The areolar strain concept applied to elasticity

I. D. Kotchergenko
Instituto Militar de Engenharia, Rio de Janeiro, Brazil

Abstract

In contrast to the books that start with solutions of Lamé-Navier equations using complex variables, the present article starts with a presentation of the fundamentals of the complex theory of two-dimensional elasticity. A new strain expression is derived and the compatibility conditions for these strains are given. The fundamental equations for the isotropic and orthotropic plane elasticity are also presented in somewhat new complex forms. The homogeneous equilibrium equation is presented in a complex form that proved to be easily solvable. Kolosov's general solution for the isotropic case is obtained in a fairly straightforward fashion. New equilibrium equations and boundary conditions for finite rotation are also given.
Keywords: areolar strain, compatibility equation, finite rotation, complex elasticity.

1 Introduction

The development of the theory of two-dimensional elasticity hereof is grounded on the concept of areolar strain. This concept was first presented by Kotchergenko [8], in 1983, though containing some errors. In this approach the strain is obtained by the division of two complex-valued quantities associated with 2D vectors and called areolar strain owing to the fact that its real part represents a radial strain and its imaginary part represents either a circumferential strain or a rotation. The quaternion concept can probably be used to extend the areolar strain concept to the 3D vectors case. Improvements on the linear theory were recently presented, Kotchergenko [9,10]. Equilibrium equations for finite rotation and several other improvements are now included. A detailed review of the main results obtained until now is provided in order to make this article self contained.

2 The areolar strain

Let a region in the plane of the variables $z = x + iy$ and $\bar{z} = x - iy$ be mapped in a one-to-one manner onto the plane of the displacements $u(x,y)$ and $v(x,y)$ by means of the transformation $w(z,\bar{z}) = u(x,y) + iv(x,y)$. Given the direction α of the vector $z - z_0$, where z and z_0 are two neighboring points of the plane, the areolar strain is defined as the gradient of the vector field $w(z,\bar{z})$ in the direction α, through the areolar derivative:

$$\varepsilon = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{\frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial \bar{z}} d\bar{z}}{dz} \quad \text{or} \quad \varepsilon = \frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}} e^{-i2\alpha}, \tag{1}$$

where the polar form has been used for the ratio $\dfrac{d\bar{z}}{dz} = \dfrac{|d\bar{z}|e^{-i\alpha}}{|dz|e^{i\alpha}} = e^{-i2\alpha}$. This expression presupposes that z tends to z_0, maintaining the direction α. Since

$$2\frac{\partial w}{\partial z} = (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) = \theta + i2\omega ,$$

$$2\frac{\partial w}{\partial \bar{z}} = (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = \xi + i\gamma , \tag{2}$$

the areolar strain can also be written in the form

$$\varepsilon = \tfrac{1}{2}(\theta + i2\omega) + \tfrac{1}{2}(\xi + i\gamma)e^{-2i\alpha} . \tag{3}$$

When viewed in the polar form, the real part of the areolar strain represents a radial strain and the imaginary part represents either, a circumferential strain or a rotation. The second complex term is the complex shear strain. The components of the areolar strain are orthogonal as the quadratic form contained in the integrand of the work expression $U = 1/2\int C_{ij}\varepsilon_i\varepsilon_j dV$ can be converted into its canonic form

$$U = \tfrac{1}{2}\int[\tfrac{1}{2}(C_{11} + C_{12})\theta^2 + \tfrac{1}{2}(C_{11} - C_{12})\xi^2 + C_{13}\gamma^2]dV . \tag{4}$$

3 Compatibility equations

If z_0 and z are two points pertaining to the complex plane, their relative displacement is given by

$$w - w_0 = \int_C \varepsilon dz = \int_C (\frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}}\frac{d\bar{z}}{dz})dz = \int_C \frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial \bar{z}} d\bar{z} . \tag{5}$$

Since $w - w_0$ is independent of the path of integration C,

$$dw = \frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial \bar{z}} d\bar{z}$$
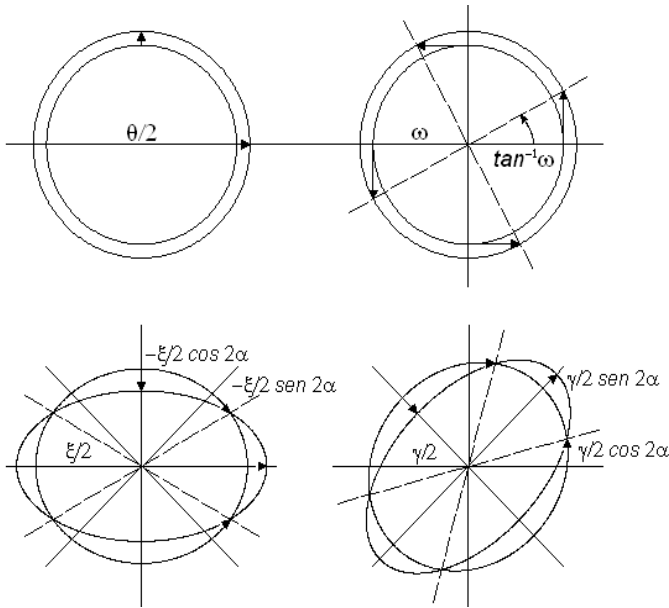
$$\tag{6}$$

Figure 1:   Fundamental modes of the areolar strain.

is a total differential. Consequentially, the displacement field must comply with the condition of continuity

$$\frac{\partial}{\partial \bar{z}}(\frac{\partial w}{\partial z}) - \frac{\partial}{\partial z}(\frac{\partial w}{\partial \bar{z}}) = 0 .$$
(7)

Separating the real and imaginary parts of this equation, the following compatibility equations are obtained:

$$\frac{\partial}{\partial x}(2\omega - \gamma) + \frac{\partial}{\partial y}(\theta + \xi) = 0 ,$$

$$\frac{\partial}{\partial y}(2\omega + \gamma) - \frac{\partial}{\partial x}(\theta - \xi) = 0 .$$
(8)

Saint-Venant's compatibility equation is obtained from these equations through elimination of the mode ω, by applying a cross-differentiation followed by a subtraction. Saint-Venant's compatibility equation will then be satisfied for any rotational field, which may thus violate the compatibility conditions (8).

4   Equilibrium equations

For isotropic material, $\frac{1}{2}(C_{11} + C_{12}) = \lambda + \mu$, $\frac{1}{2}(C_{11} - C_{12}) = \mu$ and $C_{13} = \mu$, thus the work expression given by eqn. (4) reduces to

$$U = \tfrac{1}{2} \int [(\lambda + \mu)\theta^2 + \mu(\xi^2 + \gamma^2)]dV \tag{9}$$

where λ and μ are Lame's elastic material constants. The Euler equations for this functional are

$$(\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu(\frac{\partial \xi}{\partial x} + \frac{\partial \gamma}{\partial y}) = 0$$
$$(\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu(\frac{\partial \gamma}{\partial x} - \frac{\partial \xi}{\partial y}) = 0 \tag{10}$$

Taking into account that

$$\frac{\partial}{\partial z}(\xi + i\gamma) = (\frac{\partial \xi}{\partial x} + \frac{\partial \gamma}{\partial y}) + i(\frac{\partial \gamma}{\partial x} - \frac{\partial \xi}{\partial y}) = \Delta u(x, y) + i\Delta v(x, y) ,$$

where the symbol Δ stands for the Laplacian operator, Lamé's homogeneous equilibrium equations can be presented in the following complex form:

$$2\frac{\partial}{\partial \bar{z}}[(\lambda + \mu)\theta] + 2\frac{\partial}{\partial z}[\mu(\xi + i\gamma)] = 0 \tag{11}$$

Multiplying each term of eqn (7) by a different complex elastic constant will result in an equation in terms of stresses. For isotropic material, undergoing small strain and small rotations, the operations required for obtaining Lame's homogeneous equilibrium equation are

$$\frac{\partial}{\partial \bar{z}}[(\lambda + \mu)(2\frac{\partial w}{\partial z} - i\omega)] - \frac{\partial}{\partial z}(\frac{\lambda}{2}2\frac{\partial w}{\partial \bar{z}}) = 0 . \tag{12}$$

Observe that only one out of the two $i\omega$, which appear in the term $i2\omega$, was removed. The term $i\omega$ represents a local rotation. Taking the derivatives of ω from eqns. (8) and substituting into eqn. (12) will give eqn. (11)

5 Boundary conditions

Applying Green's formula (e.g. Courant and Hilbert [1]) to eqn. (11), in the complex form,

$$2\iint_{\Omega} \{\frac{\partial}{\partial \bar{z}}[(\lambda + \mu)\theta] + \frac{\partial}{\partial z}[\mu(\xi + i\gamma)]\}dxdy = \frac{1}{i}\oint_C (\lambda + \mu)\theta dz - \mu(\xi + i\gamma)d\bar{z} , \tag{13}$$

the traction vector on a boundary curve C results

$$T = (\lambda + \mu)\theta - \mu(\xi + i\gamma)e^{-2i\alpha} , \tag{14}$$

where α points towards the direction of the vector element of arc dz of the closed contour curve C. Observe that if n the unit outward vector, normal to the element of arc $ds = |dz|$, then $nds = -idz = \tfrac{1}{i} dz$.

6   Other forms for the equilibrium equations

The compatibility equations (7) can also be rewritten in the following complex form

$$\frac{\partial}{\partial \bar{z}}(\theta + i2\omega) - \frac{\partial}{\partial z}(\xi + i\gamma) = 0 \tag{15}$$

.

Equation (11) can hence be reduced to the following holomorphic function (Love, [4]),

$$\frac{\partial}{\partial \bar{z}}[\theta + i\eta 2\omega] = 0 \tag{16}$$

,

where $\eta = \dfrac{\mu}{\lambda + 2\mu}$. The modes θ and ω are hence harmonic functions.

   The writer succeeded in rewriting the equilibrium equation in another form that proved to be easily solvable. The elimination of the derivatives of θ between equations (8) and (16), gives

$$2\frac{\partial \omega}{\partial x} = \frac{\lambda + 2\mu}{\lambda + \mu}(\frac{\partial \gamma}{\partial x} - \frac{\partial \xi}{\partial y}) ,$$

$$2\frac{\partial \omega}{\partial y} = -\frac{\lambda + 2\mu}{\lambda + \mu}(\frac{\partial \xi}{\partial x} + \frac{\partial \gamma}{\partial y}) . \tag{17}$$

The elimination of the derivatives of ω, from the same pair of equations, gives

$$\frac{\partial \theta}{\partial x} = -\frac{\mu}{\lambda + \mu}(\frac{\partial \xi}{\partial x} + \frac{\partial \gamma}{\partial y}) ,$$

$$\frac{\partial \theta}{\partial y} = -\frac{\mu}{\lambda + \mu}(\frac{\partial \gamma}{\partial x} - \frac{\partial \xi}{\partial y}) . \tag{18}$$

Combination of these equations yields:

$$2\frac{\partial}{\partial z}(\theta + i2\omega) = -\chi[(\frac{\partial \xi}{\partial x} + \frac{\partial \gamma}{\partial y}) - i(\frac{\partial \gamma}{\partial x} - \frac{\partial \xi}{\partial y})] , \tag{19}$$

Or after eqns. (2),

$$\frac{\partial^2 w}{\partial z^2} + \chi \overline{\frac{\partial^2 w}{\partial z \partial \bar{z}}} = 0 . \tag{20}$$

where $\chi = \dfrac{\lambda + 3\mu}{\lambda + \mu}$ for plane strain and $\chi = \dfrac{\lambda^* + 3\mu}{\lambda^* + \mu}$, for plane stress, observing

that $\lambda^* = \dfrac{2\lambda\mu}{\lambda + 2\mu}$.

7   General solution

Differentiation of eqn. (11) in z results

$$2(\lambda + \mu)\frac{\partial^2 \theta}{\partial z \partial \bar{z}} + 2\mu \frac{\partial^2}{\partial z^2}[(\xi + i\gamma)] = 0 \tag{21}$$

Since $4\dfrac{\partial^2}{\partial z \partial \bar{z}}$ is the Laplacian operator and θ is a harmonic function, this equation reduces to

$$2\mu\frac{\partial^2}{\partial z^2}[(\xi + i\gamma)] = 0 \tag{22}$$

or after eqns. (2), to

$$2\frac{\partial}{\partial z}\left(\frac{\partial^2 w}{\partial z \partial \bar{z}}\right) = 0.$$

Integration in z gives

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = -\frac{1}{2\mu}\overline{\varphi''(z)}, \tag{23}$$

where $\overline{\varphi''(z)}$ is the conjugate of an analytic function. Integrating eqn. (23) in z, results

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}(\xi + i\gamma) = -\frac{1}{2\mu}[z\overline{\varphi''(z)} + \overline{\psi'(z)}]. \tag{24}$$

Equations (20) and (23) furnish

$$\frac{\partial^2 w}{\partial z^2} = \frac{\chi}{2\mu}\varphi''(z), \tag{}$$

which after integration in z, results

$$\frac{\partial w}{\partial z} = \frac{\chi}{2\mu}\varphi'(z) + \zeta'(\bar{z}). \tag{25}$$

Differentiation of this equation with respect to $\bar{z}$ yields

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial w}{dz}\right) = \zeta''(\bar{z}). \tag{26}$$

From eqn. (23), results $\zeta''(\bar{z}) = -\dfrac{1}{2\mu}\overline{\varphi''(z)}$ and hence

$$\zeta'(\bar{z}) = -\frac{1}{2\mu}\overline{\varphi'(z)} + c, \tag{27}$$

where c is a complex constant. Substitution into eqn. (25) yields

$$\frac{\partial w}{\partial z} = \frac{1}{2}(\theta + i2\omega) = \frac{1}{2\mu}[\chi\varphi'(z) - \overline{\varphi'(z)}] + \frac{1}{2}(\theta_0 + i2\omega_0). \tag{28}$$

Using eqns. (28) and (24), the integration of the total differential, eqn. (5), gives Kolosov-Muskhelishvili's general solution, [2,3,5,7]:

$$w = w_0 + \frac{1}{2}(\theta_0 + i2\omega_0)z + \frac{1}{2\mu}[\chi\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}]. \tag{29}$$

The addition of Eqn (28) to its conjugate gives

$$\theta = \frac{\chi - 1}{2\mu}[\varphi'(z) + \overline{\varphi'(z)}] + \theta_0 \cdot \tag{30}$$

The subtraction of Eqn (28) from its conjugate gives

$$\omega = \frac{\chi + 1}{2\mu}[\frac{\varphi'(z) - \overline{\varphi'(z)}}{2i}] + \omega_0 \cdot \tag{31}$$

Kolosov´s formulas, for stresses are obtained directly from eqns. (30) and (24):

$$\sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \overline{\varphi'(z)}] + 2(\lambda + \mu)\theta_0 , \tag{32}$$

$$\sigma_{11} - \sigma_{22} + i2\sigma_{12} = -2[z\overline{\varphi''(z)} + \overline{\psi'(z)}] . \tag{33}$$

The areolar strain, in terms of analytic functions assumes the form

$$\varepsilon = \frac{1}{2}(\theta_0 + i2\omega_0) + \frac{1}{2\mu}[\chi\varphi'(z) - \overline{\varphi'(z)}] - \frac{1}{2\mu}[z\overline{\varphi''(z)} + \overline{\psi'(z)}]e^{-i2\alpha} . \tag{34}$$

8 Equilibrium equations for an orthotropic plane

The writer explored the possibility of applying to the orthotropic plane case the same approach used with eqn. (11). Taking the compatibility equation in the form of eqn. (15) and removing one out of the two $i\omega$ present into the term $i2\omega$; then multiplying each complex term of this equation by different complex elastic constants, will result in the following equation in terms of stresses:

$$(a_1 + ia_2)\frac{\partial}{\partial \overline{z}}(\theta + i\omega) - (b_1 + ib_2)\frac{\partial}{\partial z}(\xi + i\gamma) = 0 \cdot \tag{35}$$

Substitution of the derivatives of the rotation, obtained from eqn. (8), gives

$$(a_1 + ia_2)\frac{\partial \theta}{\partial \overline{z}} + [(a_1 + ia_2) - 2(b_1 + ib_2)]\frac{\partial}{\partial z}(\xi + i\gamma) = 0 \cdot \tag{36}$$

Lamé's equilibrium equations are obtained letting $a_2 = 0$, $b_2 = 0$, $a_1 = \lambda + \mu$ and $b_1 = \lambda/2$.

Application of Green's formula to eqn. (36), gives the following generalization of the traction vector formula, previously depicted in eqn. (14):

$$T = (a_1 + ia_2)\theta - [(a_1 + ia_2) - 2(b_1 + ib_2)](\xi + i\gamma)e^{-i2\alpha} . \tag{37}$$

Using the same procedure as the one used for obtaining eqn. (16) leads to the same Love's form of holomorphic function:

$$\frac{\partial}{\partial \overline{z}}[\theta + i\eta 2\omega] = 0 , \tag{38}$$

where

$$\eta = \frac{(a_1 - 2b_1) + i(a_2 - 2b_2)}{2[(a_1 - b_1) + i(a_2 - b_2)]} . \tag{39}$$

Observe that for the isotropic plane, η will be reduced to a real constant.

Separating the real and imaginary parts yields:

$$\frac{\partial \theta}{\partial x} = \eta_1 2 \frac{\partial \omega}{\partial y} + \eta_2 2 \frac{\partial \omega}{\partial x},$$

$$\frac{\partial \theta}{\partial y} = -\eta_1 2 \frac{\partial \omega}{\partial x} + \eta_2 2 \frac{\partial \omega}{\partial y}, \tag{40}$$

with $\eta = \eta_1 + i\eta_2$, where

$$\eta_1 = \frac{a_1^2 + a_2^2 - 3(a_1 b_1 + a_2 b_2) + 2(b_1^2 + b_2^2)}{2[(a_1 - b_1)^2 + (a_2 - b_2)^2]},$$

$$\eta_2 = \frac{a_2 b_1 - a_1 b_2}{2[(a_1 - b_1)^2 + (a_2 - b_2)^2]}. \tag{41}$$

Differentiations of eqns. (40), yields

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \eta_2 2 \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right). \tag{42}$$

Other differentiations of eqns. (40), yields additionally:

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{2\eta_1} \left(\eta_2 2 \frac{\partial^2 \omega}{\partial x \partial y} - \frac{\partial^2 \theta}{\partial x \partial y} \right) \quad \text{and} \quad \frac{\partial^2 \omega}{\partial y^2} = \frac{1}{2\eta_1} \left(-\eta_2 2 \frac{\partial^2 \omega}{\partial x \partial y} + \frac{\partial^2 \theta}{\partial x \partial y} \right).$$

Hence $\dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2} = 0$, and after eqn. (42), both ω and θ remain harmonics also for the orthotropic plane case.

9 Finite rotations

The gain of area during a plane deformation is given by

$$dA = \theta + \frac{\theta^2}{4} + \omega^2 - \frac{1}{4}(\xi^2 + \gamma^2) = \theta + \left| \frac{\partial w}{\partial z} \right|^2 - \left| \frac{\partial w}{\partial \bar{z}} \right|^2, \tag{43}$$

which is obtained from the determinant

$$1 + dA = \begin{vmatrix} 1 + \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & 1 + \dfrac{\partial v}{\partial y} \end{vmatrix}. \tag{44}$$

Observing fig. 1, it can be seen that a finite "rotation" ω has a significant influence into the change of area of a plane elastic body. In some bending and buckling problems, it is admitted to disregard strain terms squared, except for ω^2, reducing dA to

$$dA = \theta + \omega^2. \tag{45}$$

A rigid body rotation in this case can be approximated by the condition $\theta + \omega^2 = 0$. Then, for a point to describe a circular path of arc $\Omega = \tan^{-1}\omega$, it is required that the expansion produced by the mode ω be neutralized by the shrinkage $\theta = -\omega^2$. For the condition given by eqn. (45), the work expression assumes the form

$$U = 1/2\int (\lambda + \mu)(\theta + \omega^2)^2 + \mu(\xi^2 + \gamma^2)dxdy . \tag{46}$$

The Euler equations for this functional are

$$(\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu(\frac{\partial \xi}{\partial x} + \frac{\partial \gamma}{\partial y}) + (\lambda + \mu)[-\frac{\partial}{\partial y}(\omega\theta) + \frac{\partial}{\partial x}(\omega^2) - 3\omega^2\frac{\partial \omega}{\partial y}] = 0 ,$$

$$(\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu(\frac{\partial \gamma}{\partial x} - \frac{\partial \xi}{\partial y}) + (\lambda + \mu)[\frac{\partial}{\partial x}(\omega\theta) + \frac{\partial}{\partial y}(\omega^2) + 3\omega^2\frac{\partial \omega}{\partial x}] = 0 . \tag{47}$$

This system of equations can be written in the complex form

$$(\lambda + \mu)\frac{\partial}{\partial \bar{z}}[(1 + i\omega)(\theta + \omega^2] + \mu\frac{\partial}{\partial z}(\xi + i\gamma) = 0 . \tag{48}$$

Applying Green's formula, the traction vector on a closed curve C, in the undeformed reference frame, results

$$T = (\lambda + \mu)[(1 + i\omega)(\theta + \omega^2)] - \mu(\xi + i\gamma)e^{-i2\alpha} . \tag{49}$$

Accordingly with eqns. (I.22) from Novozhilov [6], the shearing modes due to finite strains are: $\hat{\xi} = \xi + \frac{1}{2}\theta\xi + \omega\gamma$ and $\hat{\gamma} = \gamma + \frac{1}{2}\theta\gamma - \omega\xi$, hence discarding the strains squared, other than those containing ω, results $\xi = \hat{\xi} - \omega\gamma$ and $\gamma = \hat{\gamma} + \omega\xi$. If the following approximations are used accordingly:

$$\theta + \omega^2 = \frac{\sigma_x + \sigma_y}{2(\lambda + \mu)} , \quad \xi = \frac{\sigma_x - \sigma_y}{2\mu} - \omega\frac{\tau}{\mu} \text{ and } \gamma = \frac{\tau}{\mu} + \omega\frac{\sigma_x - \sigma_y}{2\mu} , \tag{50}$$

the equilibrium equations (47) in terms of stresses, referred to the undeformed reference frame, become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} - \frac{\partial}{\partial y}(\omega\sigma_y) - \frac{\partial}{\partial x}(\omega\tau) = 0 ,$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} + \frac{\partial}{\partial x}(\omega\sigma_x) + \frac{\partial}{\partial y}(\omega\tau) = 0 . \tag{51}$$

These are exactly the equilibrium equations (II.49) given by Novozhilov, [6].

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