Transactions on Modelling and Simulation vol 10, © 1995 WIT Press, www.witpress.com, ISSN 1743-355X

New developments in the dual reciprocity method

P.W. Partridge
Departamento de Eng. Civil, Universidade de Brasília, Campus Asa Norte, 70.910-900 Brasília DF, Brazil

Abstract

In this paper recent developments in the Dual Reciprocity Boundary Element Method are briefly reviewed with particular attention to the question of which approximation function should be employed. The accuracy of each type of function is considered for boundary fluxes for two field problems, one linear, one nonlinear.

1 Introduction

The Dual Reciprocity Method is a powerful technique for taking domain integrals to the boundary in BEM analysis. The method was originally proposed by Nardini & Brebbia in 1982, [1] for elastodynamics and was soon extended to a wide range of engineering problems. A book about the method was published in 1992 [3] which collected applications up to that point.

The publication of the book provoked a discussion about aspects of the method itself including suggestions for improvement, for example: Adey & Sclhar suggested the use of an adaptive technique for determining the number of internal nodes to be employed, [4]. An important contribution was made by Yamada et al in identifying the r functions as being the radial basis functions described in the mathematical literature, [5] and demonstrating the convergence properties of the same, [6]. Zhu & Zhang have suggested a transform method for dealing with convection type problems, [7].

One of the most important aspects of the discussion about the method has concerned the approximation functions.

In [3] several different f expansions are used, however emphasis is given to $1+r$. $f = 1 + r^2 + r^3$ is used for domain integrals involving second derivatives. The use of $f = 1 + r^3$ for non second derivative terms began as a consequence of the work on radial basis functions [6] of which this function was found to have better convergence properties. The use of this function is also proposed independently in [8] in view of the fact that it avoids what the authors call a singularity in the calculation of first derivative terms. This question will be discussed in more detail in the next section.

Global functions have been proposed in the many papers by Nardini & Brebbia, however these functions have not been popular in view of difficulties in inverting the resulting **F** matrix. This problem has been recently overcome with the use of a Singular Value Decomposition algorithm, thus putting this type of function, which includes elements of the Pascal triangle, trigonometric expansions and others, on the same footing as the radial basis functions, [9,10].

The use of different expansions for different terms is done in [11], but has received little attention, however given the present availability of many different choices for f this possibility will be reconsidered.

The remainder of the paper will be dedicated to comparing the accuracy of the alternative f expansions available in order to attempt to determine the respective merits of each.

2 The Dual Reciprocity Method

The Dual Reciprocity Method is well known and is described in detail in [3]. Only a brief outline will be given here. The method involves splitting a given equation into two parts, a left hand side for which a simple fundamental solution is known, and a right hand side consisting of all other terms.

As an example, the following Poisson-type equation will be considered:-

$$\nabla^2 u = b \tag{1}$$

where b may be a function of space, of the unknown u, including any derivatives of the same, may be time dependent and/or nonlinear, and may consist of one or more expressions, including product terms, [2,3].

Putting

$$\nabla^2 \hat{u}_j = f_j \tag{2}$$

where

$$\Sigma f_j \alpha_j = b_i, \tag{3}$$

after carrying out the usual procedures, [3] the following matrix equation will be obtained:-

$$\mathbf{Hu} - \mathbf{Gq} = (\mathbf{H\hat{U}} - \mathbf{G\hat{Q}})\mathbf{F}^{-1}\mathbf{b} \tag{4}$$

The $\hat{u}_j$ are called particular solutions and are collected together in $\hat{\mathbf{U}}$. The columns of the matrix $\hat{\mathbf{Q}}$ consist of $\hat{q}_j = \partial \hat{u}_j / \partial n$ where n is the outward normal to the boundary, Γ. The matrix $\mathbf{F}$ is defined from (3). $\mathbf{H}$ and $\mathbf{G}$ have the usual meaning. Eqn (4) is written for the cases where b in (1) is a function of the problem unknowns, [3]. The functions f_j are called the approximating functions.

The vector $\mathbf{b}$ in (4) consists of nodal values of the rhs of (1) and is interpreted for each case considered as described in [3]. For the two cases considered here $\mathbf{b}$ is given below:-

In the case of the convective problem, (Problem 1),

$$\nabla^2 u = -\frac{\partial u}{\partial x} : \text{such that } \mathbf{b} = -\frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}\mathbf{u} \tag{5}$$

In the case of Burgers equation, (Problem 2)

$$\nabla^2 u = -u\frac{\partial u}{\partial x} : \text{such that } \mathbf{b} = -\mathbf{U}\frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}\mathbf{u} \tag{6}$$

In the above $\partial \mathbf{F}/\partial x$ is a matrix of derivatives of the f functions, $\mathbf{u}$ is a vector of nodal values of u. $\mathbf{U}$ is a diagonal matrix containing known values of u from a previous iteration as the second case is nonlinear.

Transactions on Modelling and Simulation vol 10, © 1995 WIT Press, www.witpress.com, ISSN 1743-355X

Thus eqn (5) leads to

$$\mathbf{Hu} - \mathbf{Gq} = -(\mathbf{H\hat{U}} - \mathbf{G\hat{Q}})\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}\mathbf{u} \tag{7}$$

which may be easily solved.

Eqn (6) leads to

$$\mathbf{Hu} - \mathbf{Gq} = -(\mathbf{H\hat{U}} - \mathbf{G\hat{Q}})\mathbf{F}^{-1}\mathbf{U}\frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}\mathbf{u} \tag{8}$$

which is solved iteratively.

3 DRM Approximation Functions

The Dual Reciprocity method was never limited to any specific function, [3],: Several possibilities were presented in the many papers by Nardini & Brebbia, but to the knowledge of this author, no subsequent study has ever gone beyond their original suggestions.

Approximating function $f = 1 + r$
This function, which may be called classical, was used by Nardini & Brebbia in the original paper on DRM [1] and in all work until recently.

The advantages of using this function are considerable: It is easy to implement, [3], its convergence has been demonstrated, [6], it has the largest literature and is employed in the largest number of computer codes.

The disadvantages are that it cannot be employed for second derivative terms and its use for first derivatives is contested by Zhang & Zhu [8] who point out that it introduces what they call singularities at collocation points.

Partridge & Brebbia, in their original paper on the use of DRM for space derivatives, [2], in which

$$\frac{\partial \mathbf{u}}{\partial x} = \frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}\mathbf{u} \tag{9}$$

was first proposed, considered that, given that the off-diagonal terms of $\partial \mathbf{F}/\partial x$ are skew-symmetric, the principal diagonal,

$$\frac{\partial r}{\partial x} = \frac{(x_j - x_i)}{r_{ij}} \quad i = j \tag{10}$$

should be zero due to the properties of this type of matrix. Thus this value was not "fixed arbitrarily" as claimed in [8]. In the opinion of this author, the diagonal terms are indeterminate rather than singular thus being able to take any value including zero. It should be noted that the use of this number permitted excellent results to be obtained, [2].

$f = 1 + r^3$ **and other approximating functions involving** r^3**.**
At least three independent groups have proposed the use of this type of function:-

Partridge & Brebbia, [2], used an expansion $f = 1 + r + r^2 + r^3$ for a test problem without noting any advantage over the simple $f = 1 + r$ function. In addition, an expansion $f = 1 + r^2 + r^3$ was used in the original work on

DRM for second derivative problems, for which $1 + r$ is unsuitable, [3].

Yamada et al, [6] investigated the convergence of the different radial basis functions described in the mathematical literature and showed that $1 + r^3$ has the best convergence properties. Note that even powers of r are not radial basis functions, in such a way that r^2, etc, *cannot be used on their own.*

Zhang & Zhu, [8] proposed $1+r^2+r^3$ and $1+r^3$ in order to avoid what they call singularities in the $1 + r$ function for first derivative problems as discussed in the last section, and also claimed better results for non-derivative problems, finding $1 + r^2 + r^3$ better than $1 + r^3$.

No author has as yet reported any disadvantages in using the r^3 functions; this author suggests, on the basis of experience, that it may be advantageous to use double precision arithmetic with them, which is rarely necessary for the $1 + r$ function.

Global Approximating Functions

The "global" functions are so called because they interpolate over the entire domain. Yamada et al [6], showed that the radial basis, or r, or "local" functions interpolate only in the neighbourhood of a given collocation point. The use of these functions was suggested in the original work on DRM by Nardini & Brebbia, however they have not been popular due to the fact that the **F** matrix thus produced, eqn (3), tends to be singular or nearly singular. Recently, Cheng et al, [10], have shown that this difficulty can be overcome with the use of a Singular Value Decomposition algorithm, and have reported extremely accurate results with the use of these functions, [9,10]. Results using two different Global type functions, A Pascal Triangle expansion and a sine expansion will be given in the next section. It should be noted however that success using the global functions depends to a large extent on the discretization, [12].

Different Approximating Functions for different terms.

In representing the derivative type terms in DRM as done in eqn (9), u is approximated using an equation similar to (3), [2,3], *ie*

$$u_i = \sum f_j^2 \beta_j \tag{11}$$

Given that α in eqn (3) is not equal to β in eqn (11), it is evident that the two f functions need not be the same. Calling f in eqn (3) f^1 then $f^1 = f^2$ was adopted in [2,3] in order to limit the number of **F** matrices to one for convenience of programing, however different f expansions may be used for derivative type terms if desired. This was done for instance in reference [11]. Before the appearance of [9,10] no advantage could be seen in doing this: the employment of the Global functions for DRM however opens the possibility of using both them and the radial basis functions in the same problem. As an example, taking f in eqn (3) as f^1 and considering the f^2 in eqn (11) to be distinct, the DRM matrix eqn (7) becomes

$$\mathbf{Hu} - \mathbf{Gq} = -(\mathbf{H\hat{U}} - \mathbf{G\hat{Q}})(\mathbf{F}^1)^{-1}\frac{\partial(\mathbf{F}^2)}{\partial x}(\mathbf{F}^2)^{-1}\mathbf{u} \tag{12}$$

Transactions on Modelling and Simulation vol 10, © 1995 WIT Press, www.witpress.com, ISSN 1743-355X

Eqn (8), or any other DRM matrix equation involving derivatives, can be treated similarly. In eqn (12), f^1 may be local and f^2 global or vice-versa, using any combination of local and global functions.

The disadvantage of eqn (12) is that more than one **F** matrix needs to be dealt with in the program: The advantage is the possibility of combining the stability and proven convergence of the r functions with the better accuracy of the global functions and at the same time avoid the serious mesh dependencies of the latter.

4 Comparison of Accuracy of Different f Functions

The DRM approximation functions discussed above will be compared for two cases: Problem 1, eqn (5), and Problem 2, eqn (6), the DRM formulation of which is given in eqns (7) and (8) respectively. In the case of the linear problem 1, a boundary condition $u = \exp^{-x}$ on Γ will be used: In the case of the nonlinear problem 2, the boundary condition $u = 2/x$ on Γ will be employed, thus enabling exact solutions for u on the domain and q on the boundary to be easily obtained.

The geometry used in both cases will be a unit circle, in the case of problem 1 the origin is at the center, in the case of problem 2 the origin is displaced to the point (-2,0) to avoid the singularity in the exact solution at $x = 0$.

The basic discretization, shown in fig. 1, contains 16 boundary nodes, N, and 17 internal nodes, L, 16 of these are at $r = 0.5$ corresponding to the boundary nodes, the remaining node is at the center.

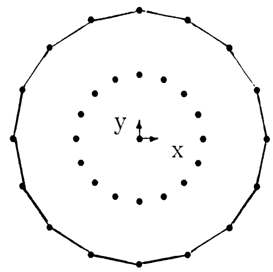

Fig. 1 Unrefined Discretization of Unit Circle

The following refinements will be introduced if necessary if satisfactory results for a given f are not obtained using the basic discretization, fig. 1:
(i) Add two rings of internal nodes: at $r = 0.25$, 8 nodes and at $r = 0.75$, 32 nodes such that there will be then 57 internal nodes, *ie* $L = 57$
(ii) Double the number of boundary nodes and elements to 32, *ie* $N = 32$
(iii)Carry out both the above refinements, *ie* $L = 57$, $N = 32$

Results will be considered satisfactory if the maximum error in q on the boundary is less than 5%. *i.e.*

$$\frac{(q_{\text{calculated}} - q_{\text{exact}})}{q_{\text{exact}}} * 100 < 5 \qquad (13)$$

at all nodes on the boundary except those for which $q_{\text{exact}} = 0$.

q is chosen in preference to u on the interior because it is more difficult to obtain good results for this function, as will be seen, the f functions considered above will produce small errors in u at interior points without mesh refinement.

Nodal results will not be given; in the tables that follow, results for maximum and mean error in q on the boundary and mean error in u on the domain will be given for each type of approximating function f, on the basic mesh (fig. 1) together with the same results for any mesh refinements introduced until eqn (13) is satisfied in such a way that the performance of the different f functions may be easily evaluated and compared for these two test problems. In the tables, the lower the values obtained, the better the performance of the function in that case.

In the case of the r^3 expansions only $f = 1 + r^3$ will be considered. In the case of problem 1 the global function that will be used is that based on elements of a Pascal triangle,

$$f = 1,\ x,\ y,\ x^2,\ xy,\ y^2, \ldots \tag{14}$$

In the case of problem 2 a global sine expansion will be employed

$$f = 1,\ \sin(x),\ \sin(y),\ \sin(2x),\ \sin(x)\sin(y), \ldots \tag{15}$$

Note that no sum is implied in eqns (14) and (15). In the case of more than one f function, $f = 1 + r$ or $f = 1 + r^3$ will be combined with the respective global function.

The transfomation method, [7], will not be employed due to its lack of generality.

Problem 1

The percentage error in q on the boundary and in u on the domain for eqn (5) on the circular geometry described above is given in table 1 for the r functions and in table 2 for the global and combined local/global functions.

Table 1: Percentage Error for Problem 1 (Local functions)

	Error % $f = 1 + r$			Error % $f = 1 + r^3$		
Discretization	q_{max}	q_{mean}	u_{mean}	q_{max}	q_{mean}	u_{mean}
$N = 16;\ L = 17$	11.35	7.95	0.19	21.33	8.86	0.48
$N = 16;\ L = 57$	7.30	4.62	0.10	4.70	1.71	0.08
$N = 32;\ L = 57$	4.39	2.62	0.09	–	–	–

Table 2: Percentage Error for Problem 1 (Single Global Pascal Triangle and with $1 + r^3$)

	Error % Global;			Error % Global/$1 + r^3$		
Discretization	q_{max}	q_{mean}	u_{mean}	q_{max}	q_{mean}	u_{mean}
$N = 16;\ L = 17$	5.73	2.15	0.03	2.73	1.27	0.23
$N = 32;\ L = 17$	3.07	0.75	0.01	–	–	–

Analysing tables 1-2, it can be seen that all results for u on the domain show mean errors of much less than 1% for all f functions and all

Transactions on Modelling and Simulation vol 10, © 1995 WIT Press, www.witpress.com, ISSN 1743-355X

discretizations examined, the error in q on the boundary being significantly larger.

In the case of the local functions on their own, $1 + r^3$ converges faster than $1 + r$ confirming the findings of [8] and [6], However it should be noted that $1 + r$ produces better results on the unrefined mesh.

The global functions produce very accurate results with few nodes as found in [9,10], however if $1 + r^3$ is combined with the Pascal Triangle expansion, the desired accuracy for q can be obtained on the unrefined mesh. There is however, some loss of accuracy in u, best results for u are those given by the single global function.

Problem 2
The percentage error in q on the boundary and in u on the domain for eqn (6) on the circular geometry described above, and using different f functions is given in tables 3-4.

Table 3: Percentage Error for Problem 2 (Local functions)

	Error % $f = 1 + r$			Error % $f = 1 + r^3$		
Discretization	q_{max}	q_{mean}	u_{mean}	q_{max}	q_{mean}	u_{mean}
$N = 16$; $L = 17$	10.96	7.87	0.30	21.48	14.64	0.43
$N = 16$; $L = 57$	5.65	4.86	0.10	6.83	2.04	0.13
$N = 32$; $L = 57$	4.45	2.95	0.10	3.81	1.12	0.04

Table 4: Percentage Error for Problem 2 (Single global sine and with $1 + r$)

	Error % Global			Error % Global/$1 + r$		
Discretization	q_{max}	q_{mean}	u_{mean}	q_{max}	q_{mean}	u_{mean}
$N = 16$; $L = 17$	5.44	2.25	0.03	3.77	2.74	0.25
$N = 32$; $L = 17$	4.81	1.62	0.02	–	–	–

Analysing tables 3-4 the conclusions are similar to those reached in considering problem 1.

In the case of the local functions, $1 + r^3$ continues to converge to a better result than $1 + r$, but the improvement is much less marked than in the first problem. Results on the unrefined mesh continue to be better with $1 + r$.

The single global sine function performs very well, and when combined with $1+r$ obtains the desired accuracy for q on the unrefined mesh, however the loss in accuracy in u is still noted.

Once again, the smallest maximum error in q is obtained combining local and global functions and the smallest error in u is given by the single global function.

5 Conclusions

In the case of the local functions, for the cases considered, $1 + r$ produced better results than some authors have reported, [8]. Although this function converges slightly slower than $1 + r^3$, it produced better results for the unrefined mesh; in the case of $1 + r^3$, the error in q on the boundary in this case was of the order of 20%, for $1 + r$ the same error was only 10%.

In the case of the global functions, very accurate results were produced with relatively few nodes, however some preliminary work was necessary in order to determine which function to use for each case: The use of an inadequate function usually produces unacceptable results, [12].

The results obtained combining local and global functions are encouraging, however much additional work is necessary as only two of a large number of possible combinations were tested.

References

1. Nardini, D. & Brebbia, C. A. A New Approach for Free Vibration Analysis using Boundary Elements, *Boundary Element Methods in Engineering*, ed C. A. Brebbia, pp 312-326, Springer-Verlag, Berlin and N. York, 1982.

2. Partridge, P. W. & Brebbia, C. A. Computer Implementation of the BEM Dual Reciprocity Method for General field Problems. *Communications in Applied Numerical Methods*, **6** pp 83-92, 1990.

3. Partridge, P. W., Brebbia, C. A. & Wrobel, L. C. *The Dual Reciprocity Boundary Element Method*, Computational Mechanics Publications & Esevier, Southampton and Boston, 1992.

4. Schclar, N. A. & Adey, R. Adaptive Dual Reciprocity Method Boundary Element Method, BEM XIV , *Proceedings*, pp 679-694, 1992.

5. Powell, M. J. D. The Theory of Radial Function Approximation, *Numerical Analysis Summer School*, Lancaster, UK, 1990.

6. Yamada, T., Wrobel, L. C. & Power, H. On the Convergence of the Dual Reciprocity Boundary Element Method, *Eng. Analysis with Boundary Elements*, in press.

7. Zhu, S. & Zhang, Y. An Improvement on the Dual Reciprocity Boundary Element Method for Equations with Convective Terms, *Communications in Applied Numerical Methods*, in press.

8. Zhang, Y. & Zhu, S. On the Choice of Interpolation Functions used in the Dual Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, **13**, pp 387-396, 1994.

9. Cheng, A. H-D., Grilli, S. & Lafe, O. Dual Reciprocity BEM based on Complete Set Global Shape Functions, BEM XV, *Proceedings*, pp 343-358, 1993.

10. Cheng, A. H-D., Lafe, O. & Grilli, S., Dual Reciprocity BEM Based on Global Interpolation Functions, *Eng. Analysis with Boundary Elements*, in press.

11. Partridge, P. W. *Transport Analysis using Boundary Elements*, Computational Mechanics Publications, Southampton and Boston, 1993.

12. Partridge, P. W. Local v. Global Approximation Functions for Diffusion, Convection and other Problems, *Engineering Analysis with Boundary Elements*, in press.