On the boundary element formulation to compute critical loads considering the effect of shear deformation in plate bending

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Abstract

A boundary element formulation to compute critical loads in static problems is analyzed. The bending model includes the effect of shear deformation that turns deflection derivatives independent of plate rotations. The effect of geometrical non-linearity (GNL) related to in-plane loading is introduced with two additional integrals in the formulation: one is performed on the domain and the other on the boundary. The boundary integral can be related to one of the natural conditions according to the boundary value problem. The inverse iteration was used to get the lowest eigenvalue. Results obtained with several boundary conditions and using derivatives of the deflection in the effect of GNL are compared and discussed with expect values from the literature.
Keywords: plate bending with in-plane forces, critical loads of plates, Reissner-Mindlin plates.

1 Introduction

A formulation to compute critical loads in plates is studied with the boundary element method (BEM). The effect of shear deformation is included in the bending model due to its benefits on plate behavior modeling as shown by Reissner on the assessment of stress concentration around holes [1] and by Mindlin on the wave propagation analysis considering short wavelengths [2]. The effect of the geometrical of non-linearity (GNL) related to in-plane loading is introduced according to the development in [3]. In spite of the fact that classical plate theory has been used in [3], the formulation for the effect of GNL is considered in bending models including the effect of shear deformation when

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thin or thin to thick plate types are analyzed (as shown in several studies in the literature).

A review on the literature until 1989 on the buckling of thin rectangular plates was done by Jones in [4]. Dawe and Roufaeil [5] discussed the effect of GNL in plate bending considering the effect of shear deformation since the first study presented by Herrmann and Armenakas [6]. Doong [7] and Matsunaga [8] studied vibration and buckling of thick plates employing a high order deformation theory for the bending model. The buckling analyses of skew plates considering the Mindlin model was done by Kitipornchai *et al.* [9] with pb-2 Rayleigh-Ritz method. Lei *et al.* [10] formulated an integral equation for buckling analysis of Reissner plates, where the domain integral related to GNL was discretized using constant triangular cells. A similar study was done by Purbolaksono and Aliabadi [11] but using constant rectangular cells.

A boundary element formulation using first derivatives of the deflection to introduce the effect of GNL in bending of plates was presented in [12, 13]. The elastodynamic solution [14] was used and critical loads were computed according to the frequency value. It is well known that the frequency value modifies the value of critical in-plane force and vice-versa [15]. The values obtained for critical loads at near zero frequencies were close to expected values from the literature for the static condition. A question appeared: Was the numerical implementation or the near zero value for the frequency that improved the formulation to compute critical loads at “the static condition”?

An analysis of the formulation using first derivatives of the deflection in the effect of GNL is done in this study. The static fundamental solution [16] is used to remove the dynamic effect whereas other features in [12, 13] remained. The effect of geometrical non-linearity (GNL) is introduced with two additional integrals in the formulation: one is performed on the domain and the other on the boundary. The boundary integral can be related to one of the natural conditions according to the boundary value problem. Isoparametric quadratic boundary elements and constant rectangular domain cells are employed. The inverse iteration and Rayleigh quotient are used to get the lowest eigenvalue. Results obtained with several boundary conditions are compared and discussed with expect values from the literature.

2 Boundary integral equations

The constitutive equations for an isotropic and homogeneous plate material are:

$$M_{\alpha\beta} = D\frac{(1-\nu)}{2}\left(\psi_{\alpha,\beta} + \psi_{\beta,\alpha} + \frac{2\nu}{1-\nu}\psi_{\gamma,\gamma}\delta_{\acute{a}\beta}\right) + \delta_{\alpha\beta}qRE \tag{1}$$

$$Q_{\alpha} = D\frac{(1-\nu)}{2}\lambda^2\left(\psi_{\alpha} + w_{,\alpha}\right) \tag{2}$$

with

$$\lambda^2 = 12\frac{\kappa^2}{h^2}; \quad RE = \frac{\nu}{\lambda^2(1-\nu)}$$

The plate has a uniform thickness h, D is the flexural rigidity, ν is the Poisson ratio, w is the deflection, ψ_α is the plate rotation in the direction α and $\delta_{\alpha\beta}$ is the Kronecker delta. The product qRE in equation (1) corresponds to the linearly weighted average effect of the normal stress component in the thickness direction and should be considered in Reissner's model [1] but not in Mindlin's model [2], in which it should be considered to be null. The shear parameter κ^2 is equal to 5/6 and $\pi^2/12$ for the Reissner and Mindlin models, respectively.

The constitutive eqns (1) and (2) employed a unified notation for the Reissner and the Mindlin model and the convention of this study was used, i.e. Latin indices take values {1, 2 and 3} and Greek indices take values {1, 2}. To match the constitutive equations used in the literature for other numerical techniques [8, 9], the linearly weighted average effect of the normal stress component in the thickness direction (qRE) is assumed null in this study and the difference between Reissner and Mindlin models will be done by the shear parameter.

The natural conditions and the equilibrium equations for the problem can be obtained with the calculus of variations [17, 18]. The energy functional of the plate is given by:

$$\Pi = \int_\Omega \left\{ \frac{D(1-\nu)}{4} \left[{\psi_{\alpha,\beta}}^2 + \psi_{\alpha,\beta}\psi_{\beta,\alpha} + \frac{2\nu}{(1-\nu)} {\psi_{\gamma,\gamma}}^2 + \lambda^2 \left(\psi_\alpha + w_{,\alpha}\right)^2 \right] \right\} d\Omega + \\ \ldots - \int_\Omega qw d\Omega - \int_{\Gamma_f} (Pw + EM_\alpha \psi_\alpha) d\Gamma + \int_\Omega \frac{1}{2} \left(N_{\alpha\beta} w_{,\alpha} w_{,\beta} \right) d\Omega \quad (3)$$

The energy functional of the plate was written in the complete form in eqn (3) and is similar to that presented in [9]. The first integral (domain integral) is the strain energy whereas the effect of GNL appeared in the last integral. The second and the third integral are the potential energy of the external loads distributed on the domain and on a portion of the boundary line (Γ_f). The out of plane loads on are q and P whereas EM_1, EM_2 are couples in directions 1, 2, respectively. The displacements w, ψ_1, ψ_2 are not prescribed on the portion of the boundary line Γ_f. The energy functional of the plate can be written in a general function to be minimized with the calculus of variations:

$$\Pi = \int_\Omega F\left(w, \psi_1, \psi_2, w_{,1}, \psi_{1,1}, \psi_{2,1}, w_{,2}, \psi_{1,2}, \psi_{2,2}\right) d\Omega \quad (4)$$

The Euler equations obtained from the minimization of eqn (4) is given by:

$$\frac{\partial F}{\partial \psi_\alpha} - \frac{\partial}{\partial X_\beta}\left(\frac{\partial F}{\partial \psi_{\alpha,\beta}}\right) = 0$$
$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial X_\beta}\left(\frac{\partial F}{\partial w_{,\beta}}\right) = 0$$

The equilibrium equations are obtained when the constitutive eqns (1) and (2) are introduced in resultant expressions from the application of Euler equations:

$$M_{\alpha\beta,\beta} - Q_\alpha = 0 \tag{5}$$

$$Q_{\alpha,\alpha} + q + \frac{\partial}{\partial x_\alpha}\left(N_{\alpha\beta}\frac{\partial w}{\partial x_\beta}\right) = 0 \tag{6}$$

The natural conditions introduce requirements on the boundary portion (Γ_f) with not prescribed displacements where the variations of displacements are not null ($\delta w \neq 0$, $\delta\psi_\alpha \neq 0$):

$$\left(\frac{\partial F}{\partial w_{,\alpha}} n_\alpha\right)\delta w = 0 \xrightarrow{yields} \left(\frac{\partial F}{\partial w_{,\alpha}} n_\alpha\right) = 0$$

$$t_3 = P - n_\alpha N_{\alpha\beta} w_{,\beta} \tag{7}$$

$$\left(\frac{\partial F}{\partial \psi_{\alpha,\beta}} n_\beta\right)\delta\psi_\alpha = 0 \xrightarrow{yields} \left(\frac{\partial F}{\partial \psi_{\alpha,\beta}} n_\beta\right) = 0$$

$$M_\alpha = EM_\alpha \tag{8}$$

Equations (1) and (2) were used to obtain t_α ($t_\alpha = M_{\alpha\beta}.n_\beta$) and t_3 ($t_3 = Q_\alpha.n_\alpha$), respectively, in natural conditions.

The general form of displacement boundary integral equations (DBIEs) with an additional domain integral containing the effect of GNL is next written with the notation proposed by Weeën:

$$\frac{1}{2}C_{ij}(x')u_j(x') + \int_\Gamma \left[T_{ij}(x',x)u_j(x) - U_{ij}(x',x)t_j(x)\right]d\Gamma(x)$$
$$= \iint_\Omega U_{i3}(x',X)\left[q + \frac{\partial}{\partial X_\alpha}\left(N_{\alpha\beta}\frac{\partial u_3}{\partial X_\beta}\right)\right]d\Omega(X) \tag{9}$$

C_{ij} is an element of the matrix C related to the boundary at the source point, which becomes the identity matrix when a smooth boundary is considered. u_α is ψ_α, u_3 is w. U_{ij} represents the rotation (j=1, 2) or the deflection (j=3) due to a unit couple (i=1, 2) or a unit point force (i=3), respectively, T_{ij} represents the moment (j=1, 2) or the shear (j=3) due to a unit couple (i=1, 2) or a unit point force (i=3), respectively.

It is well known in plate analyses that the natural condition is introduced for each generalized force t_i corresponding to the displacement not prescribed. According to eqn (7) the effect of GNL should be introduced as "an out-plane force" when the deflection is not prescribed (or, free) on the boundary portion [3, 19].

The term related to the effect of GNL in eqns (6) or (9) can be simplified with the equilibrium equations for in-plane forces ($N_{\alpha\beta,\alpha} = 0$). The second derivatives of the deflection appear as result from the simplification, as shown in several studies in the literature. The equilibrium equations for in-plane forces were not used here but an algebraic manipulation with the divergence theorem was done in the domain integral related to GNL in eqn. (6), i.e.:

$$\iint_{\Omega} U_{i3}(x',X)\left[\frac{\partial}{\partial X_{\alpha}}\left(N_{\alpha\beta}\frac{\partial u_3(X)}{\partial X_{\beta}}\right)\right]d\Omega(X) =$$

$$\int_{\Gamma} n_{\alpha}(x)N_{\alpha\beta}(x)u_{3,\beta}(x)U_{i3}(x',x)d\Gamma(x) - \iint_{\Omega} N_{\alpha\beta}(X)u_{3,\beta}(X)U_{i3,\alpha}(x',X)d\Omega(X)$$

The algebraic manipulation carried to use first derivatives of the deflection, only, when the effect of GNL is considered. The final DBIE is given by:

$$\frac{1}{2}C_{ij}(x')u_j(x') + \int_{\Gamma}\left[T_{ij}(x',x)u_j(x) - U_{ij}(x',x)t_j(x)\right]d\Gamma(x)$$
$$= \iint_{\Omega} q(X)U_{i3}\left(x',X\right)d\Omega(X) + \int_{\Gamma} n_{\alpha}(x)N_{\alpha\beta}(x)u_{3,\beta}(x)U_{i3}(x',x)d\Gamma(x) +$$
$$-\iint_{\Omega} N_{\alpha\beta}(X)u_{3,\beta}(X)U_{i3,\alpha}(x',X)d\Omega(X) \qquad (10)$$

The boundary integral containing the effect of GNL can be related to the natural condition given by eqn (7) when the boundary portion has the deflection (w or u_3) not prescribed. It can be shown by assuming the boundary Γ split into two portions: Γ_p and Γ_f where displacements are known (prescribed) and unknown (not prescribed or free), respectively.

$$\frac{1}{2}C_{ij}(x')u_j(x') + \int_{\Gamma_f} T_{ij}(x',x)u_j(x)d\Gamma(x) - \int_{\Gamma_p} U_{ij}(x',x)t_j(x)d\Gamma(x) = \cdots$$
$$\ldots = \int_{\Gamma_f} U_{i\alpha}(x',x)EM_{\alpha}(x)d\Gamma(x) + \int_{\Gamma_f} U_{i3}(x',x)P(x)d\Gamma(x) + \cdots$$
$$\ldots - \int_{\Gamma_p} T_{ij}(x',x)u_j(x)d\Gamma(x) + \iint_{\Omega} q(X)U_{i3}(x',X)d\Omega(X) + \cdots$$
$$\ldots + \int_{\Gamma_p} n_{\alpha}(x)N_{\alpha\beta}(x)u_{3,\beta}(x)U_{i3}(x',x)d\Gamma(x) + \cdots$$
$$\ldots - \iint_{\Omega} N_{\alpha\beta}(X)u_{3,\beta}(X)U_{i3,\alpha}(x',X)d\Omega(X) \qquad (11)$$

The left hand side of eqn (11) contains the unknowns i.e. displacements on Γ_f and forces on Γ_p. The external loads on the boundary portion Γ_f were introduced in the right hand side according to natural conditions shown in eqns (7) and (8). A simplification can be done on the boundary portion Γ_f due to opposite signals in the natural condition and in the boundary integral with the effect with GNL. The boundary integral with the effect of GNL computed only on the boundary portion with prescribed displacements (Γ_p) is the result from the simplification.

The BIE for the first derivative of the deflection in the direction γ at an internal point is obtained by differentiating the DBIE given by eqn (10) with respect to the coordinate of the source point (X'). The result is next written in terms of differentiation of coordinates of the field point and with direction cosines of the outward normal at the field point written off the differential operator [20]:

$$u_{3,\gamma}(X') = \int_\Gamma \left\{ n_\alpha(x) \frac{\partial}{\partial x_\gamma}\left[M_{3\alpha\beta}(X',x)\right]u_\beta(x) + \cdots \right.$$

$$\ldots + n_\beta(x) \frac{\partial}{\partial x_\gamma}\left[Q_{3\beta}(X',x)\right]u_3(x) - \frac{\partial}{\partial x_\gamma}\left[U_{3\beta}(X',x)\right]t_\beta(x) + \cdots$$

$$\left. \ldots - \frac{\partial}{\partial x_\gamma}\left[U_{33}(X',x)\right]t_3(x)\right\} d\Gamma(x) - \iint_\Omega \frac{\partial}{\partial X_\gamma}\left[U_{33}(X',X)\right]q(X)d\Omega(X) \ldots$$

$$\ldots - \int_\Gamma n_\alpha(x)N_{\alpha\beta}(x)u_{3,\beta}(x)\frac{\partial}{\partial x_\gamma}\left[U_{i3}\left(X',x\right)\right]d\Gamma(x) + \ldots$$

$$\ldots + \iint_\Omega N_{\alpha\beta}(X)u_{3,\beta}(X)\frac{\partial}{\partial X_\gamma}\left[U_{i3,\alpha}(X',X)\right]d\Omega(X)$$

3 Numerical implementation

There were employed quadratic shape functions for isoparametric boundary elements with collocation points always placed on the boundary. The same mapping function was used for conformal and non-conformal interpolations, i.e. nodes at ends of quadratic elements remain at ends when discontinuous elements were employed. The collocation points were placed at nodes in case of continuous elements and at positions (-0.67, 0.0, +0.67), in the range (-1, 1), in case of discontinuous elements, i.e. the collocation points were shifted to inside the element at the corresponding end where the discontinuity exists. The singularity subtraction [21] and the transformation of variable technique [22] were employed for the Cauchy and the weak type singularity, respectively, when integrations were performed on elements containing the collocation points. The standard Gauss–Legendre scheme was employed for integrations on elements (or, side of the cell) not containing the collocation points. Rectangular cells were used to discretize the domain integral related to the geometrical non-linearity effect. The derivatives of the deflection at the center of the cell were assumed constant on the cell. This assumption allowed to use the divergence theorem to convert the domain integral in equivalent boundary integral performed on sides of the cell. This strategy carried to a simplification on the use of integrals containing the effect of GNL because they have opposite signals, i.e.:

a) When the deflection is prescribed on the whole boundary (like a simply supported plate from all sides): The effect of GNL is computed from integrations performed only on sides of cells inside the domain but <u>not</u> on sides on the boundary of the plate;
b) When the deflection is <u>not</u> prescribed on the boundary portion of the plate (Γ_f): The effect of GNL is computed from integrations performed on sides of cells inside the domain <u>and</u> on sides of cells on the boundary portion Γ_f.

The basic inverse iteration and the Rayleigh quotient were used to perform the eigenvalue analysis. [23], i.e.:

$$Ax^{(k+1)} = x^k \tag{12}$$

$$\lambda_k = \frac{\left(x^{(k+1)}, x^k\right)}{\left(x^{(k+1)}, x^{(k+1)}\right)} \tag{13}$$

The basic inverse iteration procedure is very efficient to compute the lowest eigenvalue with corresponding eigenvector [23]. Equation (12) was not used explicitly, but starting with an eigenvector x^1 with all elements equal to 1.0, values for displacements and tractions at nodes of boundary elements were found. These values were introduced in the discretized form of equation (11), i.e. the equation written in terms of matrices, to obtain the eigenvector x^2 and the lowest eigenvalue at the first iteration step was obtained by using equation (13). The iteration procedure continued until the absolute difference between values of successive eigenvalues was less than 10^{-5}. The proof of the convergence for the lowest eigenvalue can be found in [23].

The Young modulus (E) was 206.9 GPa, the Poisson ration (ν) was 0.3 and the shear parameter was $\pi^2/12$ (Mindin). The buckling parameter k is obtained according to the boundary conditions: S=simply supported edge, C=clamped edge and F= free edge.

$$k = \frac{a^2 N_{cr}}{\pi^2 D}$$

The critical in-plane load is N_{cr} and the length of the plate side is a.

Table 1: Buckling parameter (k) of the first critical in-plane load.

Type	h/L	[24]	[25]	64 cells	100 cells
SSSS	0.001	4.0000	4.0478	4.0503	4.0325
	0.050	3.9280	3.9879	3.9928	3.9753
	0.100	3.7290	3.8227	3.8291	3.8129
	0.200	3.1190	3.2850	3.2887	3.2770
SSSC	0.001	4.8470	4.9235	4.9392	4.9069
	0.050	4.7170	4.8161	4.8343	4.8028
	0.100	4.3720	4.5248	4.5447	4.5166
	0.200	3.4180	3.6467	3.6640	3.6452
CSSS	0.001	5.7400	5.8310	5.8167	5.7897
	0.050	5.5740	5.6878	5.6710	5.6451
	0.100	5.1400	5.3116	5.2813	5.2585
	0.200	3.8760	4.1930	4.1852	4.1711
SCSC	0.001	6.7430	6.8910	6.9490	6.8776
	0.050	6.4620	6.6597	6.7181	6.6506
	0.100	5.7650	6.0566	6.1133	6.0559
	0.200	4.1090	4.4540	4.4977	4.4637
CSCS	0.001	7.6910	7.9673	7.9306	7.8484
	0.050	7.2280	7.5346	7.5166	7.4416
	0.100	6.1780	6.5342	6.5381	6.4800
	0.200	4.0560	4.4696	4.3990	4.3718

The Spline Strip Method was used in [24] and the boundary element method in [25]. The term related to the effect of GNL was simplified in [25] with equilibrium equations for in-plane forces ($N_{\alpha\beta,\alpha}$=0) and second derivatives of the deflection appeared as result from the simplification. The meshes used in [25] were: 40 isoparametric linear boundary elements with 100 constant cells for results in Table 1 and 80 isoparametric linear boundary elements with 400 constant cells for Table 2. Results in Tables 1 and 2 were obtained with: 64 quadratic boundary elements (132 nodes) with 64 constant cells and 80 quadratic boundary elements (164 nodes) with 100 constant cells

Table 2: Buckling parameter (k) of the first critical load.

Type	h/L	[24]	[25]	64 cells
FSSS	0.001	1.4020		1.4135
	0.010	1.4000		1.4137
	0.050	1.3780	1.3336	1.3958
	0.100	1.3270	1.3003	1.3546
	0.200	1.1730	1.1785	1.2260
FSCS	0.001	1.6520		1.6685
	0.010	1.6500		1.6681
	0.050	1.6200	1.5628	1.6383
	0.100	1.5560	1.5074	1.5739
	0.200	1.3700	1.3321	1.3852
FSFS	0.001	0.9523		0.9586
	0.010	0.9516		0.9586
	0.050	0.9412	0.9007	0.9499
	0.100	0.9146	0.8836	0.9287
	0.200	0.8274	0.8151	0.8562

Table 3: Buckling parameter (k) of the first critical load for a simply supported SSSS plate.

BE	Cells	[11]	Obtained	DRM	Points
20	5x5	4.241	4.122	4.189	25
24	6x6	4.173	4.085	4.141	36
28	7x7	4.143	4.063	4.060	49
32	8x8	4.079	4.048	4.032	64
36	9x9	4.068	4.038	3.985	81
40	10x10	4.041	4.030	3.999	100

Tables 1 and 2 show the results obtained with the present formulation were better than those obtained with the use of second derivatives of the deflection to introduce the effect of GNL [25] and close to those obtained by Mizusawa [24].

Furthermore, the numbers of cells were lower using the present formulation than those used in [25].

Table 3 shows a comparison between results obtained with the present formulation and those presented by Purbolaksono and Aliabadi [11]. There were used constant rectangular cells to discretize the domain and the Dual Reciprocity Method in [11]. The expected value for k was 4.000 according to [11] but no information was included on values for thickness or mechanical constants (E, ν) or length of the side of the plate. The number of boundary elements (BE) and the number of cells (Cells) used in [11] were adopted to check results with the present formulation. The numbers of points used for the DRM in [11] were the same used for cells and they were not repeated in Table 3. Instability on values for the DRM can be noted when 81 and 100 points were used when the results were lower than the expected value.

4 Conclusions

Results obtained with the use of first derivatives for the deflection to introduce the effect of GNL were apparently better than those obtained with second derivatives of the deflection. In spite of the reduction on the effort for the numerical implementation (use of first derivative, only), the number of domain cells was reduced and the convergence was monotonic according to the increase of the number of cells.

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