Application of exponential transformation coupled with adaptive subdivision technique for nearly singular boundary integrals in elasticity problems

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Abstract

In this work, exponential transformation is applied for nearly singular boundary element integrals in elasticity problems. Accurate evaluation of nearly singular integrals is an important issue in the implementation of the boundary element method (BEM). In this paper, exponential transformation is firstly developed to evaluate nearly singular boundary integrals in elasticity problems. In our work, firstly, a novel (α,β) coordinate system is introduced. Then an improved exponential transformation is constructed in the new coordinate system. Furthermore, to perform integrations on irregular elements, an adaptive integration scheme considering both the element shape and the nearest point associated with the improved transformation is employed. Compared with conventional methods, we use the nearest point instead of the projection point. So, the drawbacks of how to subdivide the integration element into subtriangles with fine shapes can be overcome. Numerical examples are presented to verify the proposed method. Results demonstrate the accuracy and efficiency of our method.
Keywords: nearly singular integrals, boundary element method, boundary face method, exponential transformation.

1 Introduction

Near singularities are involved in many boundary element method (BEM) analyses of engineering problems, such as problems on thin shell-like structures

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[1–3], the crack problems [4], the contact problems [5], as well as the sensitivity problems [6]. Accurate and efficient evaluation of nearly singular integrals with various kernel functions of the type $O(1/r^{\chi})$ is crucial for successful implementation of the boundary type numerical methods based on boundary integral equations (BIEs), such as the boundary element method (BEM), the boundary face method (BFM) [7, 8]. A near singularity arises when a source point is close to but not on the integration elements. Although these integrals are really regular in nature, they can't be evaluated accurately by the standard Gaussian quadrature. This is because the denominator r, the distance between the source and the field point, is close to zero but not zero. The difficulty encountered in the numerical evaluation mainly results from the fact that the integrands of nearly singular integrals vary drastically with respect to the distance. Effective computation of nearly singular integrals has received intensive attention in recent years. Various numerical techniques have been developed to remove the near singularities, such as semi-analytical or analytical integral formulas [9, 10], the sinh transformation [10–13], polynomial transformation [14], adaptive subdivision method [7, 15, 16], distance transformation technique [8, 17–20], the PART method [21–23], and the exponential transformation [24–27]. Most of them benefit from the strategies for computing singular integrals. Among those techniques, the exponential transformation technique seems to be a more promising method for dealing with different orders of nearly singular boundary element integrals. In this paper, we develop the exponential transformation technique for thin structures in 3D boundary element method in (α,β) coordinate system.

In our method, first a (α,β) coordinate system is introduced. This system is very similar to the polar system, but its implementation is simpler than the polar system and also performs efficiently. Then an improved exponential transformation is constructed in the new coordinate system. Using the proposed transformation, the integrals with near singularities can be accurately calculated. Furthermore, an element subdivision technique considering both the element shape and the positions of the nearest point is introduced in combination with the improved transformation to perform integrations on irregular elements. In our method, we use the nearest point instead of the projection point. This is because when the projection point is outside the integration element, how to subdivide the integration element into subtriangles with fine shapes is difficult [27]. However, the nearest point is always located in the integration element. With our method, the nearly singular boundary element integrals of regular or irregular elements can be accurately and effectively calculated. Numerical examples are presented to validate the proposed method. Results demonstrate the accuracy and efficiency of our method.

2 Boundary integral formulations

In this section, we will give a general form of the nearly singular integrals over 3D boundary elements. First we consider the boundary integral equations for 3D elasticity problems. The well-known self-regular BIE for elasticity problems in 3-D is

$$0 = \int_{\Gamma} (u_j(\mathbf{s}) - u_j(\mathbf{y})) T_{ij}(\mathbf{s},\mathbf{y}) d\Gamma - \int_{\Gamma} t_j(\mathbf{s}) U_{ij}(\mathbf{s},\mathbf{y}) d\Gamma \tag{1}$$

where **s** and **y** represent the field point and the source point in the BEM, with components s_i and y_i, $i=1,2,3$, respectively.

Eq. (1) is discretized on the boundary Γ by boundary elements $\Gamma_e (e = 1 \sim N)$ defined by interpolation functions. The integral kernels of Eq. (1) become nearly singular when the distance between the source point and integration element is very small compared to the size of integration element. And the integrals in Eq. (1) become nearly singular with different orders, namely, $U_{ij}(\mathbf{s},\mathbf{y})$ with near weak singularity, and $T_{ij}(\mathbf{s},\mathbf{y})$ with near strong singularity. In this paper, we develop the exponential transformation method for various boundary integrals with near singularities of different orders. The new method is detailed in following sections. For the sake of clarity and brevity, we take following integrals as a general form to discuss:

$$I = \int_S \frac{f(\mathbf{x},\mathbf{y})}{r^l} dS, \quad l = 1,2,3 \quad r = \|\mathbf{x} - \mathbf{y}\|_2 \tag{2}$$

where f is a smooth function, **x** and **y** represent the field point and the source point in BEM, with components x_i and y_i, respectively. S represents the boundary element. We assume that the source point is close to S, but not on it.

3 Construction of exponential transformation

3.1 New coordinate system

To construct the local (α, β) system as shown in Fig. 1, the following mapping scheme which is used for calculating weakly singular integrals [7, 8] is used:

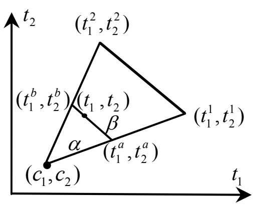

Figure 1: New system (α, β).

It is different to obtain the coordinates t_1 and t_2 in (α, β) system compared with the polar coordinate system above, which can be written as:

$$\begin{cases} t_1 = c_1 + (t_1^1 - c_1)\alpha + (t_1^2 - t_1^1)\alpha\beta \\ t_2 = c_2 + (t_2^1 - c_2)\alpha + (t_2^2 - t_2^1)\alpha\beta \end{cases} \quad \alpha, \beta \in [0,1] \tag{3}$$

The Jacobian for the transformation from (t_1, t_2) system to (α, β) system is αS_Δ

$$S_\Delta = \left| t_1^1 t_2^2 + t_1^2 c_2 + c_1 t_2^1 - t_1^2 t_2^1 - c_1 t_2^2 - t_1^1 c_2 \right| \tag{4}$$

Using Eq. (4) and following Ma's method [17–19],

$$\begin{aligned} x_k - y_k &= x_k - x_k^c + x_k^c - y_k \\ &= \left.\frac{\partial x_k}{\partial t_1}\right|_{\substack{t_1=c_1\\ t_2=c_2}} (t_1 - c_1) + \left.\frac{\partial x_k}{\partial t_2}\right|_{\substack{t_1=c_1\\ t_2=c_2}} (t_1 - c_2) + r_0 n_k(c_1, c_2) + O(\alpha^2) \\ &= \alpha A_k(\beta) + r_0 n_k(c_1, c_2) + O(\alpha^2) \end{aligned} \tag{5}$$

where

$$A_k(\beta) = \left.\frac{\partial x_k}{\partial t_1}\right|_{\substack{t_1=c_1\\ t_2=c_2}} \left[(t_1^1 - t_1^0) + (t_1^2 - t_1^1)\beta \right] + \left.\frac{\partial x_k}{\partial t_2}\right|_{\substack{t_1=c_1\\ t_2=c_2}} \left[(t_2^1 - t_2^0) + (t_2^2 - t_2^1)\beta \right] \tag{6}$$

Using Eqs. (3)–(6), we can easily obtain the distance function [17–19] in a new form:

$$r^2 = (x_k - y_k)(x_k - y_k) = A_k^2(\theta)\alpha^2 + r_0^2 + O(\alpha^3) \tag{7a}$$

$$r = \sqrt{A_k^2(\theta)\alpha^2 + r_0^2 + O(\alpha^3)} \tag{7b}$$

Using Eqs. (7a)–(7b), Eq. (2) can be written as:

$$I = \int_\Gamma \frac{f(\mathbf{x}, \mathbf{y})}{r^l} d\Gamma = \sum_m \int_0^1 \int_0^1 \frac{g(\alpha, \beta)}{(\alpha^2 + \lambda^2(\beta))^{l/2}} d\alpha d\beta \tag{8}$$

where $\lambda(\beta) = \dfrac{r_0}{A(\beta)}$, $A(\beta) = \sqrt{A_k(\beta) A_k(\beta)}$, and $g(\alpha, \beta)$ is a smooth function.

3.2 Improved exponential transformation

In order to obtain a reasonable transformation for each case, the distance r is approximated by the Taylor expansion (7a) and (7b) without considering higher orders. However, in actual computation r is still the distance from the source point to the field point. We explain how to construct different transformations. The process consists of five steps and each step is described briefly below.

From Eq. (8) we can analyze that the near singularity is essentially related to the radial variable α. So we will construct a more robust and efficient transformation for the radial variable α.

First we only consider the radial variable integral which depicts near singularity in the Eq. (8) as follows

$$I_1 = \int_0^1 \frac{g(\alpha, \beta)}{(\alpha^2 + \lambda^2(\beta))^{l/2}} d\alpha \tag{9}$$

Second we make a stretching transformation for Eq. (9)

$$\alpha = r_0 \eta_1 \tag{10}$$

Substituting Eq. (10) into Eq. (9), we have

$$I_1 = \int_0^{\frac{1}{r_0}} \frac{r_0 g(\eta_1, \beta)}{(\eta_1^2 + \lambda^2(\beta))^{l/2}} d\eta_1 \tag{11}$$

Then we make a translation transformation for Eq. (11), respectively

$$\eta_2 = \eta_1 + 1 \tag{12}$$

This step is employed to adjust the lower limit of the integration variable for the afterward logarithmic transformation.

Substituting Eq. (12) into Eq. (11), results in

$$I_1 = \int_1^{\frac{1}{r_0}+1} \frac{r_0 g(\eta_2 - 1, \beta)}{((\eta_2 - 1)^2 + \lambda^2(\beta))^{l/2}} d\eta_2 \tag{13}$$

In the four steps, we smooth out the rapid variations of the integrand by the following logarithmic transformation

$$\eta_3 = \log(\eta_2) \tag{14}$$

Substituting Eq. (14) into Eq. (13), Eq. (13) can be expressed as

$$I_1 = \int_0^{\log(\frac{1}{r_0}+1)} \frac{r_0 g(e^{\eta_3} - 1, \beta) e^{\eta_3}}{((e^{\eta_3} - 1)^2 + \lambda^2(\beta))^{l/2}} de^{\eta_3} \tag{15}$$

Using the good properties of the logarithmic function [24–27], it can easily be proved that the transformed integrand has much lower gradient.

Finally, adjusting the integration interval within [-1, 1] for performing the standard Gaussian quadrature directly, we propose the following transformation.

$$\eta_3 = k_2 \eta_4 + k_2 \tag{16}$$

where $k_2 = \frac{1}{2}\log(\frac{1}{r_0} + 1)$

We integrate all the transformations detailed above and can obtain the final transformation as

$$\alpha = r_0 (e^{k_2 \eta + k_2} - 1) \tag{17}$$

It should be noted that the exponential transformation is similar to these in Refs. [24, 25]. However, the deductions in this paper are very different from those given in Refs. [24, 25]. We construct the transformations in a general way based on the approximate distance function derived from first-order Taylor expansion. Moreover, for the first time, the exponential transformation is applied for evaluating nearly singular integrals in 3D elasticity problems.

4 Adaptive element subdivision

The element subdivision is indispensible for treating the nearly singular integrals in the 3D cases as in Refs. [7, 8, 17–20]. In this section, we subdivide an integration element in a suitable pattern considering both element shape and the position of the nearest point in the element. Adaptive integration based on element subdivision to calculate integrals is employed just as a combination for the exponential transformation [24–27]. The element subdivision technique is very similar to that discussed in Ref. [8]. However, we use the nearest point instead which in the most close point to the source point in the element instead of the projection point.

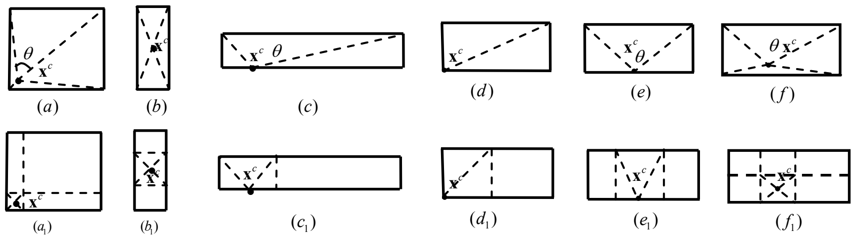

Figure 2: Subdivisions of quadrilateral element depending on the nearest point.

Note that although the original quadrangle has a fine shape, the four subtriangles may have poor shapes depending on the position of $\mathbf{x}^c$ (the nearest point) (see Fig. 2.(a)). Obtaining triangles of fine shape seems more difficult by direct subdivision for irregular initial elements as shown in Fig. 2(a) even $\mathbf{x}^c$ is located at the element center. If the angle denoted by θ , Fig. 2(b)–2(f) between two lines in common with end point $\mathbf{x}^c$ in each triangle is larger by a certain value $2\pi/3$ and even tends to π , numerical results will become less accurate.

To solve the troubles described above, we have developed an adaptive subdivision for an arbitrary quadrilateral element. The original element is divided into several triangles and additional quadrangles, which is different from these as shown in Fig. 2 (a_1)–(f_1). The adaptive subdivision consists of three main steps described briefly as follows:
First, compute the distances in the real-world-coordinate system form $\mathbf{x}^c$ to each edge of the element and obtain the minimum distance d.

Then, based on d, we construct a box defined in parametric system, but with square shape in the real-world-coordinate system as can as possible, to well cover $\mathbf{x}^c$.

Finally, triangles are constructed from the box and additional quadrangles are created outside the box in the element.

Applying the strategy above, adaptive subdivisions for the elements in Fig. 2 with suitable patterns are shown in Fig. 2(a_1)–(f_1). For each triangle, the nearly singular integrals are calculated by the scheme discussed in Section 3. However, for each quadrangle, nearly singular integrals will arise but not severe, which can be calculated by adaptive integration scheme based on the element subdivision technique discussed in Refs. [7, 8].

It should be noted that, although the element subdivision is adopted, the computational cost is reduced dramatically compared with the conventional subdivision technique to compute nearly singular integrals on the whole element.

This is because that the integrals on the local region of the element, which is more close to the source point, are calculated by the new variable transformations technique.

It is also should be noted that, compared with the conventional subdivision methods, we subdivide the element in three triangles around the nearest point $\mathbf{x}^c$ which is which is the most close point to the source point in the element instead of the projection point. So, when the projection point is located outside the element, we still can get subtriangles with fine shapes [27].

5 Numerical examples

5.1 Example 1: a thin plate problem

The geometry, boundary conditions, and the model for the problem are shown in Fig. 3. We assume the *thickness* is 0.01. The Young's modulus is 1 and the Poisson's ratio is 0.25. In order to assess the accuracy of the present method, boundary conditions of Dirichlet type are imposed on all faces corresponding to quadratic exact solutions. And the solutions are as follows:

$$\begin{cases} u_x = -2x^2 + 3y^2 + 3z^2 \\ u_y = 3x^2 - 2y^2 + 3z^2 \\ u_z = 3x^2 + 3y^2 - 2z^2 \end{cases} \tag{18}$$

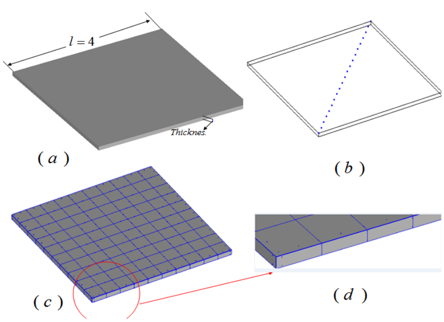

Figure 3: (a) A thin plate (b) Sample points (c) Discretization of the thin plate (d) Slender elements.

As shown in Fig. 3, the BFM model with 240 8-node quadrilateral elements and the total number of nodes is 770. For side surfaces, 8-node discontinuous quadrilateral elements are used. As illustrated in Fig. 3, the elements are slender elements in the side surfaces. To evaluate the nearly singular integrals, the improved transformation combined with the element subdivision technique is applied. The sample points are distributed on the boundary. The boundary sample points are well-distributed on isoparametric line segment from (2, -2) to (-2, 2) in the parametric space of the surface (y=0.005). Results at the sample points are illustrated in Fig. 4.

As shown in Fig. 4, with the proposed method, the near singularities are removed efficiently and accurate numerical results have been obtained. The slender element, of which the length and width ratio is larger than 10, is applied for long and narrow surfaces in this example. In our method, however, the accuracy is not influenced by elements with poor quality.

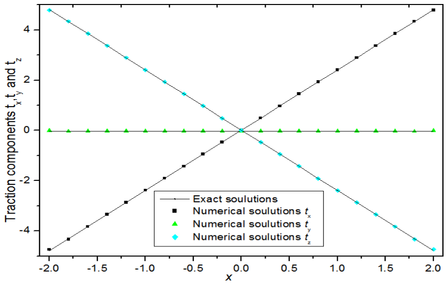

Figure 4: Results at the sample points.

5.2 Example 2: an elbow pipe problem

In order to show the advantages of our method, a problem with more complicated geometry is solved here [7]. The geometry and its main dimensions are shown in Figure 5. The meshes of the model are also shown in Fig. 5. 500 8-node quadrilateral elements are used and the total number of nodes is 1768. The boundary evaluation points are uniformly distributed along the middle ring. Dirichlet boundary conditions according to the analytical solutions Eq. (18) are imposed on all the faces of the elbow pipe. The elbow pipe is a thin shell structure, thus nearly singular integrals arises, which is evaluated by the scheme described in Section 3 and Section 4. Numerical results at the sample points are shown in Fig. 6. From Fig. 7, it can be seen that the numerical results obtained by our method is in good agreement with the exact solutions. Note that nearly singular integrals can be accurately computed by the exponential transformation, so the proposed method can be used for the analysis of thin structures.

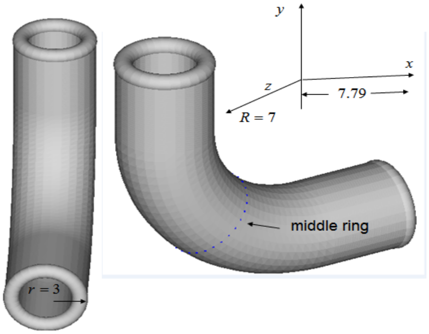

Figure 5: An elbow pipe and its main dimensions [7].

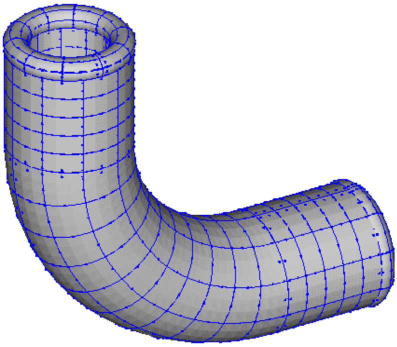

Figure 6: Discretization of the elbow pipe.

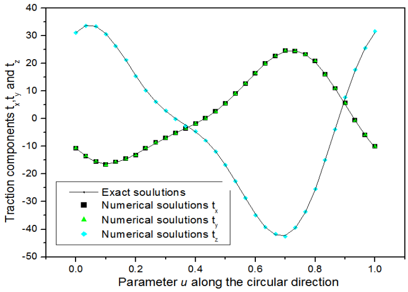

Figure 7: Results at the boundary points.

6 Conclusions

This work presented exponential transformation coupled with adaptive subdivision technique for nearly singular integrals which appear in the application of BEM for elasticity problems. By applying the proposed method in the BEM, the number of integral points in the near singular integral patches has been reduced significantly. To perform integration on irregular elements, an adaptive integration scheme considering the element shape and the nearest point in combination with the improved transformation has been introduced. Numerical examples have been presented to verify the proposed method. Results demonstrate the accuracy and efficiency of our method. For nearly hypersingular integrals or other nearly singular integrals of higher orders, however, the present method is not so effective. Using the Ma's method [18, 19], it is easy to extend our method for nearly hypersingular integrals.

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