Adaptive integration technique for nearly singular integrals in near-field acoustics boundary element analysis

Shen Qu, Hao-Ran Chen, Rui-Xiang Bai & Sheng Li
State Key Laboratory of Structural Analysis for Industrial Equipment, Dalian University of Technology, China

Abstract

The aim of this paper is to present an efficient adaptive integration technique to perform near-field acoustics boundary element analysis, in which nearly singular integrals will be encountered as the source point in integral equations is close to the boundary of the acoustic domain. At this time the integrand varies sharply, so the conventional Gaussian quadrature becomes inefficient or even inaccurate. In this paper, an adaptive integration technique is proposed, which determines required Gauss orders and the number of sub-elements according to the specified integration accuracy and the relative position from the source point in integral equations to the element under integration. Two numerical examples have been presented to demonstrate the efficiency and accuracy of the proposed approach.
Keywords: nearly singular integrals, boundary element method, adaptive integration, subdivision technique.

1 Introduction

Boundary element method has become a popular approach for solving acoustical problems by virtue of its advantages, such as semi-analytics, high accuracy and reducing dimension. Especially for the exterior radiation problem, the integral over the boundary of an unbounded fluid domain actually disappears due to the Sommerfeld radiation condition, and then only the structural boundary needs to be discretized into elements. In terms of this point, the boundary element method is more effective than the finite boundary method in calculation of infinite domain acoustical problems. Therefore, acoustics is one of the best areas to demonstrate the power of the boundary element method [1, 2]. In three-

WIT Transactions on Modelling and Simulation, Vol 56, © 2013 WIT Press
www.witpress.com, ISSN 1743-355X (on-line)
doi:10.2495/BEM360291

dimensional near-field acoustics boundary element analysis, integrals are referred to as nearly singular when the source point in integral equation is close to the element under consideration. In this case the integrand varies sharply, as is the intrinsic drawback of boundary element methods. To overcome this difficulty, several techniques have been developed, such as analytical and semi-analytical methods, translation technique, and adaptive tactics, etc.

The analytical and semi-analytical methods [3, 4] are limited to the planar elements only. When curved elements are involved, these elements must be divided into a large number of planar triangles, thus losing efficiency and accuracy. The methods proposed by Telles [5] and Telles and Oliveira [6], who is one of the pioneers in applying nonlinear transformation techniques to regularize weakly singular integrals and nearly singular integrals, include a cubic polynomial transformation depending on an optimized parameter which, in turn, depends on the position of the nearly singular point. It has been shown in reference [7] that this approach is very successful when the optimization is performed accurately, however, the method is very sensitive to the optimization procedure. In fact, Sladek *et al.* [7] conclude that an optimized parameter based on a slightly incorrect (within 1%) nearly singular point can reduce the effectiveness of the transformation significantly. Based on the sinh function, Johnston and Elliott [8] and Johnston *et al.* [9] introduced a transformation for evaluating near singular integrals, which arise in the solution of Laplace's equation in three dimensions. Distance transformation method, which belongs to a kind of non-linear transformation and has been proposed by Ma and Kamiya [10, 11], is a general strategy to deal with nearly singular integrals with various kernels in BEM. For this method, the numerical results are very sensitive to the position of the projection point of the source point because the projection point may locate inside or outside the element, as is the drawback of the method. Qin *et al.* [12] presented an improved distance transformation technique, which overcomes the conventional distance transformation technique's drawback, i.e. the accuracy is sensitive to the position of the projection point. Based on the sinh transformation and parabolic geometry elements, Gu *et al.* [13] presents an improved approach for the numerical evaluation of nearly singular integrals. Xie *et al.* [14] proposed a further development of the distance transformation technique for accurate evaluation of the nearly singular integrals arising in the two-dimensional boundary element method. More recently, this variable transformation method is extended to three-dimensional boundary element method [15]. Apart from the above methods, adaptive tactics have been widely used in tackling nearly singular integrals arising in many boundary element analyses of engineering problems, such as potential problem [16, 17], contact problems [18], plate bending [19], thermoelastic problems [20], elasto-plastic problems [21, 22], etc. Kita and Kamiya [23] gave a comprehensive review on adaptive mesh refinement schemes for boundary element methods. For acoustics boundary analysis, Chen *et al.* [24] used the Burton and Miller approach to overcome non-uniqueness difficulties for exterior acoustic problem, consequently, hypersingular integrals will be encountered in this case. Then, the

www.witpress.com, ISSN 1743-355X (on-line)

h-adaptive mesh refinement process is used to tackle singular and hypersingular integrals. The analysis is only limited in two-dimensional problem.

To the best of the authors' knowledge, there have not been papers dedicated entirely to the subject of three-dimensional near-field acoustics boundary element analysis. This paper discusses this issue and the outline is as follows. The acoustics boundary element method is reviewed in Section 2. It is pointed out that nearly singular integrals will be encountered as the source point in integral equation is close to the boundary of acoustic domain. In this case, the conventional Gaussian quadrature becomes inefficient or even inaccurate. So in Section 3, an efficient adaptive integration technique is presented. In Section 4, two numerical examples are presented to verify the efficiency and accuracy of the proposed approach. Finally, Section 5 draws conclusions.

2 Acoustics boundary element method

The governing differential equation for the acoustic pressure field in time-harmonic linear acoustics can be expressed by the Helmholtz equation:

$$\nabla^2 p + k^2 p = 0 \tag{1}$$

where p is the sound pressure, $k = \omega / c$ is the wave number, ω and c are the angular frequency and the speed of sound, respectively. A time harmonic factor of $\exp(i\omega t)$ is suppressed here and in the remaining discussion, where $i = \sqrt{-1}$ denotes the imaginary unit.

The Helmholtz integral equation is derived from Green's second identity or the weighted residual formulation. For an interior acoustic problem, the boundary integral equation [2] is

$$C^0(P)p(P) = -\int_{s(Q)} \left[i\rho\omega v_n(Q)G(Q,P) + p(Q)\frac{\partial G(Q,P)}{\partial n} \right] dS(Q) \tag{2}$$

where P is the source point and Q is the integration point on the boundary, $G(Q,P) = e^{-ikr(Q,P)} / 4\pi r(Q,P)$ is the fundamental solution in a three-dimensional free space of the Helmholtz equation, $r(P,Q)$ denotes the distance between P and Q, n is the unit outward normal of the boundary, v_n is the particle normal velocity, which is related to the sound pressure by Euler's equation, ρ is the mean density of fluid, the leading coefficient $C^0(P) = -\int_{S(Q)} \partial \psi_L(Q,P) / \partial n \, dS(Q)$, where $\psi_L(Q,P) = 1/4\pi r(Q,P)$ is the fundamental solution of the Laplace equation.

In a similar manner, the boundary integral equation for an exterior acoustic problem is

$$C(P)p(P) = -\int_{s(Q)} \left[i\omega\rho v_n(Q)G(Q,P) + p(Q)\frac{\partial G(Q,P)}{\partial n} \right] dS(Q) \tag{3}$$

www.witpress.com, ISSN 1743-355X (on-line)

Notice that the unit outward normal of the boundary for an exterior problem is actually opposite to the outward normal of the boundary for an interior problem, and the leading coefficient $C(P) = 1 - \int_{S(Q)} \partial \psi_L (Q,P) / \partial n \, dS(Q)$.

The boundary element method, which is based on a discretization procedure, is developed to approximate the integral operators. By discretizing the boundary integral equation, a system of equations is generated, which can be written in matrix form as

$$[H]\{p\} = [G]\{v_n\} \tag{4}$$

where $[H]$ and $[G]$ are matrices of influence coefficients. $\{p\}$ and $\{v_n\}$ are vectors containing the nodal values of the sound pressure and the particle normal velocity. Once the boundary conditions of the problem are applied to the equation (4), the matrices can be reordered in the form:

$$[A]\{x\} = \{b\} \tag{5}$$

in which all unknowns have been collected into the vector $\{x\}$. And the vector $\{b\}$ is known values obtained from the product of the specified boundary conditions and the corresponding matrix coefficients.

Once the boundary unknowns are determined, interior quantities at selected points can be obtained, if desired. In this process, when the source point in integral equation is close to the element under consideration, integrals are referred to as nearly singular. Special treatment is required in this case, and an efficient adaptive integration technique is presented in this paper to deal with nearly singular integrals.

3 Adaptive integration technique for nearly singular integrals

The Gaussian quadrature formula for a surface in three dimensions can be expressed in the intrinsic coordinate system by the equation

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi_1, \xi_2) d\xi_1 d\xi_2 = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} w_1^i w_2^j f(\xi_1^i, \xi_2^j) + E_1 + E_2 \tag{6}$$

where ξ_1^i, ξ_2^j are the Gauss ordinates, w_1^i, w_2^j are the weights, m_1, m_2 are the Gauss orders, and E_1, E_2 are the integration errors in the two directions. Lachat and Watson [25] firstly tackled this question in a rational manner by making use of certain analytical expressions for the bounds of the Gauss integration error. They pointed out that the error of the Gaussian quadrature depends on the number of Gauss points and the element size. An approximate formula for the upper bound of the relative error $e_i = E_i / I$ in the ith direction has been proposed by Mustoe [26]:

$$e_i \le 2\left(\frac{L_i}{4R}\right)^{2m_i} \frac{(2m_i + q - 1)!}{(2m_i)!(q-1)!} \tag{7}$$

where q is the order of singularity of the integrand r^{-q}, L_i is the length of the element in the ith direction, and R is the minimum distance from the source point to the element.

To avoid using excessively high Gauss orders m_i, elements may be further divided into subelements to reduce the L_i/R ration, as is shown in Fig. 1.

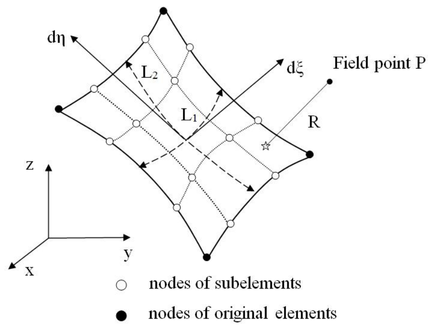

Figure 1: Element subdivision technique.

It should be pointed out that nearly singular integrals will also be encountered as source points are close to the element under consideration for the boundary integral equation. On the contrary to discrete integral equation, it is not necessary to employ extensive work in this case. Sufficient accurate results can be obtained by using normal Gauss orders (in general, four or six Gauss orders are sufficient) due to the ratio L_i/R will not be too large, except the situation where the sizes of the adjacent elements are significantly different.

For convenience, Eq. (7) may be approximated by the expression [27]

$$m_i = \sqrt{\frac{2}{3}q + \frac{2}{5}}\left[-0.1\ln\left(e_i/2\right)\right]\left[\left(\frac{8L_i}{3R}\right)^{\frac{3}{4}} + 1\right] \tag{8}$$

Rearranging this equation gives the maximum length L_i of a subelement:

$$L_i = \frac{3}{8}R\left(\frac{-10m_i}{\sqrt{2q/3 + 2/5}\,\ln\left(e_i/2\right)} - 1\right)^{\frac{4}{3}} \tag{9}$$

Using this approximation, the required Gauss order is obtained explicitly, rather than through iteration. Alternatively, given a maximal Gauss orders, the corresponding subelement dimensions can be obtained explicitly. Now, in order to implement an adaptive integration scheme based on these criteria, it is necessary to devise efficient methods for determining the geometric parameters R and L_i for each source point in integral equation and for each element or subelement. Details of derivation and implementation can be found in the literature [21, 28].

The adaptive integration technique involves increasing the number of integration points as the minimum distance between source points in integral

www.witpress.com, ISSN 1743-355X (on-line)

equation and elements decreases and subdividing the integration interval into sub-intervals if the number of required integration points is greater than a specified maximum [27]. Here, letting $m_{\max}=8$ be the specified maximum allowable order of Gauss integration, we devise the following strategy to implement it:
(a) Calculate element length L_i and minimum distance R from the source point in integral equation to elements.
(b) Calculate the Gauss order m_i with specified precision.
(c) If $m_i \le m_{\max}$ integrate using Gaussian quadrature.
(d) If $m_i \ge m_{\max}$, calculate subelement lengths, using $m_i = m_{\max}$.
(e) Divide element into equal subelements.
(f) Calculate the Jacobian of the transformation from the subelement original intrinsic coordinate system to a new Gaussian quadrature space.
(g) Integrate over all subelements using identical Gauss order $m_i = 8$.
(h) Repeat (f) for all subelements.

4 Numerical examples

In order to demonstrate the accuracy and efficiency of the method, two examples, including the interior acoustic problem and the exterior acoustic problem, are presented as follows.

4.1 One-dimensional plane wave in a box

The first example is a 1m×1m×1m box with a unit amplitude normal velocity specified on one side $x=0\text{m}$ and an anechoic termination, which means no reflection of sound, specified on the opposite side $x=1\text{m}$. The boundary condition for an anechoic termination is simply the characteristic impedance. The other four sides of the box are assumed to be rigid, i.e. zero normal velocity. The solution of this example is the one-dimensional plane wave

$$p=\rho c e^{-ikx} \tag{10}$$

We set the frequency at $f=5.45901\text{Hz}$, $\rho=1.21\text{kg/m}^3$, $c=343\text{m/s}$, and use a total of 26 nodes and 16 four-node quadrilateral elements to model the box. Six interior points in the box is selected to calculate its' quantities. The analytical solution, numerical results of conventional boundary element method and the boundary element method with adaptive integration technique are shown in Table 1.

As shown in Table 1, with higher Gauss orders, more accurate results can be obtained in conventional boundary element method. Meanwhile, the results in conventional boundary element method are poor in accuracy when the source point in integral equation is close to the surface of acoustic domain. Especially, when the distance from the source point in integral equation to the element tends to zero, the obtained results are wrong. For greater clarity, the real part of sound pressure of different source points is plotted in Fig. 2. We observe that inaccurate or incorrect results would be obtained with conventional boundary element method as the distance between source points in integral equation and the boundary of acoustic domain is less than 0.05m with the current mesh size. On the other hand, we can see that the solution of boundary element methods with adaptive integration technique agrees very well with the analytical solution.

Table 1: Sound pressure (Pa) at the interior points in the box.

Source point coordinate	Analytical solution	Conventional BEM		Adaptive integration
		Gauss orders=4	Gauss orders=8	
(0.500,0.5,0.5)	(414.511,-20.743)	(414.533,-19.584)	(414.526,-19.584)	(414.533,-19.584)
(0.800,0.5,0.5)	(413.703,-33.167)	(413.446,-31.978)	(413.743,-32.009)	(413.741,-32.009)
(0.950,0.5,0.5)	(413.159,-39.369)	(433.323,-40.231)	(412.583,-38.150)	(413.217,-38.211)
(0.975,0.5,0.5)	(413.059,-40.401)	(398.506,-37.906)	(419.657,-39.884)	(413.118,-39.244)
(0.995,0.5,0.5)	(412.977,-41.227)	(256.707,-24.760)	(362.986,-35.211)	(413.039,-40.070)
(0.999,0.5,0.5)	(412.961,-41.393)	(216.686,-20.953)	(242.609,-23.596)	(413.064,-40.239)

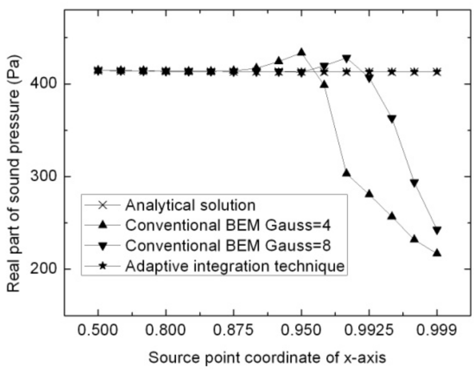

Figure 2: Real part of sound pressure of one-dimensional plate wave.

4.2 Pulsating sphere

The analytical solution of the acoustic pressure of a pulsating sphere with a radius of is given by

$$p = \left(\frac{a^2}{r}\right)\frac{i\rho\omega}{1+ika}e^{-ik(r-a)} \tag{11}$$

where the normal velocity amplitude is assumed to be unit. To run the test case, we set $a = 1\text{m}$ and set the frequency at $f = 54.5901\text{Hz}$. The surface of the sphere is modeled using 396 nodes and 394 four-node quadrilateral elements. Since this is an exterior problem, we add one CHIEF point at the center of the sphere. The analytical solution, numerical results of conventional boundary element method and the boundary element method with adaptive integration technique are shown in Table 2.

Table 2: Sound pressure (Pa) at the outer points of a pulsating sphere.

Source point coordinate	Analytical solution	Conventional BEM		Adaptive integration
		Gauss orders=4	Gauss orders=8	
(3.000,0.0, 0.0)	(34.112,-91.683)	(33.497,-91.300)	(33.497,-91.300)	(33.729,-91.317)
(1.500,0.0, 0.0)	(187.733,55.083)	(186.802,53.777)	(186.802,53.777)	(187.005,54.474)
(1.100,0.0, 0.0)	(206.541,168.874)	(205.973,166.787)	(205.988,166.806)	(205.908,167.753)
(1.050,0.0, 0.0)	(207.264,187.509)	(206.709,185.483)	(206.783,185.568)	(206.761,186.509)
(1.005,0.0, 0.0)	(207.512,205.448)	(190.320,187.788)	(214.128,211.357)	(207.053,204.501)
(1.001,0.0, 0.0)	(207.515,207.100)	(135.778,134.911)	(175.057,174.278)	(207.062,206.190)

In a similar manner, we can see from Table 2 and Fig. 3 that the boundary element method without adaptive integration fails to obtain satisfactory result when the source point in integral equation is close to the surface of acoustic domain, because nearly singular integrals are not taken into account.

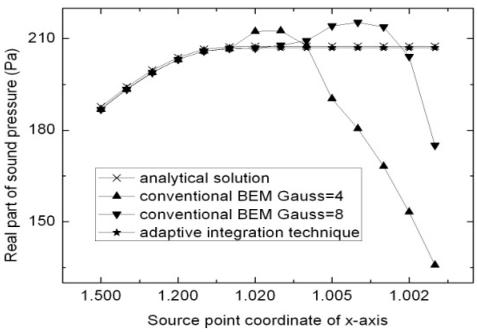

Figure 3: Real part of sound pressure of sphere wave.

5 Conclusion

In this paper, an adaptive integration technique has been presented to tackle nearly singular integrals which arise in three-dimensional near-field acoustics boundary element analysis. This method involves increasing Gauss orders as the source point in integral equation approaches the boundary of acoustic domain and subdividing the element into subelements to avoid using excessively high Gauss order according to Davies and Bu's criterion. Thus, this method takes the optimal computational cost to yield specified integration accuracy. The given numerical examples show that the proposed approach is accurate and robust.

References

[1] Ciskowski RD, Brebbia CA. Boundary element methods in acoustics. Southampton Boston: Computational Mechanics Publications; 1991.

[2] Wu TW. Boundary element acoustics fundamentals and computer codes. Southampton: WIT Press; 2000.

[3] Zhou H, Niu Z, Cheng C, Guan Z. Analytical integral algorithm applied to boundary layer effect and thin body effect in BEM for anisotropic potential problems. Computers & Structures 2008; 86(15-16): 1656-71.

[4] Niu ZR, Wendland WL, Wang XX, Zhou HL. A semi-analytical algorithm for the evaluation of the nearly singular integrals in three-dimensional boundary element methods. Computer Methods in Applied Mechanics and Engineering 2005; 194(9-11): 1057-74.

[5] Telles JCF. A self-adaptive coordinate transformation for efficient numerical evaluation of general boundary element integrals. International Journal for Numerical Methods in Engineering 1987; 24(5): 959-73.

[6] Telles JCF, Oliveira RF. 3rd degree polynomial transformation for boundary element integrals – further improvements. Engineering Analysis with Boundary Elements 1994; 13(2): 135-41.

[7] Sladek V, Sladek J, Tanaka M. Numerical integration of logarithmic and nearly logarithmic singularity in BEMs. Applied Mathematical Modelling 2001; 25(11): 901-22.

[8] Johnston PR, Elliott D. A sinh transformation for evaluating nearly singular boundary element integrals. International Journal for Numerical Methods in Engineering 2005; 62(4): 564-78.

[9] Johnston BM, Johnston PR, Elliot D. A sinh transformation for evaluating two-dimensional nearly singular boundary element integrals. International Journal for Numerical Methods in Engineering 2007; 69(7): 1460-79.

[10] Ma H, Kamiya N. Distance transformation for the numerical evaluation of near singular boundary integrals with various kernels in boundary element method. Engineering Analysis with Boundary Elements 2002; 26(4): 329-39.

[11] Ma H, Kamiya N. A general algorithm for the numerical evaluation of nearly singular boundary integrals of various orders for two- and three-dimensional elasticity. Computational Mechanics 2002; 29(4-5): 277-88.

www.witpress.com, ISSN 1743-355X (on-line)

[12] Qin X, Zhang J, Xie G, Zhou F, Li G. A general algorithm for the numerical evaluation of nearly singular integrals on 3D boundary element. Journal of Computational and Applied Mathematics 2011; 235(14): 4174-86.
[13] Gu Y, Chen W, Zhang C. The sinh transformation for evaluating nearly singular boundary element integrals over high-order geometry elements. Engineering Analysis with Boundary Elements 2013; 37(2): 301-08.
[14] Xie G, Zhang J, Qin X, Li G. New variable transformations for evaluating nearly singular integrals in 2D boundary element method. Engineering Analysis with Boundary Elements 2011; 35(6): 811-17.
[15] Xie G, Zhou F, Zhang J, Zheng X, Huang C. New variable transformations for evaluating nearly singular integrals in 3D boundary element method. Engineering Analysis with Boundary Elements 2013; 37(9): 1169-78.
[16] Kita E, Higuchi K, Kamiya N. r- and hr-adaptive boundary element method for two-dimensional potential problem. Computers & Structures 2000; 74(1): 11-19.
[17] Zhang X, An X. Exact integration and its application in adaptive boundary element analysis of two-dimensional potential problems. Communications in Numerical Methods in Engineering 2008; 24(11): 1239-50.
[18] Maischak M, Stephan EP. Adaptive hp-versions of boundary element methods for elastic contact problems. Computational Mechanics 2007; 39(5): 597-607.
[19] Sawaki Y, Kako A, Kitaoka D, Kamiya N. Plate bending analysis with hr-adaptive boundary elements. Engineering Analysis with Boundary Elements 2001; 25(8): 621-31.
[20] Miranda-Valenzuela JC, Muci-Kuchler KH, Soriano-Soriano S. Adaptive meshing for two-dimensional thermoelastic problems using Hermite boundary elements. Advances in Engineering Software 2001; 32(3): 171-88.
[21] Gao XW, Davies TG. Adaptive integration in elasto-plastic boundary element analysis. Journal of the Chinese Institute of Engineers 2000; 23(3): 349-56.
[22] Astrinidis E, Fenner RT, Tsamasphyros G. Elastoplastic analysis with adaptive boundary element method. Computational Mechanics 2004; 33(3): 186-93.
[23] Kita E, Kamiya N. Error estimation and adaptive mesh refinement in boundary element method, an overview. Engineering Analysis with Boundary Elements 2001; 25(7): 479-95.
[24] Chen JT, Chen KH, Chen CT. Adaptive boundary element method of time-harmonic exterior acoustics in two dimensions. Computer Methods in Applied Mechanics and Engineering 2002; 191(31): 3331-45.
[25] Lachat JC, Watson JO. Effective numerical treatment of boundary integral equations – formulation for 3-dimensional elastostatics. International Journal for Numerical Methods in Engineering 1976; 10(5): 991-1005.
[26] Mustoe GGW. Advanced integration schemes over boundary elements and volume cells for two- and three-dimensional non-linear analysis. In:

Banerjee PK, Mukherjee S, editor. Developments in boundary element methods. London: Elsevier; 1984. p. 213-70.
[27] Bu S, Davies TG. Effective evaluation of nonsingular integrals in 3D BEM. Advances in Engineering Software 1995; 23(2): 121-28.
[28] Gao XW, Davies TG. Boundary element programming in mechanics. Cambridge: Cambridge University Press; 2002.