The indirect boundary element method for the axisymmetric free surface Stokes flow

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Abstract

The authors present the formulation of the indirect boundary element method (IBEM) for an axisymmetric Stokes flow with a free surface in the presence of gravity. The formulae of the fundamental solutions of the Stokes equations are found for velocities and tractions in the axisymmetric case. These expressions are written in the cylindrical coordinate system and contain the elliptic integrals of the first and second kind. For the integral equations discretization the constant elements are used. The necessary integrals are evaluated numerically except for the singular ones. The analytical formulae are obtained for them. Two boundary-value problems with mixed conditions are considered. The problem of the Poiseuille flow of a viscous fluid in a round tube with the exact solution was calculated to verify the IBEM algorithm and to demonstrate its approximation convergence. Another problem of the cylindrical tube filling by a viscous fluid with a free surface was calculated to prove the IBEM in the case of a moving boundary. The simulation in a steady-state formulation showed that the stationary advancing front shapes exist in both cases when the gravity acts against the flow (Stokes number $St<0$) and aids the flow ($0<St\leq 0.94$). In the case of the Stokes parameter values greater than 1 the fountain flow was replaced by a jet flow. The stationary advancing front shapes were calculated in the range $-400\leq St\leq 0.94$ and compared with the famous data.
Keywords: boundary element method, free surface, axisymmetric flow, Poiseuille flow, cylindrical tube, filling, fountain flow, injection molding.

1 Introduction

The indirect boundary element method (IBEM) is an effective means to solve the fluid dynamics problems at low Reynolds numbers (Stokes flows) with a free surface. Its application in combination with the use of the constant elements made

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it possible to investigate a number of practically important flows [1–4]. In these papers the two-dimensional creeping flows whose characteristic feature is the presence of a free surface are considered. The basics of the method under consideration are the positions formulated by Brebbia [5]. They are realized by applying the constant elements and the analytical expressions for calculation of all the necessary integrals including those with singularities. Such approach enables to study the flows with great deformations of the free surface interacting with the solid walls. In the present paper the IBEM is applied to solve the axisymmetric problems which have much in common with the two-dimensional ones from the viewpoint of the practical realization. For example, the algorithms of modeling the motion of the free surface front practically coincide. At the same time there are essential differences. First of all one should apply the cylindrical coordinate system instead of the Cartesian one. The components of the fundamental tensors of velocities and tractions in the cylindrical coordinate system should be previously integrated with respect to the angular coordinate. As a result, one obtains the expressions containing the elliptic integrals of the first and second kind. The integration of these expressions with respect to the element is very difficult; therefore one should apply the numerical quadratures. The problem of singular integral calculation is of importance.

The application of the direct boundary element method and a similar method of the fundamental solutions to investigate the axisymmetric problems was discussed earlier in a great number of papers [6–12]. These works touch upon both the fluid dynamics and the elastic theory. They contain many useful equations to calculate the necessary integrals. For example, Pozrikidis [6] gave the expressions for the components of the fundamental tensor of velocity integrated with respect to the angle coordinate, Park's paper [9] contains some formulae which are required to calculate such expressions and their limiting values. The examined fluid dynamics problems mainly concern the modeling of the viscous fluid flow around the axisymmetric bodies.

In the present paper the application of the IBEM for the Stokes flow modeling is discussed on the two examples: the Poiseuille flow (the formulation includes specifying tractions components on a part of the boundary) and the fountain flow in the steady-state formulation. The obtained results make it possible to come to the conclusion concerning the efficiency of this numerical algorithm.

2 Governing equations and problem formulation

The creeping flow of a viscous fluid in the gravity field is described by the Stokes equation which can be represented in the dimensionless form as

$$\nabla\boldsymbol{\sigma} + \mathrm{St}\mathbf{e}_{\mathrm{g}} = 0\,, \tag{1}$$

where $\boldsymbol{\sigma} = -p\mathbf{I} + 2\dot{\mathbf{E}}$ is the stress tensor, p is the pressure, $\mathbf{I}$ is the unit tensor, $\dot{\mathbf{E}} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{*}\right)$ is the rate-of-strain tensor, $\mathbf{u}$ is the velocity vector, $\mathbf{g}$ is the acceleration vector of the gravity force, $\mathbf{e}_{\mathrm{g}} = \mathbf{g}/g$, $g = |\mathbf{g}|$, $\mathrm{St} = \dfrac{\rho g R^2}{\mu U}$ is the

Stokes number, ρ is the fluid density, μ is the dynamic viscosity coefficient, R and U are the characteristic values of the length and velocity chosen as scales. The pressure scale is $\dfrac{\mu U}{R}$.

One can remove the constant term from equation (1), this is important while transferring to the boundary integral. For this purpose it is necessary to introduce the potential of the gravity force

$$\mathbf{e}_\mathbf{g} = -\nabla\varphi ,$$

where $\varphi = -(\mathbf{r}\cdot\mathbf{e}_\mathbf{g})$, $\mathbf{r}$ is the radius-vector of the point $\mathbf{x}$. Then equation (1) becomes homogeneous

$$\nabla\boldsymbol{\sigma}' = 0 , \tag{2}$$

where $\boldsymbol{\sigma}' = -p_m\mathbf{I} + 2\dot{\mathbf{E}}$, $p_m = p + \mathrm{St}\varphi$ is the modified pressure.

Equation (1) should be considered together with the continuity equation

$$\nabla\cdot\mathbf{u} = 0 . \tag{3}$$

The boundary conditions for equations (2), (3) are specifying the velocity vector values $\mathbf{u}(\mathbf{x}_0)$ or traction vector values $\mathbf{t}(\mathbf{x}_0) = \boldsymbol{\sigma}(\mathbf{x}_0)\mathbf{n}(\mathbf{x}_0)$, where $\mathbf{x}_0$ are the points belonging to the boundary of the flow region, and $\mathbf{n}(\mathbf{x}_0)$ is an outward unit normal vector to the boundary. The velocity vector can be specified on the inlet (outlet) boundary or on a solid wall when the no-slip boundary conditions are used. The traction vector is specified mainly on the free surface, where its value should be equal to zero

$$\mathbf{t}(\mathbf{x}_0) = 0 . \tag{4}$$

Taking into account that equation (1) is used in the form (2), condition (4) is to be applied for the traction vector expressed in modified stress tensor:

$$\mathbf{t}'(\mathbf{x}_0) = \boldsymbol{\sigma}'(\mathbf{x}_0)\mathbf{n}(\mathbf{x}_0) = \mathbf{t}(\mathbf{x}_0) - \mathrm{St}\varphi(\mathbf{x}_0)\mathbf{n}(\mathbf{x}_0) .$$

Consequently, boundary condition (4) takes on the form

$$\mathbf{t}'(\mathbf{x}_0) = \mathrm{St}(\mathbf{r}(\mathbf{x}_0)\cdot\mathbf{e}_\mathbf{g}(\mathbf{x}_0))\mathbf{n}(\mathbf{x}_0) . \tag{5}$$

In addition to the dynamic condition (5) the free surface is governed by the kinematic condition which can be written down either in the Lagrangian

$$\frac{d\mathbf{r}(\mathbf{x}_0)}{dt} = \mathbf{u}(\mathbf{x}_0) ,$$

or the Euler form

$$\frac{\partial F}{\partial t} + \mathbf{u}(\mathbf{x}_0)\cdot gradF = 0 . \tag{6}$$

where $F(x,t) = 0$ is the free surface equation.

Thus, the problem of the Stokes flow with the free surface is of the quasi-stationary character. The initial conditions are to give the boundary shape at the initial momentum of time. Then the boundary-value problem is solved for equations (2), (3) with the specified values of velocity vector $\mathbf{u}(\mathbf{x}_0)$ on the one part of the flow region boundary and values of the traction vector $\mathbf{t}'(\mathbf{x}_0)$ on the other. Further, according to the found velocity distribution on the free surface one evaluates its new shape by applying the kinematic condition in the notation depending on the character of the problem.

3 Boundary integral equations

Flow is considered in the axisymmetric region Ω with the surface S obtained as a result of rotating the constituent Γ around the axis z (figure 1). The flow itself is also axisymmetric, i.e. the fluid motion velocity depends only on r, z and does not depend on the angular coordinate θ.

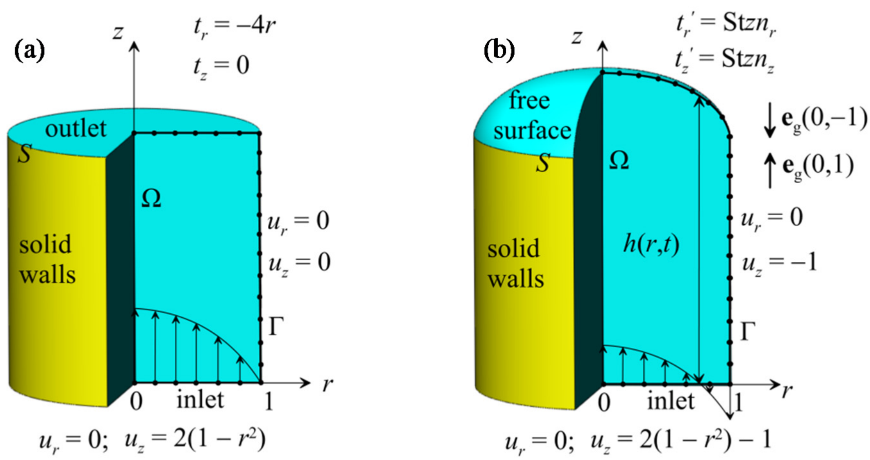

Figure 1: Boundary value problems (the solution domains and the boundary conditions) for the Poiseuille (a) and the fountain flow (b).

For the boundary integral formulation of the problem the classical concepts leading to the indirect boundary element method will be applied. The fictious sources with the density per a unit area equal to $\boldsymbol{\varphi}(\xi)$, $\xi \in S$ are considered to be distributed over the surface S. In view of the axisymmetry of the vector $\boldsymbol{\varphi}(\xi)$ has the form

$$\boldsymbol{\varphi}(\xi) = \varphi_r(\xi)\mathbf{e}_r(\xi) + \varphi_z(\xi)\mathbf{e}_z(\xi),$$

where $\mathbf{e}_r(\xi)$, $\mathbf{e}_z(\xi)$ are the unit vectors of the axes r, z in the point ξ.

Then, by applying the superposition principle one can write the following integral equations for boundary points $\mathbf{x}_0 \in S$:

$$u_r(\mathbf{x}_0) = \int_\Gamma \left[\varphi_r(\xi)\tilde{u}_r^r(\mathbf{x}_0,\xi) + \varphi_z(\xi)\tilde{u}_r^z(\mathbf{x}_0,\xi)\right] d\Gamma(\xi), \tag{7}$$

$$u_z(\mathbf{x}_0) = \int_\Gamma \left[\varphi_r(\xi)\tilde{u}_z^r(\mathbf{x}_0,\xi) + \varphi_z(\xi)\tilde{u}_z^z(\mathbf{x}_0,\xi)\right] d\Gamma(\xi), \tag{8}$$

$$t_r(\mathbf{x}_0) = \int_\Gamma \left[\varphi_r(\xi)\tilde{t}_r^r(\mathbf{x}_0,\xi) + \varphi_z(\xi)\tilde{t}_r^z(\mathbf{x}_0,\xi)\right] d\Gamma(\xi), \tag{9}$$

$$t_z(\mathbf{x}_0) = \int_\Gamma \left[\varphi_r(\xi)\tilde{t}_z^r(\mathbf{x}_0,\xi) + \varphi_z(\xi)\tilde{t}_z^z(\mathbf{x}_0,\xi) \right] d\Gamma(\xi), \tag{10}$$

where

$$\tilde{u}_r^r(\mathbf{x}_0,\xi) = r(\xi)\int_0^{2\pi} u_r^r(\mathbf{x}_0,\xi)d\theta(\xi), \ldots, \quad \tilde{t}_z^z(\mathbf{x}_0,\xi) = r(\xi)\int_0^{2\pi} t_z^z(\mathbf{x}_0,\xi)d\theta(\xi),$$

$u_r^r(\mathbf{x}_0,\xi)$,…, $t_z^z(\mathbf{x}_0,\xi)$ are the components of the fundamental tensors of velocities and tractions of the Stokes equation, they correspond to the unit concentrated forces $\mathbf{e}_r(\xi)$, $\mathbf{e}_z(\xi)$. The lower index means the vector component in the point $\mathbf{x}_0$, and the upper one points to the direction of the unit force acting in the point ξ (in this case $\xi \in S$). While writing down equations (7)–(10) it was taken into account that the surface element of S is equal to $dS(\xi) = r(\xi)d\theta(\xi)d\Gamma(\xi)$ and it is considered that in the left part of the equations $\xi \in \Gamma$. The components $u_r^r,\ldots,t_z^z$ may be obtained from the fundamental solutions of the Stokes equation in the Cartesian coordinate system [14].

The left part of equations (7)–(10) is known from the boundary conditions. Consequently, the problem consists in determining the unknown density function $\boldsymbol{\varphi}(\xi)$. Then the necessary flow characteristics may be found in any boundary or internal point.

The formulae for the integrals $\tilde{u}_r^r,\ldots,\tilde{t}_z^z$ are of the form

$$\tilde{u}_r^r = -\frac{r_\xi}{8\pi}\left(I_{11} + (r_x^2 + r_\xi^2)I_{31} - r_x r_\xi (I_{32} + I_{30})\right), \tag{11}$$

$$\tilde{u}_z^r = -\frac{r_\xi \bar{z}}{8\pi}\left(r_x I_{31} - r_\xi I_{30}\right), \tag{12}$$

$$\tilde{u}_r^z = -\frac{r_\xi \bar{z}}{8\pi}\left(r_x I_{30} - r_\xi I_{31}\right), \tag{13}$$

$$\tilde{u}_z^z = -\frac{r_\xi}{8\pi}\left(I_{10} + \bar{z}^2 I_{30}\right), \tag{14}$$

$$\tilde{t}_r^r = \frac{3r_\xi}{4\pi}\left(-cr_x r_\xi (I_{50} + I_{52}) + (r_x^2 + r_\xi^2)(cI_{51} - r_\xi n_r I_{52}) + r_x r_\xi^2 n_r (I_{51} + I_{53})\right), \tag{15}$$

$$\tilde{t}_z^r = \frac{3r_\xi}{4\pi}\left(-cr_\xi \bar{z} I_{50} + \bar{z}(cr_x + r_\xi^2 n_r) I_{51} - r_x r_\xi \bar{z} n_r I_{52}\right), \tag{16}$$

$$\tilde{t}_r^z = \frac{3r_\xi}{4\pi}\left(cr_x \bar{z} I_{50} - \bar{z}(cr_\xi + r_x r_\xi n_r) I_{51} + r_\xi^2 \bar{z} n_r I_{52}\right), \tag{17}$$

$$\tilde{t}_z^z = \frac{3r_\xi \bar{z}^2}{4\pi}\left(cI_{50} - r_\xi n_r I_{51}\right), \tag{18}$$

where r_x, z_x are the coordinates of the point $\mathbf{x}_0$, r_ξ, z_ξ are the coordinates of the point ξ, n_r, n_z are the components of the external normal in the point $\mathbf{x}_0$, $\bar{z} = z_x - z_\xi$, $c = r_x n_r + \bar{z} n_z$. Moreover

$$I_{mn} = \int_0^{2\pi} \frac{\cos^n \theta(\xi)}{\left(\overline{z} + r_x^2 + r_\xi^2 - 2 r_x r_\xi \cos\theta(\xi)\right)^{m/2}} d\theta(\xi) .$$

These integrals are expressed by means of complete elliptic integrals of the first and second kind and can be calculated using [15].

4 Numerical implementation

Equations (7)–(10) are practically similar to those for the planar case excluding the form of the fundamental solutions (11)–(18). In consequence of this, after discretization of the boundary by means of the constant elements a system of linear algebraic equations is obtained similar to that described in [4]. The extra-diagonal coefficients of the matrix may be calculated by the numerical integration (the Gauss eight-point quadrature formulae are used in this paper). When computing the diagonal coefficients one should use the analytical expressions.

The coefficients represent the integrals of the type $\left(\Delta\tilde{u}_r^r\right)^{pp} = \int\limits_{\Delta\Gamma^p} \tilde{u}_r^r(\mathbf{x}_0^p,\xi)d\Gamma(\xi),\dots,\left(\Delta\tilde{t}_z^z\right)^{pp} = \int\limits_{\Delta\Gamma^p} \tilde{t}_z^z(\mathbf{x}_0^p,\xi)d\Gamma(\xi)$, where p is the number of the element, $\Delta\Gamma^p$ is the element p, $\mathbf{x}_0^p$ is the node of the element p (the middle of the element).

The integrals $\left(\Delta\tilde{u}_r^r\right)^{pp},\dots,\left(\Delta\tilde{u}_z^z\right)^{pp}$ have the singularities which are equivalent to the logarithmic one. In a planar case the similar integrals for the constant elements may be computed exactly [4], but it is very difficult to do this for the axisymmetric problem, because the integrands contain the elliptic integrals. For this reason they are computed approximately. The approximate expressions for the functions $\tilde{u}_r^r,\dots,\tilde{u}_z^z$ are obtained by means of breaking down into the Taylor series conserving the first order terms. For the approximation of the elliptic integrals the approximate formulae of [16] are applied. As a result

$$\left(\Delta\tilde{u}_r^r\right)^{pp} = -\frac{\Delta S^p}{2\pi}\left(\ln\frac{8r_x^p}{\Delta S^p} - n_r^2 - 1\right),$$

$$\left(\Delta\tilde{u}_z^r\right)^{pp} = \left(\Delta\tilde{u}_r^z\right)^{pp} = \frac{1}{2\pi} n_r n_z \Delta S^p ,$$

$$\left(\Delta\tilde{u}_z^z\right)^{pp} = -\frac{\Delta S^p}{2\pi}\left(\ln\frac{8r_x^p}{\Delta S^p} + n_r^2 + 1\right),$$

where ΔS^p is the length of the element p, r_x^p is the radial coordinate of the node p, n_r, n_z are the components of the outward normal vector to the element p. To compute the values of integrals $\left(\Delta\tilde{t}_r^r\right)^{pp},\dots,\left(\Delta\tilde{t}_z^z\right)^{pp}$ one should use the results of the hydrodynamic potential theory. According to [14] we have

$$\left(\Delta\tilde{t}_r^r\right)^{pp} = \left(\Delta\tilde{t}_z^z\right)^{pp} = -\frac{1}{2}, \; \left(\Delta\tilde{t}_z^r\right)^{pp} = \left(\Delta\tilde{t}_r^z\right)^{pp} = 0 .$$

5 Results and discussion

To verify the computations by applying the above method and to check its accuracy the two problems were chosen. The first one was to calculate the flow of the viscous incompressible fluid in a cylindrical tube (the Poiseuille flow, figure 1(a)). The second problem consists in determining the stationary shape of a free surface while filling a vertical cylindrical tube with a viscous incompressible fluid (the fountain flow, figure 1(b)). This flow is of immediate practical value for the injection molding and has been considered by many authors in the numerical and experimental studies.

5.1 The Poiseuille flow

Setting up the problem of the flow in a cylindrical tube is illustrated in figure 1(a). The tube radius and the average flow velocity are taken as the scales of R and U. In such a form the boundary value problem has all the basic types of the boundary conditions which occur while investigating the flows with a free surface: the inlet boundary, the solid walls and the outlet boundary where the values of the traction vector components (the analog of the free surface) are specified.

As a result of the discretization of the boundary it was divided into N equal elements. In the process of computation the velocity profile on the outlet boundary was determined and compared with the famous exact solution $u_z^e = 2(1-r^2)$. The norm of space L_2 was applied to the error estimation:

$$\|u\| = \sqrt{\int_0^1 u^2(r)dr} .$$

Thus, the absolute and relative errors were calculated using the formulae:

$$E_a = \left\|u_z^c - u_z^e\right\|, \; E_r = \frac{\left\|u_z^c - u_z^e\right\|}{\left\|u_z^e\right\|} \cdot 100\% .$$

The numerical values of u_z^c were calculated in the edges of elements. The symmetry condition $\partial u_z / \partial r = 0$ was applied on the symmetry axis. To obtain the values of the non-singular integrals the Gaussian eight-point formulae were used. The results are given in figure 2. The chosen scheme of solution makes it possible to obtain a quick convergence of the approximate solution to the analytical one with increasing the number of elements. 76 elements are sufficient for the relative error not to exceed 0.1 shares of a per cent.

Thus, it was shown that the presented method of numerical solution could be applied further to simulate axisymmetric free-surface flows.

5.2 The fountain flow

The aim of the study of a vertical cylinder filling up at a constant flow rate is to obtain the stationary advancing front shapes using axisymmetric IBEM and to find out the conditions of their existence.

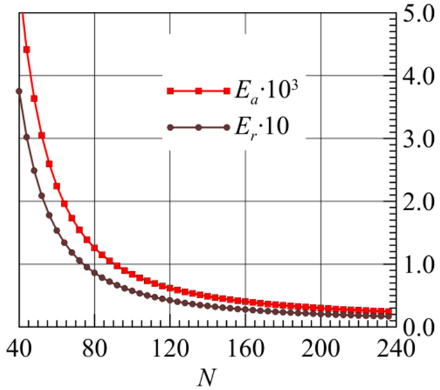

Figure 2: The dependence of E_a, E_r errors on the number of elements N.

The formulation of the corresponding boundary value problem is shown in figure 1(b). The flow behavior and, as a consequence, the shape of the free surface are completely governed by the value and the sign of the Stokes parameter $\mathrm{St} = \rho g L / \mu U$. At St > 0 the gravity force is directed along the fluid flow and aids it, and at St < 0 it acts against the flow. At St = 0 the viscosity forces are much higher than the gravitational ones and the latter do not affect the flow.

The flow is considered in the coordinate system moving together with the fluid flow at an average velocity. In this case the steady free surface is to be immobile, because it is also moving relative to the walls at an average velocity. For such a surface described by the function $h(r,t) : dh/\partial t = 0$. The initial approximation for $h(r,t)$ is the horizontal flat surface. The following approximations were calculated using the steady-state approach and in accordance with the kinematic condition at the free surface written down in the form of Euler (6). This condition gives the following equation to compute the function $h(r,t)$:

$$\frac{\partial h}{\partial t} = u_z - u_r \frac{\partial h}{\partial r} . \tag{19}$$

The discretization of the boundary is similar as in the previous problem. Condition (19) is written down in the finite-difference form as follows:

$$\frac{h_i^{n+1} - h_i^n}{\Delta t^n} = \left(u_z\right)_i^n - \frac{\Delta h_i^n}{\Delta r}\left(u_r\right)_i^n, \quad \Delta h_i^n = \begin{cases} h_i^n - h_{i-1}^n, & \left(u_r\right)_i^n > 0, \\ h_{i+1}^n - h_i^n, & \left(u_r\right)_i^n < 0, \end{cases} \tag{20}$$

where $\Delta r = 1/N$, $2 \le i \le N-1$, Δt^n is the value of a time step, n is the time step number. The time step Δt^n is chosen from the Courant condition $\Delta t^n = k \dfrac{\min\limits_i \Delta S_i^n}{\max\limits_i \left(u_z\right)_i^n}$, where ΔS_i^n is the length of the i-th element on the free surface, k is the coefficient determined by the computing experiment ($k = 0.1$ for presented results). The h value on the symmetry axis was determined from the symmetry condition $dh/\partial r = 0$. The convergence was considered to be reached when the change of the $h(0,t)$ value was smaller than 10^{-6}. The difference scheme (20) is an explicit difference scheme with the differences against the flow, it provides a stable computation of the stationary advancing front shape under the condition that such a surface exists for the given value of the Stokes number. This statement definitely holds for $\mathrm{St} \le 0$, when the gravity is absent or it acts against the flow. The steady shapes of the free surface corresponding to this case and for $\mathrm{St}>0$ are given in figure 3, where the origin of the coordinate z lies on the level of the contact line position. The dependence of the steady value of $\Delta h = h(0,t) - h(1,t)$ on the parameter St is shown in figure 4. Flow fields are presented in figure 5. Velocity magnitude $|\mathbf{u}|$ and stream lines are obtained in the coordinate system moving with the flow rate. The calculations were carried out at $N = 200$ (50 elements on the free surface) and $h(r,0) = 2$. The choice of the last value is caused by the fact that at such distance of the free surface from the inlet boundary its influence on the flow within the inlet vicinity should be excluded.

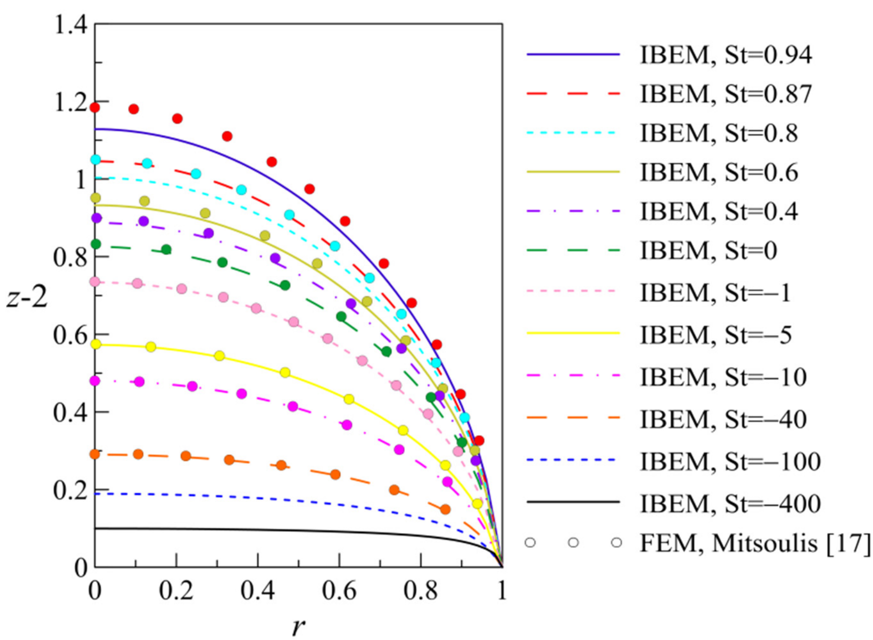

Figure 3: Stationary advancing front shapes and the comparison with Mitsoulis [17].

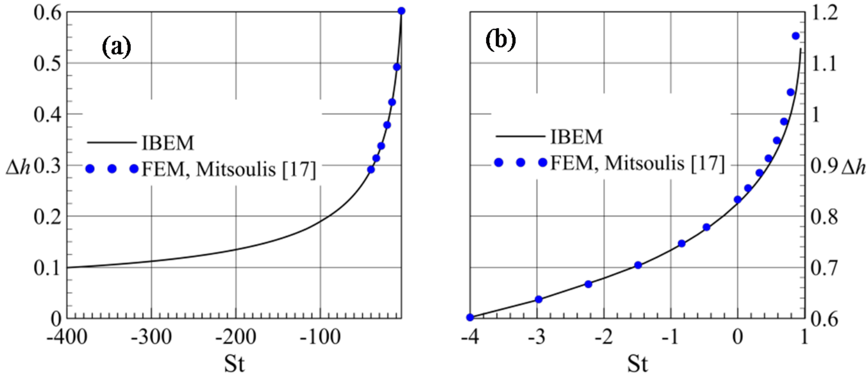

Figure 4: Steady value of $\Delta h = h(0,t) - h(1,t)$ versus St and the comparison with Mitsoulis [17] for St≤–4 (a) and St≥–4 (b).

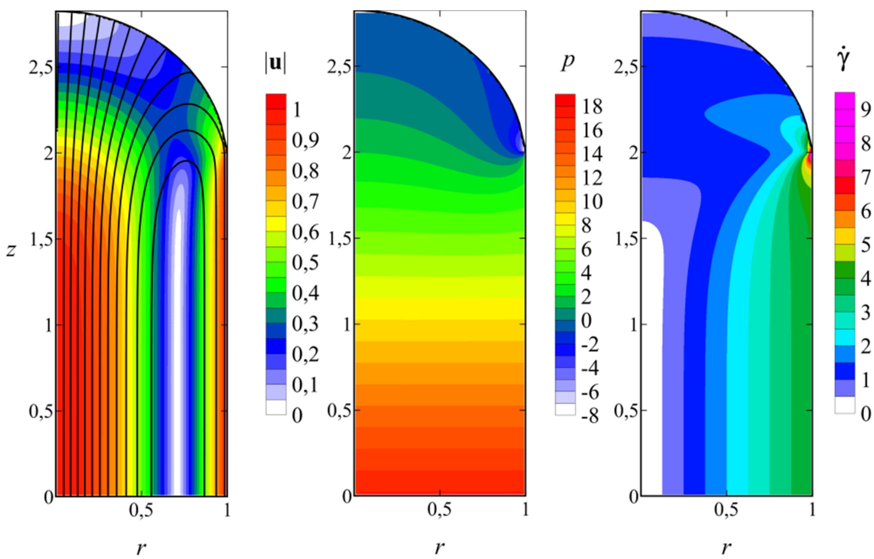

Figure 5: Flow fields at St = 0: velocity magnitude $|\mathbf{u}|$ and stream lines, pressure p and the deformation rate intensity $\dot{\gamma} = \sqrt{2\dot{\mathbf{E}} \cdot \dot{\mathbf{E}}}$.

The comparison of the obtained results with the data of Mitsoulis [17] shows their good agreement for $\mathrm{St} \le 0$ (figure 4). In [17] the problem was solved by the finite element method (FEM) for $-40 \le \mathrm{St} \le 0.87$, the main efforts were connected with carrying out a polar finite element mesh. These problems are omitted while applying the boundary element method and we can compute variants when the surfaces have practically horizontal shapes (at $\mathrm{St} < -100$).

The results for $St>0$ case, when the gravity force aids the flow somewhat differ from those of Mitsoulis [17] (figure 4(b)). The critical value of the parameter St which limits the range of existence of the stationary advancing front shapes is exactly equal to 0.94. The value $St=0.87$ was obtained by Mitsoulis [17] and the impossibility of obtaining the stationary shapes at larger St was associated with the loss of stability. In fact, as it was shown in [18] this event testifies in favor of the transfer to the filling regime with the jet formation and the gas entrainments. The fountain flow is replaced by a jet one. The numerical study in [18] was carried out by the finite-difference method for the planar channel using the kinematic condition in the Lagrangian form, and the obtained critical value of St was also close to 1.

6 Conclusions

On the basis of this investigation one can come to the conclusion that the presented version of the indirect boundary element method is an effective means to simulate the axisymmetric Stokes flows with a free surface. For its formulation the expressions for the components of the fundamental tensors are obtained for the velocities and tractions in the cylindrical coordinate system. The principal idea is to use the constant elements, the analytical expressions for the integrals of the fundamental tensors components with respect to the angle coordinate and the approximate analytical values of the integrals of velocity fundamental tensor components with respect to the singular elements. To compute numerically the extra-diagonal matrix elements of the system it is sufficient to use the eight-point Gauss quadratures. This is supported by comparing the numerical solution with the exact one (Poiseuille flow) and the numerical simulation of the fountain flow. The second problem may be considered as a benchmark for the verification of the methods of a free surface flow modeling. The results presented cover a wide range of Stokes parameter values. It is shown that the regime of a complete filling of a tube when there exist the stationary advancing front shapes is observed under the condition $St \le 0.94$. In either case at $St>0.94$ the fountain behavior of the flow is replaced by the jet one. All the computed shapes are in a good agreement with the known results.

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