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Green's function attacks geometric non-linearity in a bending of plates

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ABSTRACT

Special iterative procedure has been developed to linearize the boundary value problems modelling the geometrically non-linear bending of thin plates. Green's functions for two-dimensional biharmonic equation as well as Green's matrices for Lame's system of the displacement formulation for the plane problem in theory of elasticity are used for creating the algorithm to attack the linearized problems at each single iterative loop. The particular Green's functions and matrices needed in such a treatment had been constructed in advance by using the technique advocated by Melnikov [1] and evolved by Dolgova and Melnikov [2,3]. Numerical results are given showing a high level of an effectiveness of the approach.

INTRODUCTION

Mathematical modelling in the geometrically non-linear bending of thin plates is extremely expensive computationally. The computational techniques that have been basically utilized in this area over the years are finite differences and finite elements. In an attempt to overcome the computational disadvantages, non-traditional Green's functions approach is developed herein.

To comprehend the principal purpose of the present study, one must recall what an extremely important contribution Green's functions and matrices make in the theory of boundary value problems for ordinary and partial differential equations. This tool is impressively helpful and powerful in discussing such pure mathematical aspects as an existence and uniqueness of the solution of a problem, it also gives support to an investigation of integral and differential properties of the solution. The same time, it is well known that Green's functions unfortunately play a rather limited role in a developing numerical methods in applied mathematical physics.

Nevertheless, it should be pointed out that once a certain Green's function or matrix is successively constructed and its explicit expression is compact enough, it may readily be employed for various purposes of computing as well. Within the last two decades, a confirmation of this has been repeatedly demonstrated by Melnikov, Dolgova, Tsadikova, Davydov, Nikulin, Bajrak, Irschik, Heuer, Ziegler, Koshnarjova, Titarenko, Voloshko [4—16] in various branches of applied mechanics (steady-state heat conduction, elastic and elasto-plastic torsion, plane problem in theory of elasticity, theory of plates and shells, contact mechanics and optimal shape design in theory of elasticity). It has been discovered, in particular, that there exist some specific phases within the numerical procedures of the classical method of potential in mathematical physics, e.g. Courant and Hilbert [17], which might essentially use the properties possessed by the Green's functions or matrices in order to achieve quite visible increase in the efficiency of the original procedure.

We do believe that in the not too distant future, this computational approach will be deeply developed and widely used in applied mathematical physics providing us with new achievements in engineering and science. A strong belief arises that it will attract many followers whose present activities in science relate by any means to the method of potential. They will be enjoying unexhaustible prospectives of numerical implementations based on Green's functions and matrices formulation.

Reviewing the methods which have been traditionally used to construct Green's functions, we should like to emphasize the known fact that only the Dirichlet problem for two-dimensional Laplace's equation for a simply connected domain Ω can be, in fact, considered as a sufficiently developed case, for which, as is known, the Green's function representation may be expressed by

$$G(z,\zeta) = \frac{1}{2\pi} \ln \left| \frac{1 - w(z)\overline{w(\zeta)}}{w(z) - w(\zeta)} \right|, \tag{1}$$

where $z = x+iy$ and $\zeta = \xi + i\varsigma$ are usually referred to as the "observation" or "field" and "sourse" points respectively, the bar on ζ means complex conjugate, and function $w(z)$ maps in a one-to-one conformal manner the given domain Ω onto an interior of the unit circle. Hence, equation (1) provides an exact and rather compact expression for the Green's function to be found in the case when the mapping function $w(z)$ may be represented by a finite combination of elementary functions. Otherwise, the above equation provides us only with an approximate analytical expression for the desired Green's function.

Therefore, even for the rectangular domain which is widely applicable in science and engineering, one can not obtain an exact representation of Green's function to Dirichlet problem for Laplace's operator by operating with equation (1), because the associated mapping function $w(z)$ can not be expressed in closed form. Nevertheless, in spite of the disadvantages, equation (1) contributes greatly to the subject matter.

It creates an opportunity to obtain at least approximate values of Green's functions for quite arbitrary shapes of domains by using available numerical procedures for the conformal mapping. These can not unfortunately be easily accomplished either.

Another practical possibility to determine Green's functions results from the so called reflection (image) method. A key point of this approach may be readily introduced by an obvious and quite simple example as follows. Let $G(z,\zeta)$ be the Green's function of Dirichlet problem for the infinite strip $\Omega_S\{0 \leq \mathrm{Im}z \leq b\}$. Consider now the sum $G^*(z,\zeta) = G(z,\zeta) + G(z,-\bar{\zeta})$, with z and ζ being specified in the semi-infinite strip $\Omega_{SS}\{0 \leq \mathrm{Re}z < \infty, 0 \leq \mathrm{Im}z \leq b\}$. It is obvious then that $G^*(z,\zeta)$ does really represent the Green's function to Laplace's equation over Ω_{SS} for the specific mixed boundary value problem with Neumann boundary conditions being prescribed along the boundary line $x = 0$, while the Dirichlet conditions are prescribed along the lines $y = 0$ and $y = b$. The reflection procedure is especially helpful in some specific situations when either Dirichlet or Neumann conditions are prescribed on the boundary of a simply shaped domain.

One more a very known and probably one of the most popular ways to reach Green's functions is rooted in Fourier's method of separation of variables. This approach, in particular, is fruitful in the case of Dirichlet problem for rectangular domain. Nearly each classical publication provides an expression of the Green's function for Laplace's equation in a rectangle of sides a and b as follows

$$G(x,y;\xi,\varsigma) = -\frac{4ab}{\pi^2}\sum_{m,n=1}^{\infty}\frac{\sin\mu x\cdot\sin\mu\xi\cdot\sin\nu y\cdot\sin\nu\varsigma}{(mb)^2+(na)^2}, \tag{2}$$

where $\mu = m\pi/a$ and $\nu = n\pi/b$. The major disadvantage of the above expression is immediately seen as soon as the computational applications are needed. The point is that the series in (2) does not converge uniformly. This arises from a logarithmic nature of the singularity in the Green's function which can not be exactly evaluated by using any reasonable finite sum in the double Fourier series of that kind in equation (2). Therefore, such expressions may not be successively employed for computational applications unless some special analytical treatment is provided.

Perhaps, one of the most prospective and powerful approaches to a numerical evaluation of Green's functions for two-dimensional Laplace's equation originates directly from its well known representation

$$G(z,\zeta) = \frac{1}{2\pi}\ln\frac{1}{|z-\zeta|} + g(z,\zeta) \tag{3}$$

expressing the Green's function in terms of its logarithmically singular and regular $g(z,\zeta)$ components. If, for instance, the above expression is supposed to be used for computing the Green's function, then it leads to the corresponding a one parameter set of boundary value problems for the regular term $g(z,\zeta)$ with ζ being the mentioned parameter. Due to the last point, this approach must be treated as a very time consuming technique resulting in a massive numerical work. But, in spite of this disadvantage, the considering method should be obviously rated as one of the most challeging techniques in the discussing area. It can be readily expanded on the boundary value problems for other equations and systems of the elliptic type whose fundamental solutions are available in advance.

A successive use of each of the methods overviewed above is unfortunately limited to a specific class of problems. That is why a process for constructing the Green's function or matrix in practice is not in fact a trivial exercise, even when the problem being considered has a simple statement. It can be actually carried out in closed form for only a few classical formulations whose results are well known to all involved in this sphere of investigation. Hence, an intensive use of Green's functions for purposes of computing is essentially limited due to a lack of their appropriate compact representations available in literature.

To eliminate this omission in literature, a new kind of technique created for constructing Green's functions and matrices for equations and systems of the elliptic type has been proposed by Melnikov [1] in 1977. This has proven to be a fruitful technique for a variety of boundary value problems in computational mechanics. By the way, the particular expressions of Green's functions and matrices employed within the present study have been constructed by using that technique.

STATEMENT OF THE PROBLEM

Let us consider a pure displacement formulation for the geometrically non-linear bending of a thin elastic plate having a variable thickness. Suppose that a material of whose the plate is composed is isotropic and homogeneous. The plate occupies the domain $\Omega \subset R^2$, it is loaded with the lateral load $q = q(x,y)$. The plate thickness $h = h(x,y)$ is believed to be a two times differentiable function in Ω. Assume also that it changes continuously and smoothly over Ω allowing one to determine all internal forces and moments by using the known expressions given by Timoshenko and Woinowsky-Krieger [18] for a plate of a constant thickness.

The displacement formulation of an equilibrium state of the plate may be described by the following system of non-linear equations

$$
\begin{aligned}
D(x,y)\cdot\nabla^2\nabla^2 w(x,y) &= q(x,y) + S(u,v,w) + T(w),\\
L_1(u,v) &= P_1(w) + R_1(u,v,w), \qquad (4)\\
L_2(u,v) &= P_2(w) + R_2(u,v,w),
\end{aligned}
$$

where $u = u(x,y)$, $v = v(x,y)$, $w = w(x,y)$ are components in the displacement vector of the middle plane of the plate, ∇^2 is two-dimensional Laplacian operator, $D = Eh^3/(12(1-\sigma^2))$ is the flexural rigidity of the plate, with E, σ, and h being Young's modulus, Poisson's ratio of the material, and the thickness of the plate respectively, L_1 and L_2 are linear differential operators in Lame's system of the plane problem in theory of elasticity given by

$$L_1(u,v) \equiv 2\partial^2u/\partial x^2+(1-\sigma)\cdot\partial^2u/\partial y^2+(1+\sigma)\cdot\partial^2v/\partial x\partial y,$$

$$L_2(u,v) \equiv (1-\sigma)\cdot\partial^2v/\partial x^2+2\partial^2v/\partial y^2+(1+\sigma)\cdot\partial^2u/\partial x\partial y,$$

while S, T, P_1, P_2, R_1 and R_2 are non-linear operators expressed by

$$\begin{aligned}S(u,v,w) \equiv & \frac{Eh}{1-\sigma^2}\{[\partial u/\partial x+1/2\cdot(\partial w/\partial x)^2+\sigma(\partial v/\partial y+1/2\cdot(\partial w/\partial y)^2)]\partial^2w/\partial x^2+\\ & +[\sigma(\partial u/\partial x+1/2\cdot(\partial w/\partial x)^2)+\partial v/\partial y+1/2\cdot(\partial w/\partial y)^2]\cdot\partial^2w/\partial y^2\}+\\ & +\frac{Eh}{1+\sigma}(\partial u/\partial y+\partial v/\partial x+\partial w/\partial x\cdot\partial w/\partial y)\cdot\partial^2w/\partial x\partial y,\end{aligned}$$

$$\begin{aligned}T(w) \equiv & -\frac{E}{12(1-\sigma^2)}[2\partial(h^3)/\partial x\cdot(\partial^3w/\partial x^3+\partial^3w/\partial x\partial y^2)+2\partial(h^3)/\partial y\times\\ & \times(\partial^3w/\partial y^3+\partial^3w/\partial x^2\partial y)+\partial^2(h^3)/\partial x^2\cdot(\partial^2w/\partial x^2+\sigma\partial^2w/\partial y^2)+\\ & +\partial^2(h^3)/\partial y^2\cdot(\partial^2w/\partial y^2+\sigma\partial^2w/\partial x^2)+2(1-\sigma)\cdot\partial^2(h^3)/\partial x\partial y\cdot\partial^2w/\partial x\partial y],\end{aligned}$$

$$P_1(w) \equiv -2\partial w/\partial x\cdot(\partial^2w/\partial x^2+(1-\sigma)\cdot\partial^2w/\partial y^2)-(1+\sigma)\cdot\partial w/\partial y\cdot\partial^2w/\partial x\partial y,$$

$$P_2(w) \equiv -2\partial w/\partial y\cdot(\partial^2w/\partial y^2+(1-\sigma)\cdot\partial^2w/\partial x^2)-(1+\sigma)\cdot\partial w/\partial x\cdot\partial^2w/\partial x\partial y,$$

$$\begin{aligned}R_1(u,v,w) \equiv & -\{\partial h/\partial x\cdot[2\partial u/\partial x+(\partial w/\partial x)^2+\sigma(2\partial v/\partial y+(\partial w/\partial y)^2)]+\\ & +(1-\sigma)\cdot\partial h/\partial y\cdot(\partial u/\partial y+\partial v/\partial x+\partial w/\partial x\cdot\partial w/\partial y)\}/h,\end{aligned}$$

$$\begin{aligned}R_2(u,v,w) \equiv & -\{\partial h/\partial y\cdot[2\partial v/\partial y+(\partial w/\partial y)^2+\sigma(2\partial\partial u/\partial x+(\partial w/\partial x)^2)]+\\ & +(1-\sigma)\cdot\partial h/\partial x\cdot(\partial u/\partial y+\partial v/\partial x+\partial w/\partial x\cdot\partial w/\partial y)\}/h.\end{aligned}$$

Let a set of boundary conditions which must be prescribed on the contour Γ of the given domain Ω be written in the form

$$B_1(w)|_{(x,y)\in\Gamma} = 0, \qquad B_2(w)|_{(x,y)\in\Gamma} = 0, \tag{5}$$

$$C_1(u,v)|_{(x,y)\in\Gamma} = 0, \qquad C_2(u,v)|_{(x,y)\in\Gamma} = 0, \tag{6}$$

where B_1, B_2, C_1, and C_2 are linear differential operators specifying a certain combination of boundary conditions (clamped, free, or simply supported edges) which are actually prescribed on the contour of the plate. Thus, an analysis of an equilibrium state of the plate being considered results in the non-linear boundary value problem in equations (4)–(6). A detailed description of the iterative process developed to treat this problem as well as an experimental verification of its convergence are provided in the following sections.

ITERATIVE PROCESS

Introducing an iterative scheme which is supposed to be applied herein, let the exact solution to the problem in equations (4)–(6) be considered as limits of the functional sequences $\{u^k = u^k(x,y)\}$, $\{v^k = v^k(x,y)\}$, $\{w^k = w^k(x,y)\}$ arising from the two separate linear boundary value problems. The first of them is given by

$$D(x,y)\cdot\nabla^2\nabla^2 w^{k+1} = (1-\tau_{k+1})\cdot D(x,y)\cdot\nabla^2\nabla^2 w^k +$$

$$+ \tau_{k+1}[q + S(u^k,v^k,w^k) + T(w^k)], \tag{7}$$

$$B_1(w^{k+1})|_{(x,y)\in\Gamma} = 0, \quad B_2(w^{k+1})|_{(x,y)\in\Gamma} = 0 .$$

While the second deals with the nonhomogeneous Lame's system of the plane problem in theory of elasticity formulated as follows

$$L_1(u^{k+1},v^{k+1}) = (1-\tau_{k+1})\cdot L_1(u^k,v^k) + \tau_{k+1}[P_1(w^k + R_1(u^k,v^k,w^k)],$$

$$L_2(u^{k+1},v^{k+1}) = (1-\tau_{k+1})\cdot L_2(u^k,v^k) + \tau_{k+1}[P_2(w^k) + R_2(u^k,v^k,w^k)], \tag{8}$$

$$C_1(u^{k+1},v^{k+1})|_{(x,y)\in\Gamma} = 0, \quad C_2(u^{k+1},v^{k+1})|_{(x,y)\in\Gamma} = 0.$$

In both of the above formulations, the supscript k indicates the number of the iteration. Parameter τ_{k+1} might be varied with the number of the iteration that makes it possible to regulate the convergence of the iterative process. Such an iterative approach is called the non-stationary two-layered scheme. It is obvious that its particular case associated with $\tau_{k+1} = 1$ may be considered as the simplest scheme in the method of a direct iteration.

Hence, the limits (if any) of the sequences $\{u^k(x,y)\}$, $\{v^k(x,y)\}$, $\{w^k(x,y)\}$, which follow from the linear boundary value problems in equations (7) and (8), represent the components in the displacement vector associated with the original non-linear problem in equations (4)–(6). In this study, to arrange a computational attack on the linear problems in equations (7) and (8), a new version of the Green's functions method has been developed and applied in practice.

Advertising the major advantage in this approach, we should like to emphasize that in comparison with computational procedures of other numerical methods widely used in the discussed area (finite differences, finite elements and so on), it provides a real possibility to compute all components in the displacement vector and stress tensor with an equal level of accuracy. This arises from the fact that arranging calculations, we avoid any procedures of numerical differentiation. In fact, all differentiatings needed in this study are treated analytically. Numerical procedures which have been actually employed herein delt only with an evaluating of integrals of the form in equations (12)—(14) in the next section.

An accomplishing the proposed iterative procedure gives rise to a question of choosing values of the parameter τ_k providing a stable and fast convergence of the process. Having been worked out in the present study, the computational experiment showed that in the stationary (τ_k = const) scheme, a value of the uniform lateral loading, for which the process converges, is increased with a value of parameter τ being decreased. But the converges in this case slows down. It has been also found that if the process diverges, the sequence of the lagest differences of the two consecutive approximations of the deflection function w(x,y) is alternating.

Taking into account the information recently mentioned, the following heuristical algorithm to choose the sequence of values of the parameter τ_k has been proposed herein for the non-stationary scheme (7),(8). Assume the k-th approximation u^k, v^k, w^k of the components in the displacement vector and the associated value of the parameter τ_k are already available. Let the two following consecutive approximations be obtainted by using the scheme in equations (7),(8) with an assumption that $\tau_{k+2} = \tau_{k+1} = \tau_k$. We denote now

$$\delta_{k+1} = w^{k+1}(x^*,y^*) - w^k(x^*,y^*), \tag{9}$$

where (x^*,y^*) is the point in the middle plane of the plate at which the k-th approximation w^k of the deflection function achieves its peak in absolute value. If then the following condition

$$\delta_{k+1} \cdot \delta_{k+2} \geq - \alpha \cdot \delta^2_{k+1} \tag{10}$$

holds for $0 < \alpha < 1$ (where α is the coefficient of condensing the iterative process), then the last two approximations are successful and the iterative process proceeds. Futhermore, providing the left-hand term in the inequality (10) is positive, a value of the parameter τ, which is supposed to be used for the continuation of the process, may be increased assuming $\tau_{k+4} = \tau_{k+3} = \beta \cdot \tau_k$, where $\beta \geq 1$ is the coefficient of increasing of the step in the iterative process. Unless the condition in (10) holds, the evaluation of the (k+1)-th and (k+2)-th approximations must be restarted again getting $\tau_{k+1} = \tau_{k+2} = \gamma \cdot \tau_k$ where $\gamma < 1$ is the coefficient of decreasing of the step in the iterative process.

It has been found in the present study that a condition as written

$$(\tau_{k+2})^{-1} \cdot \max_{\Omega} | w^{k+2} - w^{k+1} | \leq \omega + \vartheta \cdot \max_{\Omega} | w^{k+2} | , \tag{11}$$

with ω and ϑ being the given parameters, may be successively used to terminate the iterative process in equations (7),(8). The left-hand side in the above inequality is the lagest difference of the two successsive approximations of the deflection function achieved by the simplest procedure in the method of a direct iteration, therefore it does not depend on a value of the parameter τ_{k+2}.

To accomplish a computation in the particular problems discussed in the present study, optimal values of the parameters τ_0, α, β, and γ have been determined by numerical experiments. It has been shown, for instance, that the variations in the coefficient of condensing α within the interval $0.1 - 0.9$ does not influence the convergence at all. The optimal values of the coefficient β have been discovered within the interval $1.3 - 1.4$, while those of the parameter γ have been found within the interval $0.4 - 0.6$. It has been also shown that the optimal values of the parameter τ_0 belong to the interval $0.005 - 0.060$. One more an important detail in the iterative algorithm (7),(8) has been discovered herein. Namely, it has been appeared that its convergence does not depend on an initial approximation, only a speed of the convergence is slightly affected.

GREEN'S FUNCTIONS FORMULATION

In order to determine each single approximation u^{k+1}, v^{k+1}, w^{k+1} to the solution of the non-linear problem in equations (4)–(6) in compliance with the formulation in equations (7) and (8), it is necessary first to evaluate the right-hand terms of these which are obtainable as the outputs of the right-hand sided operators in (7) and (8) being applied to the previous approximation u^k, v^k, w^k. Therefore, a practical convergence of the iterative process must badly depend on an accuracy of numerical evaluations achieved within any its single step. Later in this section, we are going to discuss the point again. It will be explained why the Green's functions approach in our case guarantees a high level of accuracy.

Suppose $G_B(x,y;\xi,\varsigma)$ is the Green's function of the biharmonic equation for domain Ω with the boundary conditions in equation (5) being prescribed on its contour. Then, the solution to a nonhomogeneous equation

$$\nabla^2\nabla^2 w(x,y) = -F(x,y)$$

satisfying boundary conditions in (5) can be expressed by the integral

$$w(x,y) = \iint_{\Omega} G_B(x,y;\xi,\varsigma)\, F(\xi,\varsigma)\, d\Omega(\xi,\varsigma). \tag{12}$$

As it has been alreary discussed, compact representations of such Green's functions are obtainable in the scope of our technique [1–3] for a variety of boundary value problems. Similar integral representation

$$U(x,y) = \iint_{\Omega} G_L(x,y;\xi,\varsigma)\, R(\xi,\varsigma)\, d\Omega(\xi,\varsigma) \tag{13}$$

may be readily associated with the solution of the nonhomogeneous Lame's system satisfying the boundary conditions in equations (6). Here $U(x,y)$ is a vector whose components are the tangential displacements $u(x,y)$ and $v(x,y)$ of the middle plane in the given plate, $R(\xi,\varsigma)$ is a vector of the right-hand terms in Lame's system, and finally, $G_L(x,y;\xi,\varsigma)$ is the corresponding Green's matrix of Lame's system for Ω.

Hence, to run each single loop in the iterative process (7),(8), it is necessary to be able to overcome all computational difficulties arising from an evaluating of the proper and convergent improper integrals in the form

$$\iint_{\Omega} H(x,y;\xi,\varsigma)\, Q(\xi,\varsigma)\; d\Omega(\xi,\varsigma) \tag{14}$$

whose kernel-functions or matrices $H(x,y;\xi,\varsigma)$ result from the Green's function $G_B(x,y;\xi,\varsigma)$ or Green's matrix $G_L(x,y;\xi,\varsigma)$ being affected by the right-hand sided operators in (7),(8). A factor $Q(\xi,\varsigma)$ in the integrand in the above representation is in fact obtainable by applying again integrals of the form in (14).

So, the basic computational advantage in the proposed technique originates from the fact that accomplishing all numerical evaluations, one deals here exclusively with a numerical integrating, while any numerical differentiatings are absolutely avoided. This provides a high level of accuracy within each single detail in the algorithm and ultimately in the algorithm in whole.

NUMERICAL RESULTS AND DISCUSSION

In the present study, the proposed algorithm has been employed to solve a set of problems of the described nature. The rectangular plates as well as plates having a semi-strip shape have been examined. Different kinds of boundary conditions and lateral loadings have been considered, we have also analized several problems whose statements contain certain geometrical constraints restricting in some sence a natural deflections of the plate. All these are not going to be discussed in the present paper. Some numerical results touching upon the mentioned formulations will be included in one of our comming publications.

An emphasis in this paper is different, it is made to discover a level of a non-linearity achievable by the Green's functions approach. That is why we concentrate in this section on one single numerical example from our study which does not actually present the most complicated case we are able to effort. But it does really present our calculative potential in a treating non-linearity in problems of the discussing matter. The example we exhibit deals with the rectangular plate occupying the region $\Omega_R\{0 \le x \le a,\ 0 \le y \le b\}$ subject to an uniformal lateral loading. The edges $y = 0$ and $y = b$ are supposed to be simply supported resulting in

$$w = (\partial^2 w/\partial y^2 + \sigma\cdot\partial^2 w/\partial x^2) = 0, \quad v = \partial u/\partial y = 0,$$

the edge $x = 0$ is assumed to be clamped

$$w = \partial w/\partial x = 0, \quad u = v = 0,$$

while the edge $x = a$ is free resulting in the boundary conditions as follows

$$\partial^2 w/\partial x^2 + \sigma\cdot\partial^2 w/\partial y^2 = \partial^3 w/\partial x^3 + (1-\sigma)\cdot\partial^3 w/\partial x\partial y^2 = 0,$$

$$\partial u/\partial x - \sigma\cdot\partial v/\partial y = \partial u/\partial y + \partial v/\partial x = 0,$$

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The thickness of the plate varies linearly with x coordinate and quadratically with y coordinate following an equation as written

$$h(x,y) = h_0\cdot(1+t_1\cdot x/a)\cdot[1+t_2\cdot(y/b-0.5)^2]. \qquad (15)$$

The physical properties of the material of whose the plate is composed and the geometrical parameters of the plate are given as $E = 2.1\times10^5$ MPa, $\sigma = 0.3$, $a = 2.0$ m, $b = 1.0$ m, and $h_0 = 4.0\times10^{-3}$ m , the intensity of the uniformly distributed load is $q = 5.0\times10^{-3}$ MPa. Some characteristics in a stress-strain state of this plate for the case $t_2 = -2.0$ (see equation (15)) are shown in Figure 1.

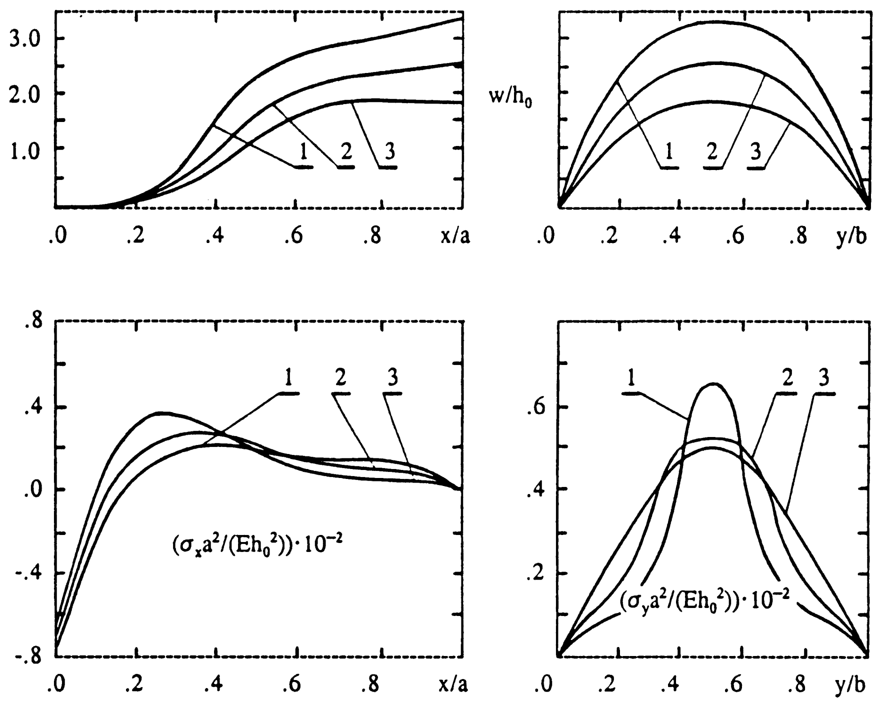

Figure 1. Deflections and bending stresses in the rectangular plate whose thickness is given by equation (15) with $t_2 = -2$

The left upper and lower fragments in Figure 1 exhibit deflections and stresses occuring along the line $y = b/2$ (the curves 1, 2, and 3 relate to $t_1 = -0.5$, 0, and 0.5 (equation (15)) respectively), while the right fragments depict displacements and stresses occuring along the edge line $x = a$, with identical labels on the curves. It is interesting to note, for example, that the maximum value of the deflection occured at point (a, b/2) in the case $t_1 = 0.5$ is only about 1.9 times larger then that in the case $t_1 = -0.5$, showing that generally speaking deflections here are not as affected by a variation of a thickness as it happens in geometrically linear problems. One more an interesting detail concerning the state of stresses follows from the lower-right fragment in the above Figure. The point is that smaller thickness of the plate ($t_1 = 0.5$) results in more changeable behaviour of the stress σ_y. This is also more typical for the non-linear problems than for linear ones.

Figure 2 depicts the same set of components in a stress-strain state of the plate with a variable thickness for $t_2 = 2$ (see equation (15)). All physical and geometrical parameters in the statement of the problem remain the same as in the previous case.

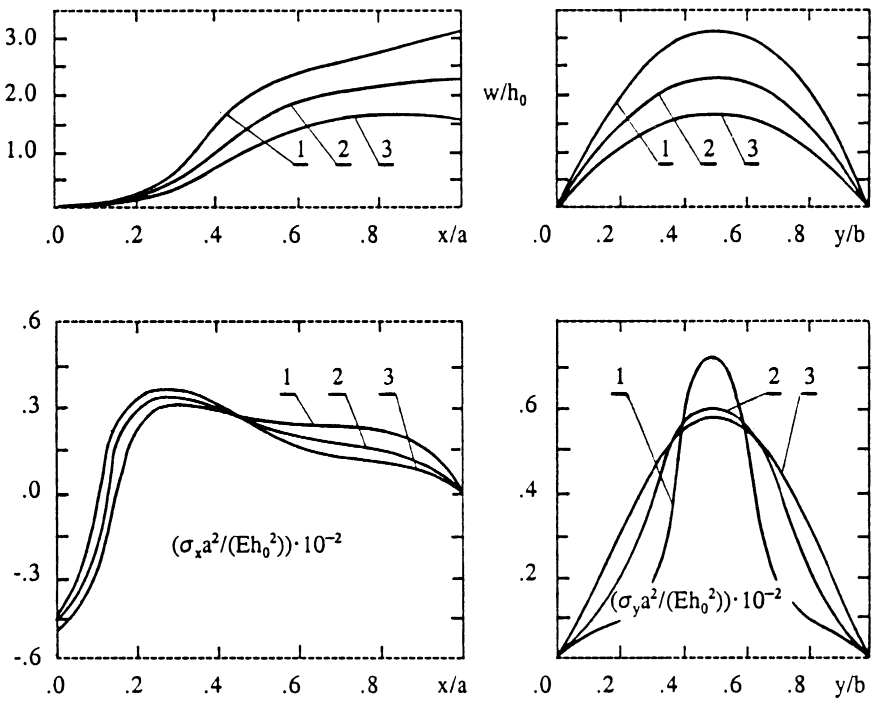

Figure 2. Deflections and bending stresses in the rectangular plate with a variable thickness given by equation (15) for $t_2 = 2$

Comparing both of the accompanying Figures, one readily discovers that each pair of the corresponding components in the stress-strain states behaves nearly likely. There is, however, a slight distinction between them which arises from the fact that each plate in the second set is a little bit more "rigid" than its mate in the first set. Hence, since both of the statements coincide exactly except for the function of thickness in equation (15), the second set of plates must basically output a lower level of displacements but an upper level of stresses. Such a relationship is merely expectable at least for linear formulations of problems. However, it is obvious that a non-linearity may affect solutions changing a situation. That happens in our case. The above mentioned relationship between the two equilibrium states does really take place almost everywhere in an interior of Ω_R, but, for example, the maximum value of the bending stress σ_y (Figure 2) at the midpoint (0, b/2) on the edge $x = 0$ is about 40% higher than in the first case. This is a direct influence of a non-linearity in a statement of the problems.

CLOSURE

Concluding the discussion in the present paper, we point out several important details in the calculating procedure used herein. First of all, the double integrals in equations (12)—(14) have been approximately evaluated by using the trapezoidal type of quadratures, with a partitioning the given domain into 100 to 200 congruent rectangular subelements. The convergent improper integrals have been treated by an approximate analytical approach based on the mean value theorem for definite integrals. Such an evaluation, as is shown in this study, guarantees in the given cases a degree of accuracy exceeding 99.9%.

Secondly, from ten to fifteen successive iterations have been accomplished for each individual non-linear problem solved in the present study, providing a practical convergence of the iterative process, with $(w^{k+1} - w^k)_{max}/w^k{}_{max} < 0.01$.

It also must be emphasized that the iterative process developed in the present study enables us to treat a high level of non-linearity ($w_{max}/h_0 \approx 4.0$) in a considered class of problems, showing eventually a great potential of the approach.

Ultimately, one more an advantageous feature of the Green's functions approach used herein must be pointed out. Namely, this approach provides a stable degree of accuracy for any parameter of a stress-strain state in the plate regardless to a location of the observation point with respect to the contour of the domain.

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