Path dependent options: the case of high water mark provision for hedge funds

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Abstract

High Water Mark (HWM) provision is an important feature in the hedge fund industry. The framework of the option pricing with HWM provision for hedge funds is developed in this paper. The closed forms of HWM look-back put option, Russian option and stop-loss option are derived. We also obtain the internal relationship between HWM look-back put and the traditional look-back option. We show that HWM look-back put is cheaper than the traditional look-back put, and the higher the incentive fee, the lower the option price.
Keywords: high water mark options, stochastic processes, hedging.

1 Introduction

Hedge funds are pooled investment vehicles; most set up as private limited partnerships and investors buy an interest into the partnership. As such, they have more freedom and flexibility than mutual funds. In the past ten years, the number of hedge funds has risen about 20% per year. Currently, there are estimated to be 4000-5000 hedge funds managing $200-$300 billion. While the number and size of hedge funds are small relative to mutual funds, their growth reflects the importance of this alternative investment category. One important feature of the hedge funds industry is the structure of the fee paid to fund managers.

The fee in a hedge fund's account mainly comprises management fee and incentive fee. Management fee is charged on the account balance whether the account is profitable or not. Management fee normally ranges between 1%-2.5% annually,

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and typically it is 2%. Hedge funds also share in profits generated in the account by charging an incentive fee based on the difference between each old and new profit high for the account. This is the so-called high water mark (HWM) provision. The up to date highest asset value in this account is the HWM. The incentive fee can only be earned by producing on-going new HWM, i.e. new profits for an account. It works this way: If the manager has an incentive fee of 20% and his/her current HWM is 10 million. Say the manager has a 50% return in one month which increases the asset value to 15 million. He/She then pockets a 1 million incentive fee and the HWM changes to 15 million. Suppose next month the asset value (15 - 1 = 14 million) shrinks to 10 million, the manager cannot collect any incentive fee. Additional incentive fees are due only to the extent the manager pushes the fund above 15 million. Moreover all commission charges and per trade transaction costs must be made up before an incentive fee is applied. Some funds have a hurdle rate provision as well, meaning that a certain level of return must be met in order to trigger the incentive allocation, or against an index such as S&P 500 or treasury rates. The incentive fee is normally between 5%-25%, with majority of 20%. These fees are usually paid from the account on a monthly or quarterly basis.

The fee structure of hedge funds has been studied intensively. Recently, Goetzmann et al. [3] use an option approach to calculate the present value of the fees charged by money managers. They show that incentive fee takes a large part. For example, for a money manager with volatility of 15%, the incentive fee can be as high as 13% of the total managed money. Fung and Hsieh [2] provide a rationale for how hedge funds are organized and they show that the incentive fee paid to successful managers can be significantly higher than the fixed management fee.

One natural question is how significant this HWM provision impacts on option pricing. Because every time a money manager charges the incentive fee, the asset value correspondingly is reduced. Option with HWM provision is clearly path dependent. And the path is changed every time a hedge fund reaches a new HWM, since a certain amount of incentive fee is charged. Surprisingly, there is not much study on this kind of option in literature. This paper sets up the framework of option pricing model with HWM provision. The closed forms of HWM look-back put option, Russian option and stop-loss option are derived. Moreover we obtain the internal relationship between HWM look-back put and the traditional look-back option. We show that HWM look-back put is cheaper than the traditional look-back put. The higher the incentive fee, the lower the option price.

2 HWM option pricing framework

We work in a continuous-time framework and assume that, in the absence of management fee, the net asset of the fund follows a lognormal diffusion process with expected rate of return μ_t and variance σ_t^2. Let g_t be the rate of the management fee. The evolution of the asset of the fund, S_t, is assumed to be the solution to a stochastic differential equation of the form

$$dS_t = (\mu_t + g_t)S_t dt + \sigma_t S_t d\omega_t, \quad S_t < H_t.$$

where ω_t is a standard Brownian motion.

In the simplest case the HWM is the highest level the asset value has reached in the past. For some incentive contracts, the HWM can be changed due to other conditions, such as some indices or treasury rates. Because they are not locally random in our model, the change of H_t is locally deterministic. So for $S_t < H_t$, the change of H_t is $dH_t = G_t H_t dt$, $S_t < H_t$ where G_t is a deterministic function defined in the HWM provision. When the asset value reaches a new high, the HWM is reset to this higher level.

Following the arguments in Black and Scholes [1] and Merton [5], by applying Ito's lemma (See Øksendal [6] and adjusting the hedge position, we can find the diffusion function of the option price. When the fund's assets are below the HWM, the option price V_t satisfies the following partial differential equation (PDE)

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + (r_t + g_t) S_t \frac{\partial V_t}{\partial S_t} + G_t H_t \frac{\partial V_t}{\partial H_t} - r_t V_t = 0, \qquad S_t < H_t \quad (1)$$

The payoff function is

$$V(S_T, H_T, T) = \Lambda(S_T, H_T) \quad (2)$$

where $\Lambda(S_T, H_T)$ is defined in the contract.

Another condition applies along the boundary $S_t = H_t$. When the asset value rises above H_t to $H_t + \epsilon$, the HWM is reset to $H_t + \epsilon$, and a incentive fee of $k\epsilon$ is paid to the manager reducing the asset value to $H_t + \epsilon(1-k)$. Therefore, the option price before any adjustments of the incentive fee and HWM is $V(H_t + \epsilon, H_t, t + \Delta t)$, and the option price after the adjustments of the incentive fee and HWM is $V(H_t + (1-k)\epsilon, H_t + \epsilon, t + \Delta t)$. Let $\epsilon \to 0$ and $\Delta t \to 0$ this gives the boundary condition

$$k\frac{\partial V_t}{\partial S_t} = \frac{\partial V_t}{\partial H_t} \qquad on\ S_t = H_t \quad (3)$$

Hence PDE (1) together with conditions (2) and (3) give the solution of the option price with the HWM provision.

3 Special cases

It is not easy to get a closed form for the HWM option. The followings are some special cases, in which some closed forms are obtained. For simplicity, we drop the subscript t for convenience when it does not cause confusion.

3.1 Look-back put HWM option

A look-back put option is an option with payoff determined by the price of the asset value and the maximum value of the underlying asset within the life of the option. Look-back options can somehow capture investor's fantasy of buying low, selling high, and minimize regrets, as Goldman et al. [4] argues.

Assume G_t is zero, i.e. the HWM doesn't change when the asset value is less than the HWM. For simplicity, let $g_t = 0$. Also assume that $r = r_t$ and $\sigma = \sigma_t$ are constant, then the following PDE gives the HWM look-back put option

$$\frac{\partial V_{HWM}}{\partial t} + \sigma^2 S^2 \frac{\partial^2 V_{HWM}}{2\partial S^2} + rS\frac{\partial V_{HWM}}{\partial S} - rV_{HWM} = 0, \qquad 0 \leqslant S < H \quad (4)$$

$$V_{HWM}(S, H, T) = max(H_T - S_T, 0), \quad (5)$$

$$V_{HWM}(0, H, t) = H, \quad (6)$$

$$k\frac{\partial V_{HWM}}{\partial S} = \frac{\partial V_{HWM}}{\partial H} \qquad \text{on } S = H. \quad (7)$$

Notice the only difference between V_{HWM} and the traditional look-back put option V_{TRD} is the boundary condition at $S = H$. The traditional look-back put option is just a special case of the HWM look-back put option when the incentive fee is zero.

Proposition 1. *The HWM look-back option $V_{HWM}(S, H, t)$ has the following relationship with the traditional look-back option $V_\theta(S, H, t)$.*

$$V_{HWM}(S, H, t) = H^{\theta-1} V_\theta(S, H, t) \quad (8)$$

where $V_\theta(S, H, t)$ is a look-back option with payoff function $H^\theta(1 - S/H)$ but without HWM provision.

Proposition 1 shows deeper relationship between HWM look-back put and a traditional look-back option. Every time when the asset value reaches a new HWM, a certain amount of incentive fee is charged. Hence there is an adjustment of the asset value that changes the path with a small downside jump. While Proposition 1 shows that a HWM look-back put is nothing but a traditional look-back option with a different payoff function, with a justification of the HWM to the power of $\theta - 1$.

Proposition 2. *The HWM look-back put option price $V_{HWM}(S_0, H_0, 0) = H_0 W(S_0/H_0, 0)$, where*

$$\begin{aligned} W = &- \frac{\theta\sigma^2}{2\nu + \theta\sigma^2} e^{-rT} \left(\frac{H_0}{S_0}\right)^{\frac{2\nu}{\sigma^2}+1} N(d_1) \\ &- \frac{(1-\theta)\sigma^2}{2\nu + (1+\theta)\sigma^2} e^{(\nu - r + \frac{1}{2}\sigma^2)T} \left(\frac{H_0}{S_0}\right)^{\frac{2\nu}{\sigma^2}} N(d_2) \\ &+ \left(\frac{2\nu + 2\theta\sigma^2}{2\nu + \theta\sigma^2} - \frac{2\nu + 2\theta\sigma^2}{2\nu + (1+\theta)\sigma^2}\right) e^{(\theta\nu - r + \frac{\theta^2}{2}\sigma^2)T} \left(\frac{S_0}{H_0}\right)^{\theta} N(d_3) \\ &- e^{(\nu - r + \frac{\sigma^2}{2})T} \left(\frac{S_0}{H_0}\right) N(d_4) \\ &+ e^{-rT} N(d_5) \end{aligned} \quad (9)$$

and

$$H_0 = \textit{high water mark at time 0} \tag{10}$$

$$S_0 = \textit{asset value at time 0} \tag{11}$$

$$d_1 = -\frac{ln(H_0/S_0) + \nu T}{\sigma\sqrt{T}} \tag{12}$$

$$d_2 = -\frac{ln(H_0/S_0) + (\nu + \sigma^2)T}{\sigma\sqrt{T}} \tag{13}$$

$$d_3 = \frac{ln(S_0/H_0) + (\nu + \theta\sigma^2)T}{\sigma\sqrt{T}} \tag{14}$$

$$d_4 = \frac{ln(H_0/S_0) - (\nu + \sigma^2)T}{\sigma\sqrt{T}} \tag{15}$$

$$d_5 = \frac{ln(H_0/S_0) - \nu T}{\sigma\sqrt{T}} \tag{16}$$

$$\nu = r + g - \frac{1}{2}\sigma^2 \tag{17}$$

$$r = \textit{riskfree rate} \tag{18}$$

$$g = \textit{management fee} \tag{19}$$

By replacing T with $T - t$, S_0 with S_t and H_0 with H_t, we verify that solution (9) satisfies PDE (4)-(7). This is exactly what we expect since both probability way and PDE way should yield the same option price.

It is neither surprising that HWM look-back put option formula (9) is more complicated than the traditional look-back put option formula . But which option is more expensive? The following proposition answers this question.

Proposition 3. *If $\theta = 1$, i.e. the incentive fee is zero, then the HWM look-back put has the same price as the traditional look-back put, i.e. $V_{HWM} = V_{TRD}$. If $\theta < 1$, i.e. the incentive fee is greater than zero, then $V_{HWM} < V_{TRD}$. Moreover, the higher the incentive fee, the lower the option price. The lower the ratio of S_0/H_0, the less impact of the HWM provision. And the price of HWM look-back put converges to the traditional look-back put option as S_0/H_0 tends to zero.*

It is a trivial case when the rate of incentive fee is zero, since it is reduced to the traditional look-back put option. Notice θ is between $\frac{1}{2}$ and 1, since the incentive fee k is between 0 and 1, and $\theta = \frac{1}{1+k}$. When the rate of incentive fee is greater than zero, every time when the asset value goes above the past HWM, a certain amount of incentive fee is charged. Hence the asset value is reduced. It is surely that it is more difficult for the asset value to reach a new high. HWM provision reduces both the asset value and the HWM. Proposition 3 reveals the different significance of the impact of HWM provision on the asset value and the corresponding HWM. It shows that the incentive fee reduces more HWM level than it does on the asset value itself. Since incentive fee only applies when S_t

reaches a new high, when S_0/H_0 is small, the probability that S_t goes above H_0 is small. Hence the less impact of the HWM provision and the price of HWM look-back put converges to the traditional look-back put option as S_0/H_0 tends to zero.

3.2 Russian HWM option

The term "Russian Option" was coined by Shepp and Shirayaev [7] to describe a perpetual American option, which, at any time chosen by the holder, pays out the maximum realised asset price up to that date. The solution of the Russion option with HWM is $V = c_1 H\xi^{\eta_1} + c_2 H\xi^{\eta_2}$. where

$$\eta_1 = \frac{-(r+g-\frac{1}{2}\sigma^2) + \sqrt{(r+g-\frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} \tag{20}$$

$$\eta_2 = \frac{-(r+g-\frac{1}{2}\sigma^2) - \sqrt{(r+g-\frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} \tag{21}$$

$$c_1 = \frac{1-(1+k)\eta_2}{(1-(1+k)\eta_2)\xi_0^{\eta_1} - (1-(1+k)\eta_1)\xi_0^{\eta_2}} \tag{22}$$

$$c_2 = \frac{1-(1+k)\eta_1}{(1-(1+k)\eta_1)\xi_0^{\eta_2} - (1-(1+k)\eta_2)\xi_0^{\eta_1}} \tag{23}$$

$$\xi_0 = \left(\frac{\eta_1(1-(1+k)\eta_2)}{\eta_2(1-(1+k)\eta_1)}\right)^{1/(\eta_2-\eta_1)} \tag{24}$$

3.3 Stop-loss HWM option

A stop-loss option may be thought of as a perpetual barrier look-back with a rebate that is a fixed proportion of the maximum realised value of the asset price. If at any time t the asset price S falls to λH, where H is the HWM and $\lambda < 1$ is fixed, the option is triggered and pays off λH.

The solution to the Stop-loss HWM option is very similar to the Russian HWM option with c_1 and c_2 defined as

$$c_1 = \frac{(1-(1+k)\eta_2)\lambda}{(1-(1+k)\eta_2)\lambda^{\eta_1} - (1-(1+k)\eta_1)\lambda^{\eta_2}}, \tag{25}$$

$$c_2 = \frac{(1-(1+k)\eta_1)\lambda}{(1-(1+k)\eta_1)\lambda^{\eta_2} - (1-(1+k)\eta_2)\lambda^{\eta_1}}. \tag{26}$$

4 Numerical example

Some numerical examples on HWM look-back are given. We also compare the price of the HWM look-back put with that of the traditional look-back put.

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Table 1 shows 3 month horizon look-back put vs. incentive fee. In this example, risk-free rate $r = 5\%$, initial HWM $H_0 = 100$, initial asset value $S_0 = 100$ and volatility $\sigma = 20\%$. When the incentive fee $k = 20\%$, the HWM look-back option price is 6.49, while the traditional look-back is 6.58. The difference is about 9 basis points. It is an interesting observation that the difference is not significant. It shows that as incentive fee goes to zero, the HWM look-back put converges to the traditional look-back put. The higher the incentive fee, the lower the option price.

Table 2 compares the HWM look-back put with the traditional look-back put with different initial asset value, assuming the HWM is 100. In this case, $r = 5\%$, $k = 20\%$, $\sigma = 20\%$. This table shows that the traditional look-back put is always more expensive than HWM look-back put. As the initial asset value tends to zero, the difference of the traditional look-back put and HWM look-back put also tends to zero.

5 Conclusion

In this paper we developed the framework of option pricing with HWM provision for hedge funds. The closed forms of HWM look-back put option, Russian option and stop-loss option are derived. We also show that HWM look-back put is cheaper than the traditional look-back put, and the higher the incentive fee, the lower the option price.

As to our knowledge, option with HWM is not traded in the market. Partly because hedge funds industry is not well known to the public until recently. Another reason is hedge funds are free from most disclosure and regulation requirements that apply to mutual funds and banks. It is difficult to hedge an option on hedge funds.

Hedge funds are not strangers to leverage and derivatives. It is quite natural to trade options on hedge funds. As hedge funds industry grows and more regulations on position reporting, we believe that it is a matter of time that derivative products on hedge funds themselves will also appear.

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