n \times n$ monomial submatrix;
- **(b)** *null-controllable* if and only if $\boldsymbol{A}$ is a nil-potent matrix;
- **(c)** *controllable* (in finite time) if and only if it is reachable and null-controllable.

The proof of Proposition 1 is given by Rumchev and James [9]. It is used in this paper to study controllability property of the vehicle population and, consequently, the controllability property of discrete time dynamic model of mobile source air-pollution (3) with (3a)-(3c) and (5).

Let us first see whether (3) with (3a)-(3c) is reachable. It is not difficult to prove that

$$\boldsymbol{A}^k = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_1\alpha_2\ldots\alpha_k & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_{n-k}\alpha_{n-k+1}\ldots\alpha_{n-1} & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{16}$$

for $k = 1,\ldots, n$-1, and

$$\boldsymbol{A}^n = \boldsymbol{0}. \tag{17}$$

Since, by hypothesis, $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, n - 1$, the first $(n - k)$ columns of $\boldsymbol{A}^k$ are monomial that is scalar multiples of the unit basis vectors $\boldsymbol{e}_{k+1} = (0,\ldots,0,1,0,\ldots, 0)^{'}$, $k = 1,\ldots, n$-1; as a matter of fact they are $(k + 1)$ – monomials.

Consider, now, the reachability matrix (15) for the system (3)-(3c). The matrix $\boldsymbol{B}$ given by (3b) consists of the unit basis vectors $\boldsymbol{e}_i$, $i = 1, 2,\ldots, p$, and, hence, the first p columns $(p \leq n)$ of reachabilility matrix $\mathcal{R}_t$ $(\boldsymbol{A}, \boldsymbol{B})$ are i – monomials, $i = 1, 2,\ldots, p$. Since the first $(n - k)$ columns of $\boldsymbol{A}^k$ are $(k + 1)$ –

monomials for k = 1,…, p,…, n-1 and the matrix $\boldsymbol{B}$ is of the form (3b), the first ($n - p$) columns of the block $\boldsymbol{A}^p\boldsymbol{B}$ in reachability matrix of (3)-(3c) are i – monomials, that is scalar multiples of $\boldsymbol{e}_i$ for $i = p+1,\ldots, n$. Therefore, the reachability matrix of dynamic model of mobile source air pollution contains n linearly independent monomial columns, which form a monomial matrix, and by Proposition 1(a) the system is reachable. On the other hand, the nil-potency of the matrix $\boldsymbol{A}$ given by (3a) implies, according to Proposition 1(b), the null-controllability of the system. So, the dynamic model of mobile source air pollution is reachable and null-controllable and, therefore, by Proposition 1(c) it is controllable. Thus, the following result has been proved.

Proposition 2. The dynamic model of mobile source air pollution (3) – (3c) (or, equivalently (1) – (2)) is controllable.

It is also not difficult to see from the proof of Proposition 2 that the system (3)-(3c) is controllable only if the matrix $\boldsymbol{B}$ contains $\boldsymbol{e}_1$, which means that *new vehicles must be allowed* into the system; it is controllable even when $\boldsymbol{B} = \boldsymbol{e}_1$ that is only new vehicles are allowed into the system. This is the necessary condition for controllability of the dynamic model of mobile source air pollution.

Proposition 3. The dynamic model of mobile source air pollution is controllable only if new vehicles are allowed into the vehicle population.

5 Final remarks

Although, in practice, desired emission levels have to be achieved first and then maintained, this paper treats first the maintainability problem because of its relative simplicity. Now, let the desired emission levels $\boldsymbol{\varepsilon}^*$ be given. Then, to determine the corresponding system state $\boldsymbol{x}^*$ the following linear programming (LP) problem can be solved

$$\text{Min } \{\boldsymbol{wx} \mid \boldsymbol{EM}\,\boldsymbol{x} = \boldsymbol{\varepsilon}^*, \boldsymbol{x} \geq \underline{\boldsymbol{x}} \geq 0\} \tag{18}$$

where the weight coefficients vector $\boldsymbol{w} \geq 0$ can be chosen as $\boldsymbol{w} = (0,\ldots, 0, w_{p+1},\ldots,w_n)'$ with $\sum_{i=p+1}^{n} w_i = 1$ to minimise the number of old vehicles in the population. The solution $\boldsymbol{x}^*$ to LP (18) is the distribution of vehicles among the cohorts that produces the desired emission levels $\boldsymbol{\varepsilon}^*$. The state $\boldsymbol{x}^* \geq 0$ then is to be reached first from the current (initial) state $\boldsymbol{x}(0) \geq 0$ and then maintained; the state $\boldsymbol{x}^*$ can always be achieved only if there is an inflow (purchases) of new cars into the vehicle population and maintained if it is in the maintainable cone. To determine the corresponding decision policies, the implementation of which leads to the desired emission levels, one has to formulate and solve some specific decision (control) problems; these problems will be discussed in another paper.

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