Numerical evaluation of high-order singular boundary integrals using third-degree B-spline basis functions

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Abstract

A novel method is presented for numerical evaluation of high-order singular boundary integrals that exist in the Cauchy principal value sense in two- and three-dimensional problems. In this method, three-dimensional boundary integrals are transformed into a line integral over the contour of the surface and a radial integral which contains singularities by using the radial integration method. The analytical elimination of singularities condensed in the radial integral formulas can be achieved by expressing the non-singular parts of the integration kernels as a series of cubic B-spline basis functions in the local distance ρ of the intrinsic coordinate system and using the intrinsic features of the radial integral. Some examples are provided to verify the correctness and robustness of the presented method.
Keywords: singular integrals, boundary element method, radial integration method, Cauchy principal value, B-spline basis function.

1 Introduction

Once the boundary element method (BEM) [1, 2] is used to solve potential and mechanical problems, various orders of singular integrals appear in the basic boundary integral equations and their gradient computation equations [2]. The accurate evaluation of general high-order singular boundary integrals is very important and has obtained extensive research in recent years [3–8].

Recently, for evaluating arbitrarily high-order 2D and 3D singular boundary integrals in a unified way, Gao [9, 10] proposed an efficient approach by

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expressing the non-singular part of a singular integrand and the global distance r as power series in the local distance ρ of the intrinsic coordinate system and then eliminating the singularities analytically.

In this paper, the global distance r is expressed as power series in ρ, and the non-singular part of a singular integrand is expressed as a series of third-degree B-spline basis functions [11] in ρ. One advantage of this method is that the coefficients calculation becomes much simpler because of the local support and the endpoint interpolatory properties of B-spline basis functions with open knot vectors [12] compared to coefficients calculation in reference [10]. For three-dimensional boundary integrals, the radial integration method (RIM) [13, 14] is employed and the singularities condensed in the radial integral are removed analytically in the intrinsic coordinate system. A number of numerical examples will be given to verify the correctness and robustness of the presented method.

2 B-spline basis functions

Let $U=\{u_0,\ldots,u_m\}$ be a nondecreasing sequence of real numbers, i.e., $u_i \le u_{i+1}, i=0,1,\ldots,m-1$. u_i are called knots, and U is the knot vector. The ith B-spline basis function of p-degree, denoted by $N_{i,p}(u)$, is defined as [11, 15]:

$$N_{i,0}=\begin{cases}1 & \text{if} \quad u_i \le u < u_{i+1} \\ 0 & \text{otherwise}\end{cases}, \tag{1a}$$

$$N_{i,p}(u)=\frac{u-u_i}{u_{i+p}-u_i}N_{i,p-1}(u)+\frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}}N_{i+1,p-1}(u), \tag{1b}$$

where i=0,1,…,n; m=n+p+1. If knots are equally-spaced in the parametric space, they are said to be uniform [11, 12]. A knot vector is said to be open if its first and last knots appear $p+1$ times. An initial example of the results of applying (1a) and (1b) to a uniform knot vector $U=\{0,1,2,3,4,5,6\}$ is presented in figure 1.

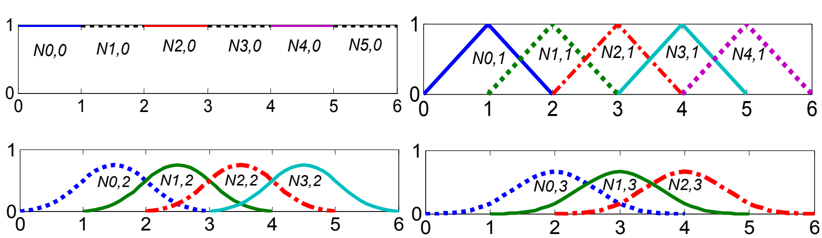

Figure 1: B-spine basis functions of order 0,1,2,3 for uniform knot vector $U=\{0,1,2,3,4,5,6\}$.

It should be noted that the support of each $N_{i,p}$ is compact and contained in the interval $[u_i, u_{i+p+1}]$. For an open, non-uniform knot vector, the basis functions are interpolatory at the ends of the interval.

3 Discretization of boundary integrals

Consider the following boundary integral:

$$I(\boldsymbol{x}^p) = \int_\Gamma \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})} d\Gamma(\boldsymbol{x}), \tag{2}$$

In this paper, the singular integrals are evaluated in the Cauchy principal value sense, i.e.,

$$I(\boldsymbol{x}^p) = \int_\Gamma \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})} d\Gamma(\boldsymbol{x}) = \lim_{\Gamma_\varepsilon \to 0} \int_{\Gamma - \Gamma_\varepsilon} \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})} d\Gamma(\boldsymbol{x}). \tag{3}$$

After discretizing the boundary Γ into N_{elem} elements, the global coordinate vector $\boldsymbol{x}$ within each element is interpolated through the coordinates of the element nodes, i.e.,

$$\boldsymbol{x}(\xi) = \sum_{\alpha=1}^{N_{node}} N_\alpha(\xi) \boldsymbol{x}^\alpha, \tag{4}$$

and eqn. (2) can be written as

$$I(\boldsymbol{x}^p) = \sum_{e=1}^{N_{elem}} I^e(\boldsymbol{x}^p), \tag{5a}$$

where

$$I^e(\boldsymbol{x}^p) = \int_{\Gamma_e} \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})} d\Gamma(\boldsymbol{x}) = \int_{\Gamma_\xi} \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})} |J_e| d\Gamma(\xi) \tag{5b}$$

in which Γ_e is the global integration region of element e, Γ_ξ is a straight line from –1 to +1 for 2D and a $[-1,1] \times [-1,1]$ square for 3D problems, and $|J_e|$ is the transformation Jacobian from the global coordinates to the intrinsic coordinates.

For regular integrals, eqn. (5b) can be evaluated using Gaussian quadrature for each element [10]. For singular integrals, a particular integration technique should be used.

4 Evaluation of 2D singular boundary integrals

Consider integration over a line element. Eqn. (5b) becomes

$$I^e(\boldsymbol{x}^p) = I_1^e(\boldsymbol{x}^p) + I_2^e(\boldsymbol{x}^p), \tag{6}$$

where

$$I_s^e(\boldsymbol{x}^p) = \int_0^{\rho(\xi_p, \xi_s)} \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})} |J_e| d\rho, \tag{7}$$

in which s=1 and 2, $\xi_1 = -1$, $\xi_2 = 1$ and $\rho(\xi_p, \xi_1)$ and $\rho(\xi_p, \xi_2)$ are the local distances from the source point to node 1 and node 2, respectively.

Considering a limiting process, eqn. (7) yields

$$I_s^e(\boldsymbol{x}^p) = \lim_{\rho_s(\varepsilon)\to 0} \int_{\rho_s(\varepsilon)}^{\rho(\xi_p,\xi_s)} \frac{\bar{f}(\boldsymbol{x}^p,\boldsymbol{x})|J_e|}{r^\beta(\boldsymbol{x}^p,\boldsymbol{x})} d\rho, \tag{8}$$

where $\rho_s(\varepsilon)$ is an infinitesimal local distance determined by an infinitesimal global distance ε for the integration region formed by the source point and element nodes.

4.1 Express r as power series

In order to evaluate the singular integrals in eqn. (8), the global distance r needs to be expressed as a power series in the local distance ρ. To do this, the following expansion is proposed:

$$r = \rho\bar{\rho} = \rho\sqrt{\sum_{n=0}^{M} G_n \rho^n}, \tag{9}$$

where M is the order of the power series, and G_n are coefficients. These quantities can be exactly determined by comparing equal order terms of ρ on the left- and right-hand sides of the following equation

$$\sum_{i=1}^{D}\left\{\sum_{\alpha=1}^{N_{node}} N_\alpha(\xi(\rho))x_i^\alpha - x_i^p\right\}^2 = \rho^2 \sum_{n=0}^{M} G_n \rho^n, \tag{10}$$

where

$$\xi(\rho) = \xi_p + \xi_s \rho. \tag{11}$$

4.2 Express non-singular part of the integrand as a series of third-degree B-spline basis functions

Substituting eqn. (9) into eqn. (8) results in

$$I_s^e(\boldsymbol{x}^p) = \lim_{\rho_s(\varepsilon)\to 0} \int_{\rho_s(\varepsilon)}^{\rho(\xi_p,\xi_s)} \frac{\bar{f}(\boldsymbol{x}^p,\boldsymbol{x})|J_e|}{\bar{\rho}^\beta(\rho)\rho^\beta} d\rho. \tag{12}$$

In order to evaluate the singular integral in the above equation, the non-singular part of the integrand is expressed as a series of third-degree B-spline basis functions in the local distance ρ, i.e.,

$$\frac{\bar{f}(\boldsymbol{x}^p,\boldsymbol{x})|J_e|}{\bar{\rho}^\beta(\rho)} = \sum_{n=0}^{L} B_n N_{n,3}(\rho), \tag{13}$$

where $N_{n,3}(\rho)$ are third-degree B-spline basis functions and B_n are coefficients. Once knot vector $U = \{\rho_0, \rho_1, ..., \rho_{L+3+1}\}$ is defined, the third-degree B-spline basis functions $N_{n,3}(\rho)$ can be determined. In this paper, the following seven types of knot vectors are used, corresponding to L=3,4,…,9 , respectively.

$$U_1 = \{0,0,0,0,\hat{\rho},\hat{\rho},\hat{\rho},\hat{\rho}\}, \tag{14a}$$

$$U_2 = \{0,0,0,0,\hat{\rho}/2,\hat{\rho},\hat{\rho},\hat{\rho},\hat{\rho}\}\ , \tag{14b}$$

$$U_3 = \{0,0,0,0,\hat{\rho}/3,2*\hat{\rho}/3,\hat{\rho},\hat{\rho},\hat{\rho},\hat{\rho}\}\ , \tag{14c}$$

$$U_4 = \{0,0,0,0,\hat{\rho}/4,\hat{\rho}/2,3*\hat{\rho}/4,\hat{\rho},\hat{\rho},\hat{\rho},\hat{\rho}\}\ , \tag{14d}$$

$$U_5 = \{0,0,0,0,\hat{\rho}/5,2*\hat{\rho}/5,3*\hat{\rho}/5,4*\hat{\rho}/5,\hat{\rho},\hat{\rho},\hat{\rho},\hat{\rho}\}\ , \tag{14e}$$

$$U_6 = \{0,0,0,0,\hat{\rho}/6,\hat{\rho}/3,\hat{\rho}/2,2*\hat{\rho}/3,5*\hat{\rho}/6,\hat{\rho},\hat{\rho},\hat{\rho},\hat{\rho}\}\ , \tag{14f}$$

$$U_7 = \{0,0,0,0,\hat{\rho}/7,2*\hat{\rho}/7,3*\hat{\rho}/7,4*\hat{\rho}/7,5*\hat{\rho}/7,6*\hat{\rho}/7,\hat{\rho},\hat{\rho},\hat{\rho},\hat{\rho}\}, \tag{14g}$$

where $\hat{\rho} = \rho(\xi_p, \xi_s)$. It should be noted that the open knot vectors listed in eqns. (14a)–(14g) are C^2-continuous [11, 12] except at the endpoints of the interval and that makes the calculation more accurate. Besides, basis functions formed from open knot vectors are interpolatory at the ends of the parametric space interval $[0, \rho(\xi_p, \xi_s)]$, i.e.,

$$B_0 = \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x}^p)|J_e|}{(C_0)^\beta},\ B_L = \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x}^s)|J_e|}{\bar{\rho}^\beta(\rho(\xi_p, \xi_s))}. \tag{15}$$

The other coefficients B_n are determined by collocating L+1 equally spaced points ($0, \rho'_1, \cdots, \rho'_L$) over the integration region $[0, \rho(\xi_p, \xi_s)]$, in which $\rho'_L = \rho(\xi_p, \xi_s)$. These coefficients can be solved using

$$[R(L-1)]\{B\} = \{Y\}, \tag{16}$$

where $[R(L-1)]$ is a square matrix of order L-1, i.e.,

$$[R(L-1)] = \begin{bmatrix} N_{1,3}(\rho'_1), & N_{2,3}(\rho'_1), & \cdots, & N_{L-1,3}(\rho'_1) \\ N_{1,3}(\rho'_2), & N_{2,3}(\rho'_2), & \cdots, & N_{L-1,3}(\rho'_2) \\ \vdots & \vdots & & \vdots \\ N_{1,3}(\rho'_{L-1}), & N_{2,3}(\rho'_{L-1}), & \cdots, & N_{L-1,3}(\rho'_{L-1}) \end{bmatrix} \tag{17}$$

and $\{B\}$ and $\{Y\}$ are vectors as follows

$$\{B\} = \begin{Bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{L-1} \end{Bmatrix}, \{Y\} = \begin{Bmatrix} \bar{f}(\boldsymbol{x}^p, \boldsymbol{x}^1)|J_e|/\bar{\rho}^\beta(\rho'_1) - B_0 N_{0,3}(\rho'_1) - B_L N_{L,3}(\rho'_1) \\ \bar{f}(\boldsymbol{x}^p, \boldsymbol{x}^2)|J_e|/\bar{\rho}^\beta(\rho'_2) - B_0 N_{0,3}(\rho'_2) - B_L N_{L,3}(\rho'_2) \\ \vdots \\ \bar{f}(\boldsymbol{x}^p, \boldsymbol{x}^{L-1})|J_e|/\bar{\rho}^\beta(\rho'_{L-1}) - B_0 N_{0,3}(\rho'_{L-1}) - B_L N_{L,3}(\rho'_{L-1}) \end{Bmatrix} \tag{18}$$

Because the local support property [11, 12] of B-spline basis functions, the coefficient matrix $[R(L-1)]$ is banded and eqn. (16) is easy to be solved.

4.3 Evaluation of singular boundary integrals

Once the coefficients B_n are obtained from above equations, substituting eqn. (13) into eqn. (12) yields

$$I_s^e(\boldsymbol{x}^p) = \sum_{n=0}^{L} B_n \lim_{\rho_s(\varepsilon)\to 0} \int_{\rho_s(\varepsilon)}^{\rho(\xi_p,\xi_s)} N_{n,3}(\rho)/\rho^\beta d\rho = \sum_{n=0}^{L} B_n Q_n \ , \tag{19}$$

where

$$Q_n = \lim_{\rho_s(\varepsilon)\to 0} \int_{\rho_s(\varepsilon)}^{\rho(\varepsilon_p,\varepsilon_s)} N_{n,3}(\rho)/\rho^\beta d\rho . \tag{20}$$

The formulation of third-degree B-spline basis functions $N_{n,3}(\rho)$ can be deduced from eqns. (1a) and (1b) as

$$N_{n,3}(\rho) = \begin{cases} \dfrac{(\rho-\rho_n)^3}{(\rho_{n+3}-\rho_n)(\rho_{n+2}-\rho_n)(\rho_{n+1}-\rho_n)} \overset{\Delta}{=} M_{n1}, \quad \rho \in [\rho_n, \rho_{n+1}) \\ \dfrac{(\rho-\rho_n)^2(\rho_{n+2}-\rho)}{(\rho_{n+3}-\rho_n)(\rho_{n+2}-\rho_n)(\rho_{n+2}-\rho_{n+1})} + \\ \dfrac{(\rho-\rho_n)(\rho_{n+3}-\rho)(\rho-\rho_{n+1})}{(\rho_{n+3}-\rho_n)(\rho_{n+2}-\rho_{n+1})(\rho_{n+3}-\rho_{n+1})} + \\ \dfrac{(\rho_{n+4}-\rho)(\rho-\rho_{n+1})^2}{(\rho_{n+4}-\rho_{n+1})(\rho_{n+3}-\rho_{n+1})(\rho_{n+2}-\rho_{n+1})} \overset{\Delta}{=} M_{n2}, \quad \rho \in [\rho_{n+1}, \rho_{n+2}) \\ \dfrac{(\rho-\rho_n)(\rho_{n+3}-\rho)^2}{(\rho_{n+3}-\rho_n)(\rho_{n+3}-\rho_{n+2})(\rho_{n+3}-\rho_{n+1})} + \\ \dfrac{(\rho_{n+3}-\rho)(\rho_{n+4}-\rho)(\rho-\rho_{n+1})}{(\rho_{n+3}-\rho_{n+2})(\rho_{n+4}-\rho_{n+1})(\rho_{n+3}-\rho_{n+1})} + \\ \dfrac{(\rho_{n+4}-\rho)^2(\rho-\rho_{n+2})}{(\rho_{n+4}-\rho_{n+1})(\rho_{n+3}-\rho_{n+2})(\rho_{n+4}-\rho_{n+2})} \overset{\Delta}{=} M_{n3}, \quad \rho \in [\rho_{n+2}, \rho_{n+3}) \\ \dfrac{(\rho_{n+4}-\rho)^3}{(\rho_{n+4}-\rho_{n+1})(\rho_{n+4}-\rho_{n+2})(\rho_{n+4}-\rho_{n+3})} \overset{\Delta}{=} M_{n4}, \quad \rho \in [\rho_{n+3}, \rho_{n+4}) \end{cases} . \tag{21}$$

Substituting eqn. (21) into eqn. (20) yields

$$\begin{aligned} Q_n = \lim_{\rho_s(\varepsilon)\to 0} (&\int_{\rho_n}^{\rho_{n+1}} M_{n1}/\rho^\beta d\rho + \int_{\rho_{n+1}}^{\rho_{n+2}} M_{n2}/\rho^\beta d\rho + \int_{\rho_{n+2}}^{\rho_{n+3}} M_{n3}/\rho^\beta d\rho + \\ &\int_{\rho_{n+3}}^{\rho_{n+4}} M_{n4}/\rho^\beta d\rho) = \sum_{j=1}^{4} \left(\lim_{\rho_s(\varepsilon)\to 0} \left(\int_{\rho_{n+j-1}}^{\rho_{n+j}} M_{nj}/\rho^\beta d\rho \right) \right) \overset{\Delta}{=} \sum_{j=1}^{4} T_{nj} \end{aligned} . \tag{22}$$

From eqn. (21), it can be seen that M_{nj} is cubic polynomial functions of independent variable ρ. So M_{nj} can be expressed as

$$M_{nj} = \sum_{k=0}^{3} A_{njk}\rho^k \quad , \quad j = 1,2,3,4 \ , \ n = 0,1,\ldots,L \ , \tag{23}$$

where A_{njk} can be determined by polynomial arithmetic of eqn. (21).

Then T_{nj} from eqn. (22) can be expressed as

$$T_{nj} = \sum_{k=0}^{3} A_{njk} \left(\lim_{\rho_s(\varepsilon)\to 0} \int_{\rho_{n+j-1}}^{\rho_{n+j}} \rho^{k-\beta} d\rho \right) = \sum_{k=0}^{3} A_{njk} E_{njk} \ , \tag{24}$$

where

$$E_{njk}=\begin{cases}0, & \text{for } \rho_{n+j-1}=\rho_{n+j};\\ \dfrac{\rho_{n+j}^{k-\beta+1}-\rho_{n+j-1}^{k-\beta+1}}{k-\beta+1}, & \text{for } \rho_{n+j-1}\neq\rho_{n+j} \text{ and } \rho_{n+j-1}\neq\rho_s(\varepsilon);\\ \dfrac{1}{k-\beta+1}\left[\dfrac{1}{\rho_{n+j}^{\beta-k-1}}-\lim\limits_{\rho_s(\varepsilon)\to 0}\dfrac{1}{\rho_s^{\beta-k-1}(\varepsilon)}\right], & \text{for } \rho_{n+j-1}\neq\rho_{n+j} \text{ and}\\ & \rho_{n+j-1}=\rho_s(\varepsilon) \text{ and } k\neq\beta-1;\\ \ln\rho_{n+j}-\lim\limits_{\rho_s(\varepsilon)\to 0}\ln\rho_s(\varepsilon), & \text{for } \rho_{n+j-1}\neq\rho_{n+j}, \rho_{n+j-1}=\rho_s(\varepsilon) \text{ and } k=\beta-1.\end{cases} \quad (25)$$

The limiting terms in eqn. (25) can be expressed as summations of finite parts and infinite terms [10], i.e.,

$$\lim_{\rho_s(\varepsilon)\to 0}\ln\rho_s(\varepsilon)=\ln H_0+\mathit{Infinite\ terms}, \quad (26a)$$

$$\lim_{\rho_s(\varepsilon)\to 0}\frac{1}{\rho_s^{\beta-k-1}(\varepsilon)}=H_{\beta-k-1}+\mathit{Infinite\ terms} \quad (0\le k\le\beta-2). \quad (26b)$$

The results and detailed derivation of coefficients H_0 and $H_{\beta-k-1}$ can be seen in [10]. For a physical problem, the infinite terms in eqn. (26) should be cancelled out by free terms [8]. Thus, substituting eqn. (26) into eqn. (25), it follows that

$$E_{njk}=\begin{cases}0, & \text{for } \rho_{n+j-1}=\rho_{n+j}\\ \dfrac{\rho_{n+j}^{k-\beta+1}-\rho_{n+j-1}^{k-\beta+1}}{k-\beta+1}, & \text{for } \rho_{n+j-1}\neq\rho_{n+j} \text{ and } \rho_{n+j-1}\neq\rho_s(\varepsilon)\\ \dfrac{1}{k-\beta+1}\left[\dfrac{1}{\rho_{n+j}^{\beta-k-1}}-H_{\beta-k-1}\right], & \text{for } \rho_{n+j-1}\neq\rho_{n+j}, \rho_{n+j-1}=\rho_s(\varepsilon)\\ & \text{and } 0\le k\le\beta-2\\ \ln\rho_{n+j}-\ln H_0, & \text{for } \rho_{n+j-1}\neq\rho_{n+j},\ \rho_{n+j-1}=\rho_s(\varepsilon) \text{ and } k=\beta-1\\ \dfrac{\rho_{n+j}^{k-\beta+1}}{k-\beta+1}, & \text{for } \rho_{n+j-1}\neq\rho_{n+j},\ \rho_{n+j-1}=\rho_s(\varepsilon) \text{ and } k>\beta-1\end{cases} \quad (27)$$

Substituting eqn. (27) into eqn. (24), the result into eqn. (22), and finally applying eqn. (19) yields

$$I_s^e(\boldsymbol{x}^p)=\sum_{n=0}^{L}B_n\sum_{j=1}^{4}\sum_{k=0}^{3}A_{njk}E_{njk}. \quad (28)$$

Equation (28) is the regularized expression for evaluating 2D singular boundary integrals specified by eqns. (6) and (7).

5 Evaluation of 3D singular boundary integrals

5.1 Radial integration formulas

For 3D boundary integrals, eqn. (5b) can be written for the e-th quadrilateral element as

$$I^e(\boldsymbol{x}^p)=\int_{-1}^{1}\int_{-1}^{1}\frac{\bar{f}(\boldsymbol{x}^p,\boldsymbol{x})}{r^{\beta}(\boldsymbol{x}^p,\boldsymbol{x})}\left|J_e\right|d\xi d\eta. \quad (29)$$

Using the Radial Integration Method by Gao [13, 14], the integral in eqn. (29) can be transformed into a contour integral along the four sides of the element, i.e.,

$$I^e(\boldsymbol{x}^p) = \int_L \frac{1}{\rho(p,q)} \frac{\partial \rho(p,q)}{\partial n'} F\ (\boldsymbol{x}^p, \boldsymbol{x}^q) dL(q), \tag{30}$$

where

$$F\ (\boldsymbol{x}^p, \boldsymbol{x}^q) = \lim_{\rho_\alpha(\varepsilon)\to 0} \int_{\rho_\alpha(\varepsilon)}^{\rho(p,q)} \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})}{r^\beta(\boldsymbol{x}^p, \boldsymbol{x})} |J_e| \rho d\rho, \tag{31}$$

in which q represents the field point which takes on a value from the boundary of the squares; $n' = (n'_\xi, n'_\eta)$ is the outward normal to the boundary of the integration square in the intrinsic coordinate system (ξ, η); and ρ is the local distance defined as

$$\rho = \sqrt{(\xi - \xi_p)^2 + (\eta - \eta_p)^2}\ . \tag{32}$$

5.2 Express r as power series

To evaluate the radial integral (31), as in the previous section, we need to express the global distance r as a power series in the local distance ρ. Equation (9) is still available. For quadrilateral element, eqn. (9) can be written as

$$r^2 = \sum_{i=1}^{D} \left\{ \sum_{\alpha=1}^{N_{node}} N_\alpha \big(\xi(\rho), \eta(\rho)\big) x_i^\alpha - x_i{}^p \right\}^2 = \rho^2 \sum_{n=0}^{M} G_n \rho^n\ . \tag{33}$$

Substituting the following relationships

$$\xi = \xi_p + \rho\rho_{,\xi}; \quad \eta = \eta_p + \rho\rho_{,\eta} \tag{34}$$

into eqn. (33) and comparing equal order terms of ρ on the left- and right-hand sides can obtain the results in principle and the value of M is 2 for 4-noded, 4 for 8-noded and 6 for 9-noded shape functions, respectively.

Once the coefficients G_n are obtained, substituting eqn. (9) into eqn. (31) yields

$$F(\boldsymbol{x}^p, \boldsymbol{x}^q) = \lim_{\rho_\alpha(\varepsilon)\to 0} \int_{\rho_\alpha(\varepsilon)}^{\rho(p,q)} \frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})|J_e|}{\bar{\rho}^\beta(\rho)\rho^{\beta-1}} d\rho\ . \tag{35}$$

5.3 Express non-singular part of the integrand as a series of third-degree B-spline basis functions

As in Section 4.2, the singular integral in the above equation is evaluated by expressing the non-singular part of the integrand as a series of third-degree B-spline basis functions in ρ, i.e.,

$$\frac{\bar{f}(\boldsymbol{x}^p, \boldsymbol{x})|J_e|}{\bar{\rho}^\beta(\rho)} = \sum_{n=0}^{L} B_n N_{n,3}(\rho)\ . \tag{36}$$

Eqns. (14)–(18) are applied to determine the coefficients B_n. Once coefficients B_n are obtained, substituting eqn. (36) into eqn. (35) yields

$$F(x^p, x^q) = \sum_{n=0}^{L} B_n \lim_{\rho_s(\varepsilon)\to 0} \int_{\rho_s(\varepsilon)}^{\rho(\xi_p,\xi_s)} N_{n,3}(\rho)/\rho^{\beta-1} d\rho = \sum_{n=0}^{L} B_n Q_n \,, \qquad (37)$$

where

$$Q_n = \lim_{\rho_s(\varepsilon)\to 0} \int_{\rho_s(\varepsilon)}^{\rho(\varepsilon_p,\varepsilon_s)} N_{n,3}(\rho)/\rho^{\beta-1} d\rho. \qquad (38)$$

5.4 Evaluation of singular boundary integrals

Following the procedure in section 4.3 (as Eqns. (21)–(25) do) yields the following equation

$$Q_n = \sum_{j=1}^{4} \sum_{k=0}^{3} A_{njk} E_{njk} \,, \qquad (39)$$

where A_{njk} is defined as eqn. (23) and determined by polynomial arithmetic of eqn. (21) . The regularized E_{njk} has the following form

$$E_{njk} = \begin{cases} 0, & \text{for } \rho_{n+j-1} = \rho_{n+j} \\ \dfrac{\rho_{n+j}^{k-\beta+2} - \rho_{n+j-1}^{k-\beta+2}}{k-\beta+2}, & \text{for } \rho_{n+j-1} \neq \rho_{n+j} \text{ and } \rho_{n+j-1} \neq \rho_s(\varepsilon) \\ \dfrac{1}{k-\beta+2}\left[\dfrac{1}{\rho_{n+j}^{\beta-k-2}} - H_{\beta-k-2}\right], & \text{for } \rho_{n+j-1} \neq \rho_{n+j}, \\ & \rho_{n+j-1} = \rho_s(\varepsilon) \text{ and } 0 \le k \le \beta-3 \\ \ln \rho_{n+j} - \ln H_0, & \text{for } \rho_{n+j-1} \neq \rho_{n+j},\ \rho_{n+j-1} = \rho_s(\varepsilon) \text{ and } k = \beta-2 \\ \dfrac{\rho_{n+j}^{k-\beta+2}}{k-\beta+2}, & \text{for } \rho_{n+j-1} \neq \rho_{n+j},\ \rho_{n+j-1} = \rho_s(\varepsilon) \text{ and } k > \beta-2 \end{cases}, \qquad (40)$$

where coefficients H_n are determined using Eqns. (26a) and (26b).

Substituting eqn. (40) into eqn. (39), the result into eqn. (37) yields

$$F(x^p, x^q) = \sum_{n=0}^{L} \sum_{j=1}^{4} \sum_{k=0}^{3} B_n A_{njk} E_{njk} \,, \qquad (41)$$

which can be used to tackle 3D singular boundary integrals.

6 Numerical examples

Example 1: (2D Singular Boundary Integrals $I_i(\boldsymbol{x}^p) = \int_\Gamma r_{,i}/r^\beta d\Gamma$). In the integrand, $r_{,i} = \partial r/\partial x_i$. The integration is over a straight line with a length of 3 and a tilt angle of 30º (figure 2). The source point considered is located at 1/3 along the length of the line. For a straight line, $r_{,i}$ is a constant.

Tables 1 and 2 show the analytical and computed results where the two sets of results are almost the same. The current computed results for L=3,4,…,9 , which

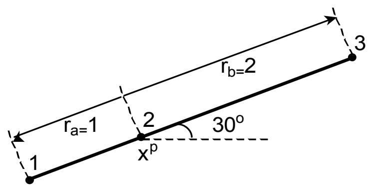

Figure 2: Integration boundary.

are corresponding to knot vectors defined as eqns. (14a)–(14g) are the same. That's because the non-singular part of the integrand is a constant.

Table 1: Computational results for $I_1(\boldsymbol{x}^p)$.

	β=2	β=3	β=4	β=5
Analytical	0.4330128	0.3247596	0.2525908	0.2029748
Current	0.4330127	0.3247595	0.2525907	0.2029747

Table 2: Computational results for $I_2(\boldsymbol{x}^p)$.

	β=2	β=3	β=4	β=5
Analytical	0.2499998	0.187499866	0.1458332	0.1171874
Current	0.2500000	0.187500000	0.1458333	0.1171875

Example 2: (Highly Singular Integral over a Curved 3D Boundary Element). The curved element is a quarter frustum of a cone panel (up radius=1, bottom radius=1.5, height=1) as shown in Figure 3. $I\ \ (\boldsymbol{x}^p\ ) = \int_\Gamma \frac{-1}{r^\beta}\left[3r_{,3}\frac{\partial r}{\partial n} - n_3\right]d\Gamma$.

Three source points are computed corresponding to intrinsic coordinates ξ=0.5 and η=0.5, 0.75, and 0.99, respectively.

Table 3 lists the computed results for singularity orders of β from 1 to 6 for these points using L=9. While E in Table 3 denotes the computed results using L=6 and B denotes the computed results in the case that the integration interval $[\rho_\alpha(\varepsilon), \rho(p,q)]$ in eqn. (35) is equally divided into 20 parts and the non-singular part of the integrand in each sub-interval is expressing as a series of three-order B-spline basis functions in ρ using L=9. The problem is also analyzed by the method in [10] for comparison. From Table 3 it can be seen that, good results can be obtained using L=9 for β =1, 2 and 3, while for β =4, 5 and 6, they are moderately close. However, for β =4 or 5, the results can be improved by expressing the non-singular part of the integrand as multi-segment series of three-order B-spline basis functions (results denoted by B). While for β =6, the results can be improved by using smaller L (results using L=6 are denoted by E). Good results can still be achieved even if the source point is very close to the boundary (the case of η=0.99). This is attributed to the use of the subdivision technique [2, 16, 17] in the evaluation of the contour integral shown in eqn. (30).

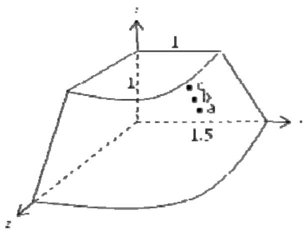

Figure 3: Curved 3D Boundary Element.

Table 3: Computed results for various values of β at three source points.

η	0.5 (Point a)		0.75 (Point b)	
β	Ref. [10]	Current	Ref. [10]	Current
1	0.112867	0.112869	0.105666	0.105667
2	-0.094925	-0.094955	-0.1362621	-0.1362744
3	-0.342384	-0.343038	-0.5232094	-0.5237431
4	-0.362838	-0.355644 -0.357311,B	-1.160997	-1.155234 -1.156895,B
5	-0.586423	-0.618487 -0.601696,B	-4.423292	-4.455555 -4.43633,B
6	-1.464013	-1.427682 -1.459975,E	-21.04375	-21.01116 -21.04945,E

η	0.99 (Point c)	
β	Ref. [10]	Current
1	0.0909436	0.0909439
2	-0.392686	-0.392679
3	-9.887196	-9.887706
4	-687.6359	-687.7020
5	-69731.26	-69725.60
6	-8.277E+06	-8.279E+06

7 Concluding discussions

A novel method has been presented for evaluating high order of 2D and 3D singular boundary integrals based on the uses of power series and B-spline basis functions expansion technique and the RIM. The distinct feature of this method is that various orders of singular integrals are treated in a simple and unified way. And the coefficients calculation becomes much simpler because of the local support and the endpoint interpolatory properties of B-spline basis functions with open knot vectors compared to the method in [10]. The provided numerical examples demonstrate that the presented method is accurate and robust.

It is emphasized that this work only deals with singular integrals that exist in the Cauchy principal value sense and the derived H_n can treat singularity orders of β up to 5 for 2D and 6 for 3D boundary integrals. This is sufficiently high for general physical problems. However, if needed, coefficients for higher singularity values can be derived following the procedure described in [10].

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