The areolar strain approach for grazing waves

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Abstract

Authors are almost unanimous regarding the ratios between reflected and incident plane wave amplitudes obtained through the condition of zero traction at a free surface, except for grazing waves. Goodier and Bishop showed that the traditional approach fails in cases of wave travels grazing the surface. The writer argues that the traditional theory fails also for incidence angles beyond that where the total mode conversion from P-wave to S-wave occurs, and also shows that after this angle, phase inversion no longer occurs. The reason for this behavior is that plane waves are characterized by some constraints between their fundamental modes of strain. These constraints are set up by the waveguide equations presented by the writer. It then happens that it does not exist a set of incident and reflected plane waves that can fulfils the condition of zero traction at the reflecting surface. Therefore some scattering of the incident wave is supposed to occur near the free boundary. The equation of conservation of flux of energy together with the equations of zero traction at surface, are not sufficient for determining the nature of this scattering. The writer hence appeals also to the strain compatibility equation at the free surface and obtains an expression for the reflecting P-wave that agrees with the results of Goodier and Bishop. It is remarkable that the angle of total mode conversion is preserved.

New curves of reflections are proposed. The P-wave reflection curve little differs from the traditional up to the total mode conversion angle but completely differs between this up to the grazing angle. The S-wave to P-wave reflection ratio almost follows the traditional.

Waves are represented through the superposition of the four fundamental modes of the plane areolar strain theory of elasticity. This theory was presented at WIT Transactions on Modeling and Simulation, Vol. 46, 2007, WIT Press, (free), with application to orthotropic materials and finite rotations.
Keywords: elastic waves, wave reflection, waveguides, grazing waves.

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1 Introduction

Ewing *et al*. [1] Eringen and Suhubi [2], Kolsky [3], Achenbach [4] and Harris [6], are almost unanimous with regard to the ratios between reflected and incident plane waves amplitudes obtained through the condition of zero traction at a reflecting free surface. Goodier and Bishop [8] and Miklowitz [7] showed that this approach fails for the case the wave travels grazing the surface. Graff [5] commented that the S-wave amplitudes obtained considering only the zero traction condition at the free surface seems to violate the principle of energy conservation.

The writer argues that the traditional theory fails also for incidence angles beyond that where the total conversion from P-wave to S-wave occurs, and also shows that after this incidence angle, phase inversion no longer occurs.

The reason for this disagreement originates from the fact that when a plane wave approaches a free surface, with an acute angle, there is no sufficient restraint at the side turned to the boundary, to maintain undisturbed the *in situ* strain in the direction normal to the wave path, a condition required to maintain the planarity of the wave. The writer presents waveguide equations that set up the constraints the plane waves must comply with. It then happens that it does not exist a set of incident and reflected plane waves that can fulfils the condition of zero traction at the reflecting surface. Therefore we should consider that some scattering of the incident wave is supposed to occur near the free boundary. The equation of conservation of flux of energy together with the equations of zero traction at surface, are not sufficient for determining the nature of this scattering. The writer hence appeals also to the strain compatibility equation at the free surface and obtains an expression for the reflecting P-wave that seems to be physically more plausible than the traditional ones. It is remarkable that the angle of total conversion of P-wave to S-wave is preserved.

Elastic waves are represented throughout the superposition of the four fundamental modes of the plane areolar strain theory. This theory was presented at Kotchergenko [9–11], with application to finite rotations, orthotropic materials and finite element models, but a summary review is here presented. The areolar strain approach does not distinguish finite from infinitesimal strain because besides the traditional “forward” strain it incorporates the “sidelong” strain into its imaginary part. Instead of comparing the change in distance between two contiguous points, this concept describes the complete state of strains on an areola that surrounds a given point. The strain must comply with two conditions: 1) first derivatives only should be used, as expected from the physical meaning of strain; 2) the relative displacement between two arbitrary points of the strained plane should be obtained through a line integral of the strain, along any path joining these points. The areolar strain fulfills both these conditions. In this approach, the strain is obtained by the division of two complex-valued quantities, associated with 2D vectors. The real part of the areolar strain is a radial strain while the imaginary part is either a circumferential strain or a rotation (see Fig. 1). By superimposing the fundamental modes of the areolar strain, a dipper insight can be achieved about the nature of the different types of elastic waves (see Fig. 2).

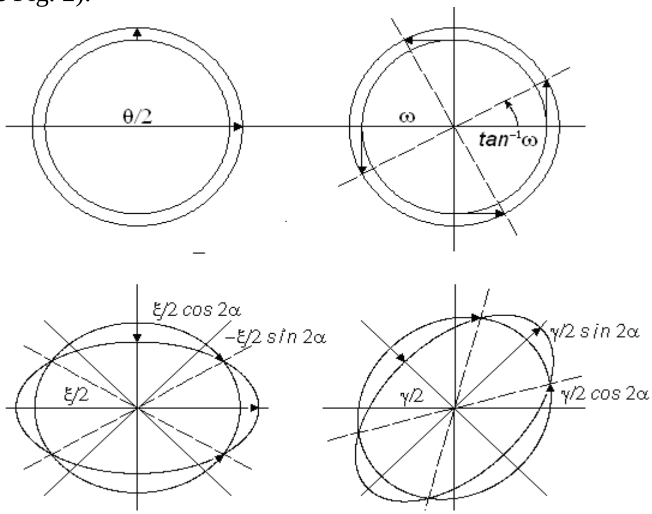

Figure 1: Fundamental modes of the plane strain.

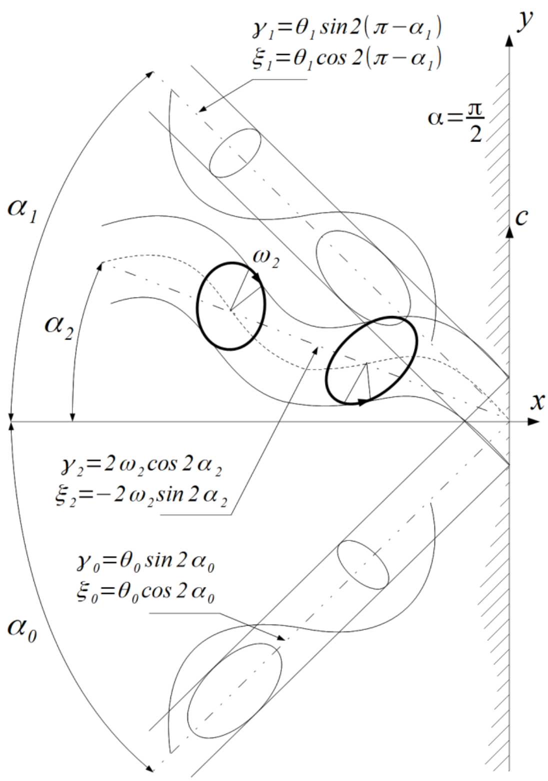

Figure 2: Reflection of a P-wave.

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2 The areolar strain concept

Let a region in the plane of the variables $z = x + iy$ and $\bar{z} = x - iy$ be mapped in a one-to-one manner onto the plane of the displacements $u(x, y)$ and $v(x, y)$ by means of the transformation $w(z,\bar{z}) = u(x, y) + iv(x, y)$. The areolar strain is defined as the gradient of the vector field $w(z,\bar{z})$, through the Riemann derivative [12]:

$$\varepsilon = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{\frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial \bar{z}} d\bar{z}}{dz}$$

$$\varepsilon = \frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}} e^{-i2\alpha} \tag{1}$$

where the polar form has been used for the ratio

$$\frac{d\bar{z}}{dz} = \frac{|d\bar{z}|\, e^{-i\alpha}}{|dz|\, e^{i\alpha}} = e^{-i2\alpha}$$

The last expression presupposes that z tends to z_0, maintaining the direction α. Equation (1) is obtained making

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \;,\; dv = \frac{\partial v}{dx} dx + \frac{\partial v}{dy} dy$$

$$du + i\,dv = \frac{\partial w}{\partial z}(dx + i\,dy) + \frac{\partial w}{\partial \bar{z}}(dx - i\,dy)$$

with

$$2\frac{\partial w}{\partial z} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \theta + i\,2\omega$$

$$2\frac{\partial w}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \xi + i\,\gamma \tag{2}$$

Equations (2) are the Kolosov-Wirtinger [12] derivatives. The areolar strain can hence be written in the form

$$\varepsilon = \tfrac{1}{2}(\theta + i\,2\omega) + \tfrac{1}{2}(\xi + i\,\gamma)e^{-2i\alpha} \tag{3}$$

When viewed in the polar form (see Fig. 1), the real part of the areolar strain represents a radial strain and the imaginary part represents either, a circumferential strain or a rotation. In this conception, the plane strain is decomposed into two orthogonal complex strains shown at fig. 1. The second complex term is the complex shear strain.

3 Compatibility equations

At reference [9] the following compatibility equations were obtained:

$$\frac{\partial}{\partial x}(2\omega-\gamma)+\frac{\partial}{\partial y}(\theta+\xi)=0$$

$$\frac{\partial}{\partial y}(2\omega+\gamma)-\frac{\partial}{\partial x}(\theta-\xi)=0 \tag{4}$$

Saint-Venant's compatibility equation is obtained from these equations through elimination of the mode ω, by applying a cross-differentiation followed by a subtraction. Saint-Venant's compatibility equation will then be satisfied for any field of rotations ω, which may thus violate the compatibility conditions established by eqn. (4). Saint-Venant's mistake was to admit this mode as a rigid body rotation instead of a field of rotations

4 Waves in plane infinite media

Plane waves in homogeneous infinite media are shortly commented, using the approach of superimposing the fundamental strain modes shown in Fig. 1. The addition of the inertial term to Lame's equation 11 in ref. [9], gives

$$(\lambda+\mu)2\frac{\partial\theta}{\partial\bar{z}}+\mu\,2\frac{\partial}{\partial z}(\xi+i\,\gamma)-\rho\frac{\partial^2(u+i\,v)}{\partial t^2}=0 \tag{5}$$

The compatibility equations (4) can be cast into the following complex form

$$\frac{\partial}{\partial\bar{z}}(\theta+2i\,\omega)-\frac{\partial}{\partial z}(\xi+i\,\gamma)=0 \tag{6}$$

Substitution in the second term of eqn. (5) the second term of eqn. (6), gives

$$(\lambda+2\mu)2\frac{\partial\theta}{\partial\bar{z}}+i\mu\,4\frac{\partial\omega}{\partial\bar{z}}-\rho\frac{\partial^2(u+i\,v)}{\partial t^2}=0 \tag{7}$$

Derivative in z results

$$(\lambda+2\mu)2\frac{\partial^2\theta}{\partial z\partial\bar{z}}+i\,\mu\,4\frac{\partial^2\omega}{\partial z\partial\bar{z}}-\rho\frac{\partial^2}{\partial t^2}\frac{\partial w}{\partial z} \tag{8}$$

As $4\frac{\partial^2}{\partial z\partial\bar{z}}=\nabla^2$ and $\frac{\partial w}{\partial z}=\frac{1}{2}(\theta+i\,2\omega)$, separating the real and imaginary parts, we obtain the following irrotational and equivoluminal wave equations:

$$\nabla^2\theta-\frac{\rho}{\lambda+2\mu}\frac{\partial^2\theta}{\partial t^2}=0 \tag{9}$$

$$\nabla^2\omega - \frac{\rho}{\mu}\frac{\partial^2\omega}{\partial t^2} = 0 \quad (10)$$

As the two other fundamental modes ξ and γ must vibrate together with either mode θ or ω, in order to comply with the compatibility equation, similar wave equations governs those modes. It is easy to prove that by taking from the compatibility equation

$$\frac{\partial\theta}{\partial\bar{z}} = \frac{\partial}{\partial z}(\xi + i\,\gamma) - 2\frac{\partial\omega}{\partial\bar{z}} \quad (11)$$

and after grafting $\omega = 0$ for the irrotational wave case, substitution of $\frac{\partial\theta}{\partial\bar{z}}$ into the equilibrium equation gives

$$(\lambda + 2\mu)2\frac{\partial}{\partial z}(\xi + i\,\gamma) - \rho\frac{\partial^2 w}{\partial t^2} = 0 \quad (12)$$

Now taking the derivative in $\bar{z}$, as $\frac{\partial w}{\partial\bar{z}} = \frac{1}{2}(\xi + i\,\gamma)$, results

$$\begin{aligned} \nabla^2\xi_\theta - \frac{\rho}{\lambda + 2\mu}\frac{\partial^2\xi_\theta}{\partial t^2} = 0 \\ \nabla^2\gamma_\theta - \frac{\rho}{\lambda + 2\mu}\frac{\partial^2\gamma_\theta}{\partial t^2} = 0 \end{aligned} \quad (13)$$

Now starting with the condition $\theta = 0$ for the equivoluminal wave condition and following the same procedures, results

$$\begin{aligned} \nabla^2\xi_\omega - \frac{\rho}{\mu}\frac{\partial^2\xi_\omega}{\partial t^2} = 0 \\ \nabla^2\gamma_\omega - \frac{\rho}{\mu}\frac{\partial^2\gamma_\omega}{\partial t^2} = 0 \end{aligned} \quad (14)$$

5 Plane waves in infinite media

In a plane wave displacing in the X direction, the *in situ* strain in the Y direction must not be disturbed in order to avoid wave spreading in this direction. Looking at Fig. 1, we see that the amplitude of vibration of mode θ must be neutralized by the amplitude of mode ξ. Grafting $\alpha = \pi/2$ into eqn. (1) and equating to zero, will result

$$\frac{\partial w}{\partial z} - \frac{\partial w}{\partial\bar{z}} = 0 \quad (15)$$

This would be a Beltrami's equation, if the Beltrami coefficient could be equal to 1. For $\alpha = \pi/2$, eqn. (3) will give $\theta = \xi$ and $2\omega = \gamma$. Actually these equations simply mean that $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = 0$ respectively, but an inspection of Fig. 1 allows a deeper insight. The first condition gives zero radial strain in the Y direction while the second condition gives zero circumferential strain in that same direction as it implies that the amplitude $\gamma / 2$ of the rotation with respect to the X-axis, resulting from the γ shear mode, is neutralized by the amplitude of the rotation ω. As a result, the strain amplitudes in the radial and circumferential Y directions remains zero while the strain amplitude in the radial X direction will be $1/2\,\theta + 1/2\,\xi = \theta$ while the strain amplitude in the circumferential X direction reaches $\gamma / 2 + \omega = \gamma$. The condition $\theta = \xi$ gives rise to a P-wave while the condition $2\omega = \gamma$ gives rise to a S-wave, both traveling in the X direction.

Plane waves, displacing in a direction forming an angle α_0 with the X-axis comply with the condition

$$\frac{\partial w}{\partial z} + \frac{\partial w}{\partial \overline{z}} e^{-i2(\alpha_0 + \frac{\pi}{2})} = 0 \tag{16}$$

For a P-wave, as $\omega = 0$, results

$$\begin{aligned} \gamma_\theta &= \theta \sin 2\alpha_0 \\ \xi_\theta &= \theta \cos 2\alpha_0 \end{aligned} \tag{17}$$

and for S-wave, as $\theta = 0$,

$$\begin{aligned} \gamma_\omega &= 2\omega \cos 2\alpha_0 \\ \xi_\omega &= -2\omega \sin 2\alpha_0 \end{aligned} \tag{18}$$

Equations (17) and (18) can be called the in-plane waveguide equations for plane waves.

6 Flux of energy

The flux of energy of P-waves is exposed on several books, however the same does not occurs with the S-waves. The reflected S-wave, having amplitude $2\tilde{\omega}_2$ and traveling at the direction α_2 with the same frequency $\Omega = \kappa_2 \, c_T$ of the P-wave, with the velocity $c_T = \sqrt{\frac{\mu}{\rho}}$, and wave number $\kappa_2 = \kappa_0 \sqrt{\frac{\lambda + 2\mu}{\mu}}$, has the form

$$\omega_2 = \tilde{\omega}_2 \cos[\kappa_2 \, (s - c_T \, t)] \tag{19}$$

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The velocity of the P-wave is $c_L = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ and $\Omega = \kappa_0\, c_L$. Equation (18) gives

$$\xi_2 = -2\tilde{\omega}_2 \cos[\kappa_2\,(s - c_T\, t)]\sin 2\alpha_2 \tag{20}$$

$$\gamma_2 = 2\tilde{\omega}_2 \cos[\kappa_2\,(s - c_T\, t)]\cos 2\alpha_2 \tag{21}$$

As for S-wave, $\theta = 0$, the areolar strain in the direction α_2, according to eqn. (3), will be

$$\varepsilon_2 = \frac{1}{2}[2\,i\,\omega_2 + (\xi_2 + i\,\gamma_2)e^{-2i\alpha 2}] \tag{22}$$

Substitution of equations (20) and (21) gives

$$\varepsilon_2 = i\;2\tilde{\omega}_2 \cos[\kappa_2(s - c_T t)] \tag{23}$$

The imaginary number shows that it is a circumferential strain in the direction of the wave path. Integration of eqn. (23) along *s*, gives its displacement

$$w_2 = i\;2\tilde{\omega}_2 \sin[\kappa_2(s - c_T\, t)]/\,\kappa_2 \tag{24}$$

and derivative in time, the velocity

$$\dot{w}_2 = -i\;2\tilde{\omega}_2 c_T \cos[\kappa_2(s - c_T t)] \tag{25}$$

At Ref [9] it was shown that the traction at a surface is given by

$$T = (\lambda + \mu)\theta - \mu(\xi + i\,\gamma)e^{-2i\,\alpha} \tag{26}$$

where α is the angle the counterclockwise tangent to the surface makes with the X axis; the real part is the normal stress and the imaginary part is the shearing stress. The traction at the wave front is then

$$\sigma s_2 = -\mu(\xi_2 + i\,\gamma_2)e^{-2i(\alpha_2 + \frac{\pi}{2})} \tag{27}$$

Substitution of eqn. (18), rewritten in the form

$$\begin{aligned} \gamma_2 &= 2\omega_2 \cos 2\alpha_2 \\ \xi_2 &= -2\omega_2 \sin 2\alpha_2 \end{aligned} \tag{28}$$

gives

$$\sigma s_2 = -i\,\mu\,2\tilde{\omega}_2 \cos[\kappa_2(s - c_T t)] \tag{29}$$

Multiplying by the front wave velocity given by eqn. (25), yields

$$\Pi_2 = (2\tilde{\omega}_2)^2\, c_T\,\mu \cos^2[\kappa_2(s - c_T t)] \tag{30}$$

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The mean fluxes of energy of incident and reflected waves will hence be

$$\Pi_0 = \frac{1}{2} c_L \tilde{\theta}_0^2 (\lambda + 2\mu) \tag{31}$$

$$\Pi_1 = \frac{1}{2} c_L \tilde{\theta}_1^2 (\lambda + 2\mu) \tag{32}$$

$$\Pi_2 = \frac{1}{2} (2\tilde{\omega}_2)^2 \, c_T \mu \tag{33}$$

where $2\tilde{\omega}_2$ represents the amplitude of the of the S-wave.

7 Reflection of plane waves in semi-infinite media

The sum of the mean reflected fluxes of energy must be equal to the mean flux of energy of the incident wave. Then

$$\Pi_0 cos\,\alpha_0 = \Pi_1 cos\,\alpha_0 + \Pi_2 cos\,\alpha_2 \tag{34}$$

where the cosines represent the wideness of the beams of the waves, referred to a unit length of the boundary, (see fig. 2). Solving for $\tilde{\omega}_2$ gives

$$2\tilde{\omega}_2 = \frac{\sqrt{(\tilde{\theta}_0^2 - \tilde{\theta}_1^2)}\sqrt{c_L(\lambda + 2\mu)\cos\alpha_0}}{\sqrt{c_T \mu \cos\alpha_2}} \tag{35}$$

The tractions at the free surface, positioned at $\alpha = \pi / 2$, as shown in Fig. 2, are $T = \sigma_{xx} + i\,\tau_{xy}$. It is impossible to find a set of incident and reflected plane waves, represented by equations (17) and (18) that will comply with the condition of zero traction at the surface. We will then suppose that plane waves are suddenly scattered when approaching the reflecting surface and no longer remain plane. Hence we no longer will use the constraints given by the waveguide equations (17) for the P-waves. However for the S-wave, equation (18) will be needed due to the following reason: the change in area due to a plane strain, as given in Ref. [9], is

$$dA = \theta + \frac{\theta^2}{4} + \omega^2 - \frac{1}{4}(\xi^2 + \gamma^2) = \theta + \left|\frac{\partial w}{dz}\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}\right|^2 = \theta + J \tag{36}$$

where J stands for the Jacobian of the mapping $w(z,\bar{z}) = u(x,y) + iv(x,y)$. Taking into account that for S-wave, $\theta = 0$, eq. (36) gives

$$(2\omega_2)^2 = \xi_2^2 + \gamma_2^2 \tag{37}$$

Equation (18) gives a plausible solution for this equation and will be adopted. Therefore, the stresses at the free surface will not be subjected to all the constraints the plane waves are subjected to and will read

$$\sigma_{xx0} = \tilde{\theta}_0(\lambda + \mu) + \tilde{\xi}_0\ \mu \tag{38}$$

$$\tau_{xy0} = \tilde{\gamma}_0\ \mu \tag{39}$$

$$\sigma_{xx1} = \tilde{\theta}_1(\lambda + \mu) + \tilde{\xi}_0\ \mu \tag{40}$$

$$\tau_{xy1} = \tilde{\gamma}_1\ \mu \tag{41}$$

$$\sigma_{xx2} = \tilde{\xi}_2\ \mu \tag{42}$$

$$\tau_{xy2} = \tilde{\gamma}_2\ \mu \tag{43}$$

As for a free surface, the resulting traction in null, then

$$\sigma_{xx0} + \sigma_{xx1} + \sigma_{xx2} = 0 \tag{44}$$

$$\tau_{xy0} + \tau_{xy1} + \tau_{xy2} = 0 \tag{45}$$

Following Ewing, Jardetzky and Press [1] we seek for a solutions of the form

$$\theta = f[y]e^{ik(ct-x)},\ \omega_2 = g[y]e^{ik(ct-x)} \tag{46}$$

Substituting θ in equation (9) gives

$$\frac{\partial^2 f}{\partial y^2} + (\frac{c^2}{cL^2} - 1)k^2 f = 0 \tag{47}$$

Integration of this equation is

$$f[y] = \tilde{\theta}_0 \exp[i\,k\sqrt{\frac{c^2}{cL^2} - 1}\ y] + \tilde{\theta}_1 \exp[-i\,k\sqrt{\frac{c^2}{cL^2} - 1}\ y] \tag{48}$$

As explained in Ref. [1], c is an apparent velocity of the wave along surface and

$$\sqrt{\frac{c^2}{cL^2} - 1} = \cot\alpha_0 \tag{49}$$

where α_0 is the angle of incidence of the incoming P-wave, (see fig 2). The solutions (46) will then be

$$\theta_0 = \tilde{\theta}_0 \exp[i\,k(c\,t - x + y\cot\alpha_0)] \tag{50}$$

$$\theta_1 = \tilde{\theta}_1 \exp[i\,k(c\,t - x - y\cot\alpha_0)] \tag{51}$$

$$\omega_2 = \tilde{\omega}_2 \exp[i\,k(c\,t - x - y\cot\alpha_2)] \tag{52}$$

where $\tilde{\theta}_0$ is the amplitude of the incident P-wave, $\tilde{\theta}_1$ is the amplitude of the reflected P-wave and $2\tilde{\omega}_2$ is the amplitude of the reflected S-wave. Equations (42) and (43) furnish

$$\begin{aligned} \gamma_1 &= -\gamma_0 - \gamma_2 \\ \theta_1 &= -\frac{(\lambda+\mu)\theta_0 + \mu(\xi_0 + \xi_1 + \xi_2)}{\lambda+\mu} \end{aligned} \tag{51}$$

From this system of equations we obtain γ_0 of ξ_0 and from equations (35) and (37) we obtain γ_2.

The second compatibility equation (4) requires that

$$\frac{\partial}{\partial y}(2\omega_2 + \gamma_0 + \gamma_1 + \gamma_2) - \frac{\partial}{\partial x}(\theta_0 + \theta_1 - \xi_0 - \xi_1 - \xi_2) = 0 \tag{52}$$

When these derivatives are calculated, the derivatives of the unknowns functions ξ_1, ξ_2 and γ_1 cancel, as a consequence of eq. (52) being an identity. Putting x=0 for the free surface and also y=0, t=0, as this equation must be valid for any point of the free surface and at any time, results

$$\frac{\tilde{\theta}_1}{\tilde{\theta}_0} = \frac{-cL\,\mu\cos^2\alpha_0 - 2\,cT(\lambda+2\mu)\cos\alpha_0\cos\alpha_2 + cL\,\mu(1+\sin^2\alpha_0)}{cL\,\mu\cos^2\alpha_0 - 2\,cT(\lambda+2\mu)\cos\alpha_0\cos\alpha_2 - cL\,\mu(1+\sin^2\alpha_0)} \tag{53}$$

This ratio, with $\lambda = \mu$, is plotted bellow with solid line. The dashed line represents the traditional ratio. The solid line is more realistic and is in agreement with the results of Goodier and Bishop for the grazing condition. As the point of total mode conversion and the point of grazing condition are joined by a smooth curve, it is expected that the solid line better represents this ratio.

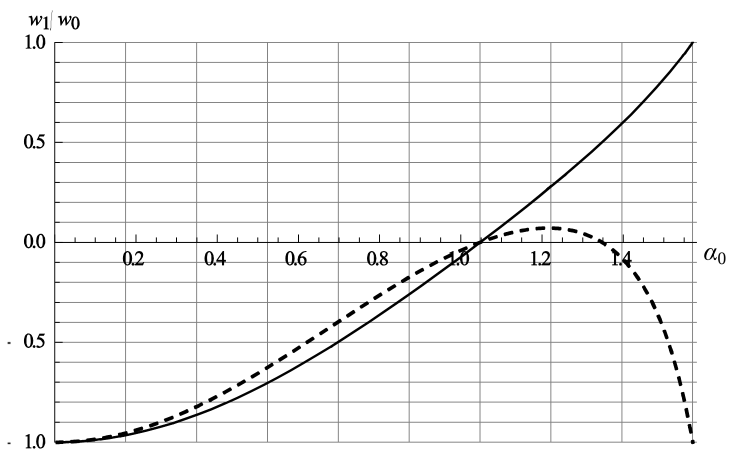

Figure 3: Reflection ratios of a P-wave amplitudes.

The amplitude of the S-wave strain is given by the modulus of the complex shear strain. From eq. (28)

$$\sqrt{(\xi_2 + i\gamma_2)\overline{(\xi_2 + i\gamma_2)}} = \sqrt{\xi_2^2 + \gamma_2^2} = 2\tilde{\omega}_2 \tag{54}$$

From eq. (24) we get the amplitude of displacement in the direction transversal to the S-wave path

$$\tilde{w}_2 = 2\tilde{\omega}_2 / \kappa_2 \tag{55}$$

Using for the P-wave, the same procedure used from eq. (19) to eq. (24), we get the displacement amplitudes of the P-waves

$$\tilde{w}_0 = \tilde{\theta}_0 / \kappa_0 \;, \tilde{w}_1 = \tilde{\theta}_1 / \kappa_0 \tag{56}$$

Therefore the ratio between reflected and incident P-waves amplitudes is equal to the ratio between their strain amplitudes, as a consequence of having the same wave length. The ratio of the S-wave to the incident P-wave, calculated for a material with $\lambda = \mu$, is plotted below.

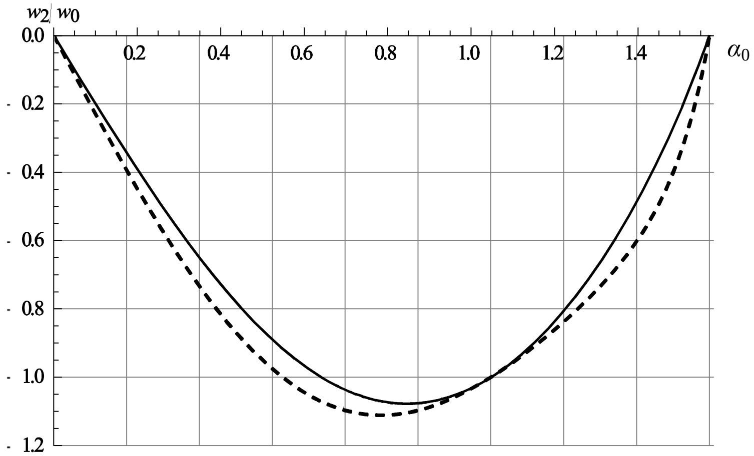

Figure 4: Amplitude ratio of S-wave.

8 Conclusion

The problem of the reflection of a P-wave grazing a surface has been the concern since long. It was assumed that the waves maintain their planarity even when approaching the free surface. But when a plane wave approaches a free surface, with an acute angle, there is no sufficient restraint at the side turned to the boundary, to maintain undisturbed the *in situ* strain in the direction normal to the wave path. However this was the condition set for the waveguide equations (17) and (18). At the present approach this condition was relaxed and the tractions assumed at equations (36) to (41) are of a general character. This change makes it necessary to appeal for three conditions in order to solve the problem: conservation of the flux of energy, two free boundary traction conditions and one equation of compatibility of strains at free boundary. It was demonstrated that the complex shear mode $\xi_0 + i\,\gamma_0$ of the incoming wave drastically changes near the surface, compared to that of the plane wave given at equations (17). However the complete picture of the scattering was not achieved as several strain modes could not be determined. The obtained results differ a little from the traditional ones up to the incidence angle of total mode conversion. From this angle up to the grazing angle, the behavior of the reflecting P-wave changes significantly and phase inversion no longer occurs. This behavior complies with the results obtained by Goodier and Bishop.

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