An application of the Hanks stability parameter for the macroviscous-inertial transition in the surface flow of a neutrally buoyant suspension

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Abstract

This paper focuses on the evaluation of the regime transition in the surface flow of a neutrally buoyant suspension. The revisiting of the 1954 Bagnold paper made by Hunt *et al.* (2002) showed that the choice of the non-dimensional parameter afterwards known as Bagnold number for evaluating the transition between macroviscous and grain-inertial regime is at least doubtful. Here, a different approach, based on the universal stability parameter of Hanks (1963a, b), is proposed. The application of this transition criterion to the case of the incompressible uniform surface flow of a liquid-solid mixture shows that the Reynolds number is the parameter that really controls the transition between the flow regimes. The critical value of the Reynolds number is a function of the solid volume fraction of the mixture. This value is derived assuming that the critical value of the Hanks stability parameter does not change when referring to the ensemble averaged momentum equations of the continuous phase rather than to the classical Navier-Stokes equations. The present analysis shows that highly concentrated neutrally buoyant suspensions can be characterized by the macroviscous regime even if the Reynolds number is not small, up to an order of 10^5. Thus, one should pay attention in applying at real scale laws obtained through experiments performed at laboratory scale.

Keywords: neutrally buoyant suspension, transition, stability parameter, uniform surface flow.

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1 Introduction

An important aspect for a deeper understanding of the mechanical behaviour of a debris flow concerns the evaluation of the flow regime. Most of experimental data regarding the hyperconcentrated flows indeed are obtained through laboratory scale models because of the extreme difficulties in predicting the location where a real scale debris flow will develop and because of the complex and expensive set-up needed for performing measurements. Hence, it is of fundamental interest to understand if and when the laws obtained through these experiments can be generalized and applied also to the real scale phenomena.

Although the real debris flows are composed of a disperse phase much heavier than the continuous phase, it can be convenient initially focusing on the simpler case of a mixture with constant density. Bagnold [1] found that a neutrally buoyant suspension of rigid spheres was characterized by two different flow regimes: macroviscous and grain-inertial. The Author proposed a critical value of a non-dimensional parameter, afterwards known as the Bagnold number, for the transition between these regimes. This choice appears not completely satisfactory (see § 2). In § 3 we introduce the stability parameter suggested by Hanks [6, 7] for evaluating the laminar-turbulent transition of a Newtonian fluid. This parameter does not depend on the geometrical configuration of the flow domain. Then we apply this stability criterion to the continuous phase of an incompressible solid–liquid mixture, which flows in uniform condition over a fixed bed in order to obtain information about the actual non-dimensional parameter, which governs the macroviscous-inertial transition. Finally, some conclusions are given in § 4.

2 Bagnold criterion for the transition

The Bagnold [1] paper regarding the experiments performed in a Couette device with a neutrally buoyant suspension of spheres in water is still the basis of a lot of works on debris flow.

The annulus between the two coaxial cylinders was filled with a mixture of water or glycerine and molten plastic drops (50% mixture of paraffin wax and lead stearato with diameter $d = 0.13$ cm) and then the outer cylinder was rotated at constant angular velocity, while the inner was kept fixed. Bagnold was able to measure both the shear and the normal stress in this condition. The Author found that there were two distinctive flow regimes and that the transition was governed by a non-dimensional number, afterwards known as the Bagnold number, N_{Bag}, representative of the ratio between the inertia of the grains and the viscous force exerted by the fluid:

$$N_{\mathrm{Bag}} = \frac{\lambda^{\frac{1}{2}} \rho_s d^2}{\mu} \gamma \tag{1}$$

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In eqn (1), ρ_s is the density of the grains, μ is the fluid viscosity, γ is the shear rate, imposed through the angular velocity of the external cylinder, and λ is the linear concentration, defined as:

$$\lambda = \frac{1}{\left(\frac{c_{\max}}{c}\right)^{\frac{1}{3}} - 1} \tag{2}$$

where c is the solid volume fraction of the mixture and $c_{\max}$ is the maximum packing volume fraction (for perfect spheres $c_{\max} = 0.74$).

Bagnold observed that for $N_{\text{Bag}} < 40$ the viscous forces prevail on the grain inertia and that both the normal and the shear stress were proportional to the shear rate. Bagnold suggested to adopt the following expression for the shear stress, τ, in this regime, called macroviscous, on the basis of some preliminary theoretical considerations:

$$\tau = (1 + \lambda)\left(1 + \frac{\lambda}{2}\right)\mu\gamma \tag{3}$$

He found, then, on the basis of a regression over the experimental results, that the shear stress obeys to the following expression:

$$\tau = 2.25\lambda^{\frac{3}{2}}\,\mu\gamma \tag{4}$$

Eqns (3) and (4) are almost coincident in the range of the linear concentration examined by Bagnold, but eqn (3) has the advantage of being consistent with the expression valid for a Newtonian fluid when the linear concentration vanishes. The presence of the solid phase has the effect of increasing the apparent viscosity of the mixture.

When $N_{\text{Bag}} > 450$, instead, Bagnold noticed that both the normal and the shear stress were proportional to the square of the shear rate and called grain-inertial this regime, because he found that the main dissipative mechanism in the flow was due to the collisions between the grains. In this grain-inertial regime, the shear stress can be expressed as:

$$\tau = a_b \rho_s \lambda f(\lambda) d^2 \sin\delta\,\gamma^2 \tag{5}$$

with a_b an empirical parameter, δ a dynamic friction angle and $f(\lambda)$ a not well defined function of the linear concentration. Eqn (5) shows that the inertial terms are characterized by a local length scale that is proportional to the diameter of the grains. The Bagnold number is an index of the ratio between the inertial shear stress of eqn (5) and the macroviscous shear stress of eqn (4).

Finally, in the range $40 < N_{Bag} < 450$, the flow behaviour is intermediate between macroviscous and grain-inertial.

It must be underlined that the non-dimensional parameter introduced by Bagnold (eqn 1) is a local parameter. Hence, the evaluation of the global flow regime of a current through this parameter, *e.g.* into depth-integrated mathematical models such as those based on the Saint Venant equations [2, 3, 4], is not clear.

Hunt *et al.* [8] revisited the Bagnold paper [1] and noticed that only the experiments performed with glycerine as interstitial fluid showed the linear relation between shear stress and shear rate typical of the macroviscous regime. All the other experimental data, instead, were characterized by a power law between shear stress and shear rate with an exponent intermediate between one and two, so that no grain-inertial regime is effectively observed. The choice of the value of the critical Bagnold number indicating the transition between macroviscous and grain-inertial regime seems therefore at least doubtful. Moreover, even the choice of the Bagnold number as the parameter governing the behaviour of the mixture seems not completely justified.

Hunt *et al.* [8] criticized also the configuration of the Bagnold experimental set-up. The small ratio between the height h of the apparatus and the gap $t = r_2 - r_1$ between the cylinders (r_2 and r_1 being, respectively, the radius of the outer and of the inner cylinder), indeed, caused that the Taylor vortices due to the presence of the end plates of the apparatus had a not negligible effect on the measurements of the shear stress. Moreover, it is well known that, in the case of clear water, these vortices increase the instability of the flow, so that the critical Reynolds number that indicates the transition from laminar to turbulent regime is smaller when the ratio h/t diminishes.

Hunt *et al.* [8] suggested therefore that the transition observed in Bagnold's experiments was the transition between a classical Couette flow ($h/t \to \infty$) and a flow disturbed by the presence of Taylor vortices, rather than a transition between macroviscous and grain-inertial regime.

3 Application of Hanks stability parameter

Hanks [6] [7] proposed and tested a criterion for the laminar-turbulent transition based on a universal stability parameter valid for Newtonian and Bingham fluids flowing in pipes, concentric annuli and parallel plates. The momentum equations for the fluid are the following:

$$\rho\partial_t\vec{v} + \frac{1}{2}\rho\nabla(\vec{v}\cdot\vec{v}) - \rho\vec{v}\times(\nabla\times\vec{v}) = \vec{F} - \nabla p - \nabla\cdot\vec{T} \qquad (6)$$

where ρ is the density, $\vec{v}$ the velocity vector, $\vec{F}$ the gravity, p the pressure and $\vec{T}$ the stress tensor. Hanks suggested that the transition is controlled by the ratio K between the absolute value of the force originated from the vorticity of the flux and the absolute value of the viscous force:

$$K = \frac{\left|\rho\vec{v} \times \left(\nabla \times \vec{v}\right)\right|}{\left|\nabla \cdot \vec{T}\right|} \tag{7}$$

The stability parameter K is a function of the position in the flow domain and vanishes at the solid boundaries and along the symmetry lines of the flow field. Hence, somewhere there must be a maximum value $\overline{K}$. When $\overline{K}$ is equal to a certain critical value κ, which is independent on the geometry and equal to 404, the perturbations can not be damped by the viscous forces and the flow becomes turbulent.

We chose to apply the Hanks stability criterion to the planar uniform surface flow of a neutrally buoyant suspension (Fig. 1) of constant depth H over a fixed bed. The flow is in x-direction, making an angle θ with horizontal. We assumed that for such a mixture the ensemble averaged velocities of the continuous and of the disperse phase are the same, so that even the inertial terms are the same. If this is the case, the transition between the macroviscous and the grain-inertial regime can be interpreted as the transition between the laminar (ruled by the viscous forces) and the turbulent (ruled by the inertial forces) regime of a Newtonian fluid.

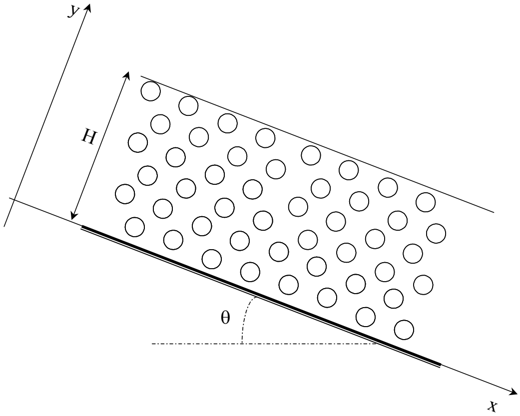

Figure 1: Sketch of the uniform surface flow configuration.

The momentum equations for the continuous phase of the mixture can be obtained from (6) through an ensemble averaging over all the possible microscopic configurations of the grains [10]. If we assume a constant solid volume fraction c in the mixture, the momentum equations of the continuous phase in uniform flow condition can be written as:

$$\frac{1}{2}\rho\nabla\left(\vec{v} \cdot \vec{v}\right) - \rho\vec{v} \times \left(\nabla \times \vec{v}\right) = \vec{F} - \nabla p - \nabla \cdot \vec{T} \tag{8}$$

where for sake of simplicity we used the same symbol of eqn (6) for indicating the ensemble averaged variables. Provided that:

$$\begin{aligned} &\frac{1}{2}\rho\nabla(\vec{v}\cdot\vec{v}) = \rho\vec{v}\times(\nabla\times\vec{v}) \\ &\vec{F} - \nabla p = \nabla\cdot\vec{T} \end{aligned} \tag{9}$$

the stability parameter K is equal to:

$$K = \frac{\left|\rho\vec{v}\times(\nabla\times\vec{v})\right|}{\left|\nabla\cdot\vec{T}\right|} = \frac{\left|\frac{1}{2}\rho\nabla(\vec{v}\cdot\vec{v})\right|}{\left|\vec{F}-\nabla p\right|} = \frac{1}{2}\rho\frac{\partial_y u^2}{\rho g\sin\theta} \tag{10}$$

Here, u is the ensemble averaged component along x-direction of the velocity of the continuous phase. If the shear stress in the macroviscous regime has the expression of eqn (3) the velocity profile is that of a Poiseuille flow for a Newtonian fluid, with an apparent viscosity $\mu' = (1+\lambda)(1+0.5\lambda)\mu$. Hence:

$$u = \frac{\rho g}{\mu'}\left(Hy - \frac{y^2}{2}\right)\sin\theta \tag{11}$$

and the integration of eqn (11) leads to the following flow rule:

$$\sin\theta = 3\frac{\mu'}{\rho g}\frac{V}{H^2} \tag{12}$$

with V mean velocity of the continuous phase of the mixture.

Substituting eqns (11) and (12) in eqn (10) leads to:

$$K = 3\frac{\rho VH}{\mu'}\left(\xi - \frac{\xi^2}{2}\right)(1-\xi) \tag{13}$$

where the non-dimensional variable $\xi = y/H$ is introduced. The parameter K reaches its maximum when $\xi = \dfrac{3-\sqrt{3}}{3}$. Thus:

$$\overline{K} = \frac{\sqrt{3}}{3}\frac{\rho VH}{\mu'} = \frac{\sqrt{3}}{12}\frac{1}{(1+\lambda)(1+0.5\lambda)}N_{\text{Re}} \tag{14}$$

Here we have introduced the classical global Reynolds number $N_{\text{Re}} = \frac{\rho V(4H)}{\mu}$. The critical value of the Reynolds number $N_{\text{Re}c}$ that indicates the transition from macroviscous to grain-inertial regime can therefore be obtained by substituting $\overline{K} = \kappa$ in eqn (14). The relation between $N_{\text{Re}c}$ and the normalized solid volume fraction $c/c_{\max}$ for $\kappa = 404$, obtained by substituting eqn (2) into eqn (14), is depicted in fig. 2. It can be noticed that the increasing of the solid volume fraction causes the flux to be more stable ($N_{\text{Re}c}$ greatly increases), as noted by many authors [8, 9]. Hence, laboratory scale high-concentrated neutrally buoyant suspensions could be characterized by the macroviscous regime and this must be taken into account in the modelling of debris flow.

It must be underlined that we used the value $\kappa = 404$ suggested by Hanks [6, 7] and valid if the ratio between the inertial and the viscous forces is evaluated at the scale of the classical Navier-Stokes equations (eqn 6). Its application even at the scale of the ensemble averaged momentum equations (eqn 8) will be subject of further investigations.

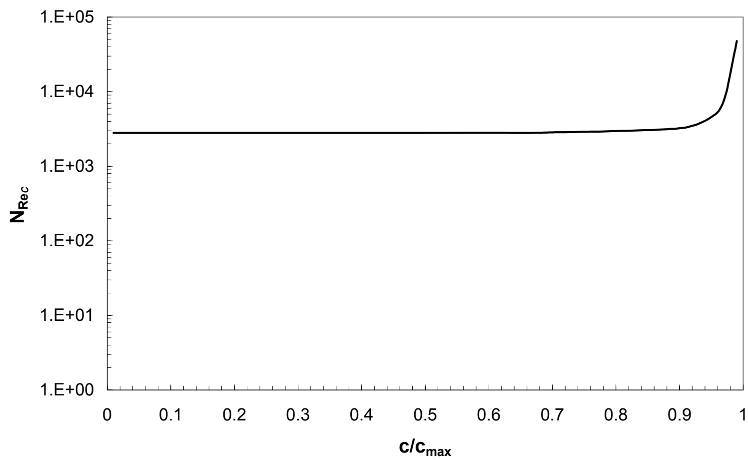

Figure 2: Relation between the critical Reynolds number and the normalized solid volume fraction.

We can introduce a global Bagnold number for the flux related to the global Reynolds number through the following expression:

$$N_{\text{Bag}} = \frac{\lambda^{\frac{1}{2}} \rho_s d^2}{\mu} \frac{V}{4H} = \frac{\lambda^{\frac{1}{2}}}{16} \left(\frac{d}{H} \right)^2 N_{\text{Re}} \tag{15}$$

provided that in a neutrally buoyant suspension $\rho_s = \rho$. Therefore it is possible to obtain also the value of the critical Bagnold number indicating the transition from macroviscous to grain-inertial regime. This critical value though depends not only on the solid volume fraction but also on the ratio *d*/*H*, because the diameter is forced to appear in the expression of the Bagnold number. Recent experiments regarding dry granular flow [5] allowed to raise some doubts about the choice of this local length scale for characterizing the inertial stress of the grains. Eqn (14) shows indeed that the Bagnold number is not the fundamental non-dimensional parameter that governs the transition from the macroviscous to the inertial regime.

4 Concluding remarks

This work deals with the criterion for evaluating the flow regime of a neutrally buoyant solid-liquid mixture. This represents a first step towards the comprehension of the non-dimensional parameters that controls the dynamics of the debris flows and is needed also for determining if general laws to be applied at real scale can be inferred from the experimental results obtained through laboratory set-up. The widest used criterion is up-to-date based on a non-dimensional number introduced by Bagnold [1] on the basis of experiments carried out in a rheometer, which should govern the transition from a macroviscous to a grain-inertial regime. The analysis performed by Hunt et al. [8] though showed that the configuration of the Bagnold experimental set-up invalidates this conclusion. Here, we adopted a transition criterion based on the Hanks stability parameter [6, 7]. We treated the continuous phase of the mixture as a Newtonian fluid, with an apparent viscosity different from the viscosity of the interstitial fluid because of the presence of the solid grains and we applied the Hanks criterion to the case of a surface uniform flow due to the practical interest of such a configuration. We found that the actual non-dimensional parameter, which controls the transition from the macroviscous to the grain-inertial regime, here assimilated to the turbulent regime of the continuous phase, is the classical Reynolds number. The critical value of the Reynolds number though depends on the assumed constant solid volume fraction of the mixture, so that the higher is the solid concentration the more stable is the flux. Hence, it is possible for high-concentrated flows, with a solid fraction near to the maximum packing concentration, to be characterized by macroviscous regime even for Reynolds number of order 10^5, a value that is quite common for laboratory experiments. A carefully analysis is therefore needed before using empirical relations derived from laboratory tests to predict the behaviour of real scale phenomena.

It must be highlighted anyhow that the obtained critical values of the Reynolds number depend on the critical value of the Hanks stability parameter, which we assumed to be the same tested for Newtonian fluid, without the presence of solid grains. Further investigations concerning the critical value of the Hanks parameter when applied to the scale of the ensemble averaged momentum equations of the continuous phase are therefore needed.

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