LOCAL MOVING LEAST SQUARES METHOD FOR GRANULAR DEBRIS FLOW

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ABSTRACT
This study investigates the potential applications of local meshless methods in modelling rapid slope movements dynamics. These geohazards pose significant threats to mountainous infrastructure, necessitating the development of robust numerical models for predictive purposes. Contemporary research has seen an increased focus on meshless methods across various disciplines due to their potential advantages over traditional numerical approaches. In this investigation, the local weighted residual method was employed to develop a debris flow model, as it demonstrated suitability for addressing the relevant differential equations. To assess the efficacy of this methodology, the research centred on dry granular soil movement models, with the resultant solutions being compared against several established case studies. This comparative analysis aims to validate the viability of the proposed method in the context of geohazard modelling.
Keywords: debris flow, meshless method, moving least squares.

1 INTRODUCTION

Debris flows, avalanches and rapid mass movements represent significant geohazards in mountainous regions worldwide. These phenomena, characterised by the rapid downslope movement of granular materials mixed with varying amounts of water, pose substantial threats to infrastructure and human lives. The complexity of these flows, arising from their nonlinear dynamics, variable rheology, and interaction with complex topographies, necessitates advanced numerical modelling techniques for accurate prediction and risk assessment. The foundation for many contemporary debris flow models lies in the seminal work of Savage and Hutter [1], who proposed a theoretical framework for describing the rapid creep of granular materials along inclined surfaces in 1989. This theory and its subsequent extensions [2], [3] form the basis of depth-averaged mathematical models that capture the essential physics of debris flows while remaining computationally tractable. The governing equations for debris flows are similar to the shallow water equations (SWEs) but with additional terms for internal friction and variable basal topography. These equations form a system of hyperbolic partial differential equations (PDEs) that can exhibit shock-like phenomena and require careful numerical treatment. Traditionally, finite volume methods [4] have been employed to solve these equations. These methods offer several advantages, including ease of implementation for complex geometries, the natural handling of large deformations, and the ability to adapt to resolution dynamically. Those based on radial basis functions (RBFs) have gained popularity among meshless [5] techniques. However, RBF methods often lead to ill-conditioned matrices, particularly for large-scale problems. In this study, we propose and investigate the application of the local moving least squares (LMLS) method [6] to debris flow modelling. The LMLS approach offers several key advantages.

It provides a robust framework for spatial discretisation without a structured mesh. It allows for high-order approximations, which are crucial for capturing the complex dynamics of debris flows. It maintains good conditioning even for large numbers of computational points. It naturally handles irregular point distributions, enabling adaptive resolution in areas of interest.

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Our numerical scheme combines the LMLS spatial discretisation with a high-order Runge–Kutta time integration method. This approach is designed to capture the debris flow nonlinear behaviour accurately while maintaining numerical stability and efficiency. The remainder of this paper is structured as follows.

Section 2 presents the governing equations based on the extended Savage–Hutter theory, detailing the depth-averaged mass and momentum conservation laws with Mohr–Coulomb rheology. Section 3 describes our numerical methodology, including the temporal discretisation scheme (Section 3.1), the LMLS spatial approximation technique (Section 3.2) and a solution stabilisation process to enhance robustness (Section 3.3).

Section 4 demonstrates the efficacy of our approach through a comprehensive numerical example, simulating the flow of dry cohesionless soil down an inclined slope. Section 5 concludes the paper with a summary of our findings and directions for future research.

Through this work, we aim to contribute to the ongoing development of accurate and efficient numerical tools for debris flow modelling, ultimately enhancing our ability to assess and mitigate related geohazards in mountainous regions.

2 GOVERNING EQUATIONS

This study's mathematical model for debris flows is based on the extended Savage–Hutter theory, which combines principles from SWEs with a Mohr–Coulomb rheology for granular materials. This approach simulates rapid, gravity-driven flows of cohesionless granular materials over complex topographies.

The soil body can be modelled at yield as a Mohr–Coulomb plastic material that slides over a rigid basal topography. The depth integration decreases the solution's dimension to two-dimensional planar movement.

2.1 Definition of the curvilinear coordinate system

The curvilinear coordinate system expresses the governing equations (Fig. 1). The depth-integrated mass balance equation can be written as:

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y}=0 \tag{1}$$

and the moment balance equations are:

$$\frac{\partial (hu)}{\partial t}+\frac{\partial \left(hu^2\right)}{\partial x}+\frac{\partial (huv)}{\partial y}=h\left(s_x-\beta_x\frac{\partial h}{\partial x}\right) \tag{2}$$

$$\frac{\partial (hv)}{\partial t}+\frac{\partial (huv)}{\partial x}+\frac{\partial \left(hv^2\right)}{\partial y}=h\left(s_y-\beta_y\frac{\partial h}{\partial y}\right) \tag{3}$$

where h is the thickness of the avalanche, and u and v are the depth-averaged velocities in the directions x and y, respectively. The terms s_x and s_y represent the accelerations in the downslope and lateral directions. If the basal surface is curved downslope and laterally planar, these terms are defined as [4]:

$$s_x=g\left(\sin\xi-\cos\xi\frac{u}{|\mathbf{u}|}\tan\delta\right) \tag{4}$$

$$s_y=-g\cos\xi\frac{v}{|\mathbf{u}|}\tan\delta \tag{5}$$

$$\mathbf{u} = [u, v]^T, \quad |\mathbf{u}| = \sqrt{u^2 + v^2} \tag{6}$$

where δ is the basal Coulomb dry-friction angle.

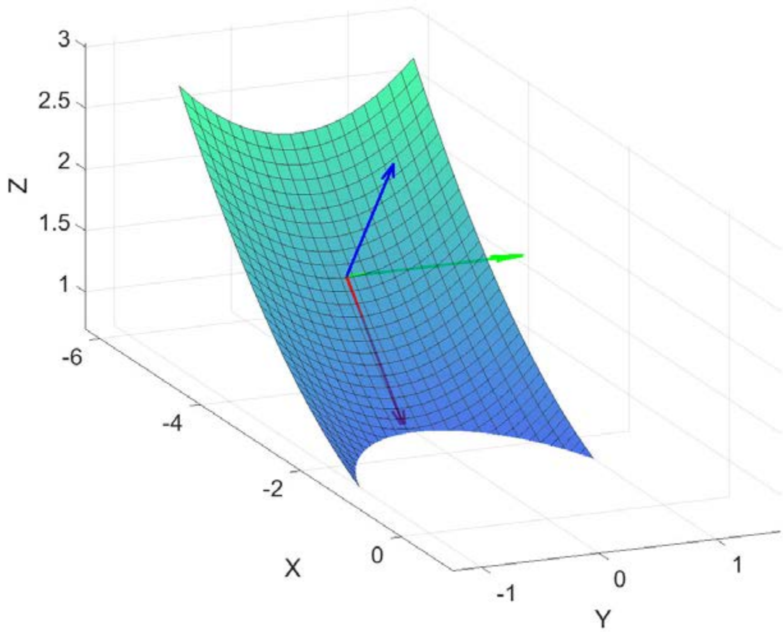

Figure 1: Definition of the curvilinear coordinate system.

The factors β_x and β_y are defined as [3]:

$$\beta_x = g\cos\xi K_x, \quad \beta_y = g\cos\xi K_y \tag{7}$$

where ξ is the inclination and K_x, and K_y are the down- and cross-slope earth pressure coefficients defined by the Mohr–Coulomb criterion:

$$K_{xa/xp} = \frac{2}{\cos^2\phi}\left(1 \mp \sqrt{1 - \sin^2\phi/\cos^2\delta}\right) - 1 \tag{8}$$

$$K_{ya/yp}^{xa/xp} = \frac{1}{2}\left(K_{xa/xp} + 1 \mp \sqrt{\left(K_{xa/xp} - 1\right)^2 + 4\tan^2\delta}\right) \tag{9}$$

where ϕ is the internal friction angle, and the subscripts 'a' or 'p' denote active or passive stress states [7]. The active stress states are generated in the case of soil dilation, and the passive stress states are joined with compression. The choice of active or passive state coefficients in a numerical model depends on the acceleration in an appropriate direction [4]:

$$K_x = \begin{cases} K_{xa} & \text{if } \partial u / \partial x \geq 0 \\ K_{xp} & \text{if } \partial u / \partial x < 0 \end{cases} \tag{10}$$

$$K_y = \begin{cases} K_{ya}^{xa} & if \partial u/\partial x \geq 0, \partial v/\partial y \geq 0 \\ K_{yp}^{xa} & if \partial u/\partial x \geq 0, \partial v/\partial y < 0 \\ K_{ya}^{xp} & if \partial u/\partial x < 0, \partial v/\partial y \geq 0 \\ K_{yp}^{xp} & if \partial u/\partial x < 0, \partial v/\partial y < 0 \end{cases} \tag{11}$$

The usual boundary conditions of eqns (1), (2) and (3) are:

- Given depth, where the value of the depth h on the part of boundary Γ_1 is given, i.e., $h = h_0$,
- No-flux condition, where the flux perpendicular to the boundary Γ_2 is zero $q_n = 0$,
- No-slip condition, where the velocities u and v are zero on this part Γ_3 of the boundary.

and the whole boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. The initial condition is defined by the prescribed values of the depth and velocities at time $t_0 = 0$.

3 NUMERICAL SOLUTION METHODOLOGY

3.1 Temporal discretisation scheme

The governing equations for debris flow, derived from the extended Savage–Hutter theory, form a system of hyperbolic PDEs. In vector form, these can be expressed as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = \mathbf{S}(\mathbf{U}) \tag{12}$$

where $\mathbf{U}$ is the vector of conserved variables, $\mathbf{F}(\mathbf{U})$ and $\mathbf{G}(\mathbf{U})$ are the flux tensors, and $\mathbf{S}(\mathbf{U})$ represents the source terms. Specifically:

$$\mathbf{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} hu \\ hu^2 \\ huv \end{bmatrix}, \quad \mathbf{G}(\mathbf{U}) = \begin{bmatrix} hv \\ huv \\ hv^2 \end{bmatrix}, \quad \mathbf{S}(\mathbf{U}) = \begin{bmatrix} 0 \\ h\left(s_x - \beta_x \frac{\partial h}{\partial x}\right) \\ h\left(s_y - \beta_y \frac{\partial h}{\partial y}\right) \end{bmatrix} \tag{13}$$

Here, h is the flow depth, u_x and u_y are depth-averaged velocities in the x and y directions, respectively, g is the gravitational acceleration, and κ_x and κ_y are earth pressure coefficients.

To solve this system numerically, we employ an explicit third-order Runge–Kutta (RK3) method for time integration. This choice balances computational efficiency with the accuracy required for capturing the complex dynamics of debris flows. The RK3 scheme is formulated as follows:

$$\begin{aligned} \mathbf{U}^{(0)} &= \mathbf{U}^n \\ \mathbf{U}^{(1)} &= \mathbf{U}^{(0)} + \Delta t \mathbf{L}\left(\mathbf{U}^{(0)}\right) \\ \mathbf{U}^{(2)} &= \frac{3}{4}\mathbf{U}^{(0)} + \frac{1}{4}\left[\mathbf{U}^{(1)} + \Delta t \mathbf{L}\left(\mathbf{U}^{(1)}\right)\right] \\ \mathbf{U}^{(3)} &= \frac{1}{3}\mathbf{U}^{(0)} + \frac{2}{3}\left[\mathbf{U}^{(2)} + \Delta t \mathbf{L}\left(\mathbf{U}^{(2)}\right)\right] \\ \mathbf{U}^{n+1} &= \mathbf{U}^{(3)} \end{aligned} \tag{14}$$

where Δt is the time step, and the operator $\mathbf{L}(\mathbf{U})$ represents the spatial discretisation. The spatial derivatives in $\mathbf{L}(\mathbf{U})$ are approximated using the LMLS method, as detailed in Section 3.2. This meshless approach allows for flexible handling of complex topographies often encountered in debris flow scenarios.

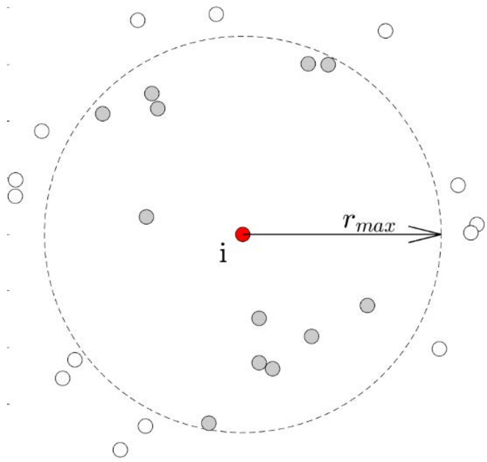

Figure 2: Support domain of the *i*th point.

3.2 Spatial derivative approximation

We employ the LMLS method to calculate spatial derivatives in our debris flow model. This meshless approach offers enhanced accuracy and stability compared to traditional methods, particularly in preserving mass conservation.

In the LMLS method, we approximate the function $\phi(x, y)$ at any point (x, y) within the domain as:

$$\phi(x, y) \approx \sum_{j=1}^{m} \alpha_j (x, y) p_j(x, y) \tag{15}$$

where $p_j(x, y)$ are polynomial basis functions, and $\alpha_j(x, y)$ are unknown coefficients that depend on the spatial coordinates. The choice of m corresponds to the polynomial degree, with the 2D polynomial set typically defined as:

$$\{p_j(x, y)\} = \{1, x, y, x^2, xy, y^2, \dots\} \tag{16}$$

To determine the unknown coefficients $\alpha_j(x, y)$, we minimise the following weighted least squares functional:

$$J(\mathbf{x}) = \sum_{i=1}^{n} W(\mathbf{x} - \mathbf{x}_i)\left[\phi_i - \sum_{j=1}^{m} \alpha_j (\mathbf{x}) p_j(\mathbf{x}_i)\right]^2 \tag{17}$$

where $\mathbf{x} = (x, y)$, $\mathbf{x}_i = (x_i, y_i)$ are the scattered data points, ϕ_i are the known function values at these points, and $W(\mathbf{x} - \mathbf{x}_i)$ is a weight function.

A common choice for the weight function is the exponential function:

$$W(\mathbf{x} - \mathbf{x}_i) = \exp(-\beta \parallel \mathbf{x} - \mathbf{x}_i \parallel^2) \tag{18}$$

where β is a shape parameter controlling the influence of neighbouring points.

Minimising the functional $J(\mathbf{x})$ with respect to $\alpha_j(\mathbf{x})$ leads to the following system of equations:

$$\sum_{j=1}^{m} \alpha_j (\mathbf{x}) \sum_{i=1}^{n} W(\mathbf{x} - \mathbf{x}_i) p_j(\mathbf{x}_i) p_k(\mathbf{x}_i) = \sum_{i=1}^{n} W(\mathbf{x} - \mathbf{x}_i) \phi_i p_k(\mathbf{x}_i) \tag{19}$$

for $k = 1, \dots, m$. This can be written in matrix form as:

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$$\mathbf{A}(\mathbf{x})\boldsymbol{\alpha}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\boldsymbol{\phi} \tag{20}$$

where:

$$\begin{aligned} A_{kj}(\mathbf{x}) &= \textstyle\sum_{i=1}^{n} W\,(\mathbf{x}-\mathbf{x}_i)p_j(\mathbf{x}_i)p_k(\mathbf{x}_i) \\ B_k(\mathbf{x}) &= \textstyle\sum_{i=1}^{n} W\,(\mathbf{x}-\mathbf{x}_i)p_k(\mathbf{x}_i) \end{aligned} \tag{21}$$

The solution for the unknown coefficients is then:

$$\boldsymbol{\alpha}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\boldsymbol{\phi} \tag{22}$$

Once we have determined $\boldsymbol{\alpha}(\mathbf{x})$, we can approximate the function and its derivatives at any point $\mathbf{x}$ as:

$$\begin{aligned} \phi(\mathbf{x}) &\approx \textstyle\sum_{j=1}^{m} \alpha_j\,(\mathbf{x})p_j(\mathbf{x}) \\ \frac{\partial\phi}{\partial x}(\mathbf{x}) &\approx \textstyle\sum_{j=1}^{m} \alpha_j\,(\mathbf{x})\frac{\partial p_j}{\partial x}(\mathbf{x}) \\ \frac{\partial\phi}{\partial y}(\mathbf{x}) &\approx \textstyle\sum_{j=1}^{m} \alpha_j\,(\mathbf{x})\frac{\partial p_j}{\partial y}(\mathbf{x}) \end{aligned} \tag{23}$$

3.3 Solution stabilisation process

While the LMLS method described in Section 3.2 provides a robust framework for spatial discretisation, the time integration process may still introduce numerical instabilities, particularly in regions with steep gradients or discontinuities standard in debris flows. To address this, we implement a solution stabilisation process that leverages the inherent properties of the LMLS approximation.

Our stabilisation approach focuses on the depth variable h, critical for mass conservation in debris flow simulations. At each time step $n+1$, we have two values for depth at each point i:

1. h_i^{n+1}: obtained directly from the Runge–Kutta time integration (as described in Section 3.1).
2. h'^{n+1}_i: the LMLS approximation computed using the coefficients $\alpha_j(\mathbf{x}_i)$ from Section 3.2.

We define a local error measure ε_i at each point i:

$$\varepsilon_i = \frac{|h_i^{n+1} - h'^{n+1}_i|}{h'^{n+1}_i} \tag{24}$$

We compare this local error to a threshold value $\varepsilon_{threshold}$ to determine whether to apply smoothing. We use the following adaptive smoothing algorithm:

1. If $\varepsilon_i > \varepsilon_{threshold}$:

$$h_i^{n+1} = \omega h_i^{n+1} + (1-\omega)h'^{n+1}_i \tag{25}$$

2. If $\varepsilon_i \leq \varepsilon_{threshold}$: No smoothing is applied.

Here, ω is a relaxation parameter $(0 < \omega < 1)$ that controls the degree of smoothing. This stabilisation process takes advantage of the LMLS approximation's continuity and smoothness properties while preserving sharp features when the local error is below the threshold. It helps maintain numerical stability without overly diffusing the solution, which is crucial for accurately capturing the complex dynamics of debris flows.

The adaptive nature of this stabilisation technique is particularly beneficial in handling the varying flow regimes encountered in debris flow simulations, from rapid, turbulent flows to slower, more viscous behaviours. By applying smoothing selectively, we can maintain the sharp flow fronts and discontinuities characteristic of debris flows while dampening numerical oscillations in smoother flow regions.

It's important to note that this stabilisation process is applied after each Runge–Kutta substep, ensuring stability is maintained throughout the time-integration process. The LMLS coefficients $\alpha_j(\mathbf{x}_i)$ are recomputed at each time step using the updated depth values, maintaining consistency between the stabilised solution and the LMLS approximation.

This approach balances numerical stability and solution accuracy, crucial for reliable long-term simulations of complex geophysical flows. It complements the spatial discretisation provided by the LMLS method, resulting in a robust numerical scheme capable of handling the challenges posed by debris flow modelling.

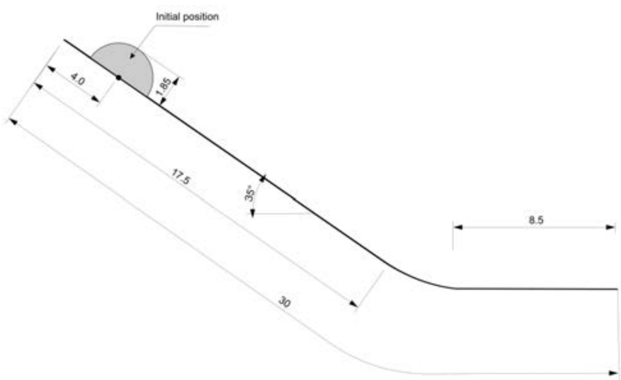

Figure 3: Longitudinal profile of computational area at $y = 0$ with initial position of the soil material [5].

4 NUMERICAL CASE STUDY: DRY COHESIONLESS SOIL FLOW ON AN INCLINED SLOPE

This section presents a comprehensive numerical example that demonstrates the capabilities and efficacy of the LMLS method in simulating a complex debris flow scenario. The example focuses on modelling the movement of a given volume of dry cohesionless soil down an inclined slope [5].

4.1 Problem setup

The computational domain consists of a planar slope inclined at an angle $\xi(x)$ that varies with the x-coordinate according to:

$$\xi(x) = \begin{cases} 35^\circ & \text{for } 0 \leq x \leq 17.5 \\ 35^\circ\left(1 - \left(\frac{x-17.5}{4}\right)^2\right) & \text{for } 17.5 < x < 21.5 \\ 0^\circ & \text{for } x \geq 21.5 \end{cases} \tag{26}$$

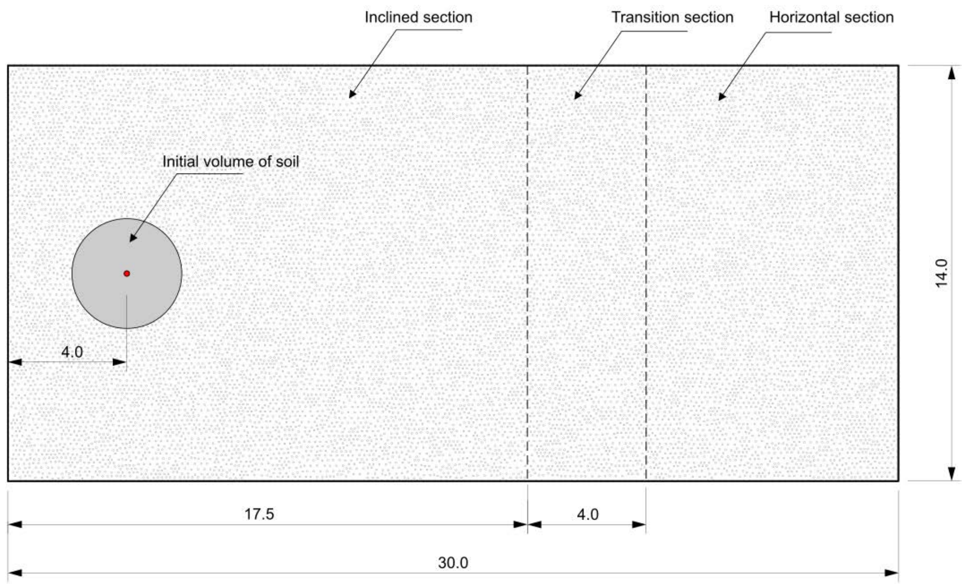

Figure 4: Dimensions of computational area, network of points and initial position of the soil material [5].

This configuration represents a steep upper slope transitioning smoothly to a horizontal runout zone, mimicking typical topographies encountered in debris flow hazard areas.

The initial soil mass is modelled as a hemisphere with radius R = 1.85, centred at coordinates $(x_0, y_0) = (4{,}0)$. The material properties are defined as follows:

- Internal friction angle: $\phi = 30°$.
- Bed friction angle: $\delta = 30°$.

4.2 Numerical implementation

We employ two distinct point distributions to discretise the computational domain [5]:

1. A regular rectangular grid comprising 10,721 points (151 × 71).
2. An irregular distribution containing 15,700 points, generated using the Poisson disc algorithm [5].

In the LMLS approximation, we use quadratic basis functions (m = 6) for both distributions. The support size for each point is adaptively determined to include approximately 50 neighbouring points. The simulation is run for a dimensionless time of T = 24.

4.3 Results and discussion

Figs 5 and 6. illustrates the evolution of the flow depth contours at dimensionless times t = 3, 6, 9, 12, 15, 18, 21 and 24 for both the regular and irregular point distributions.

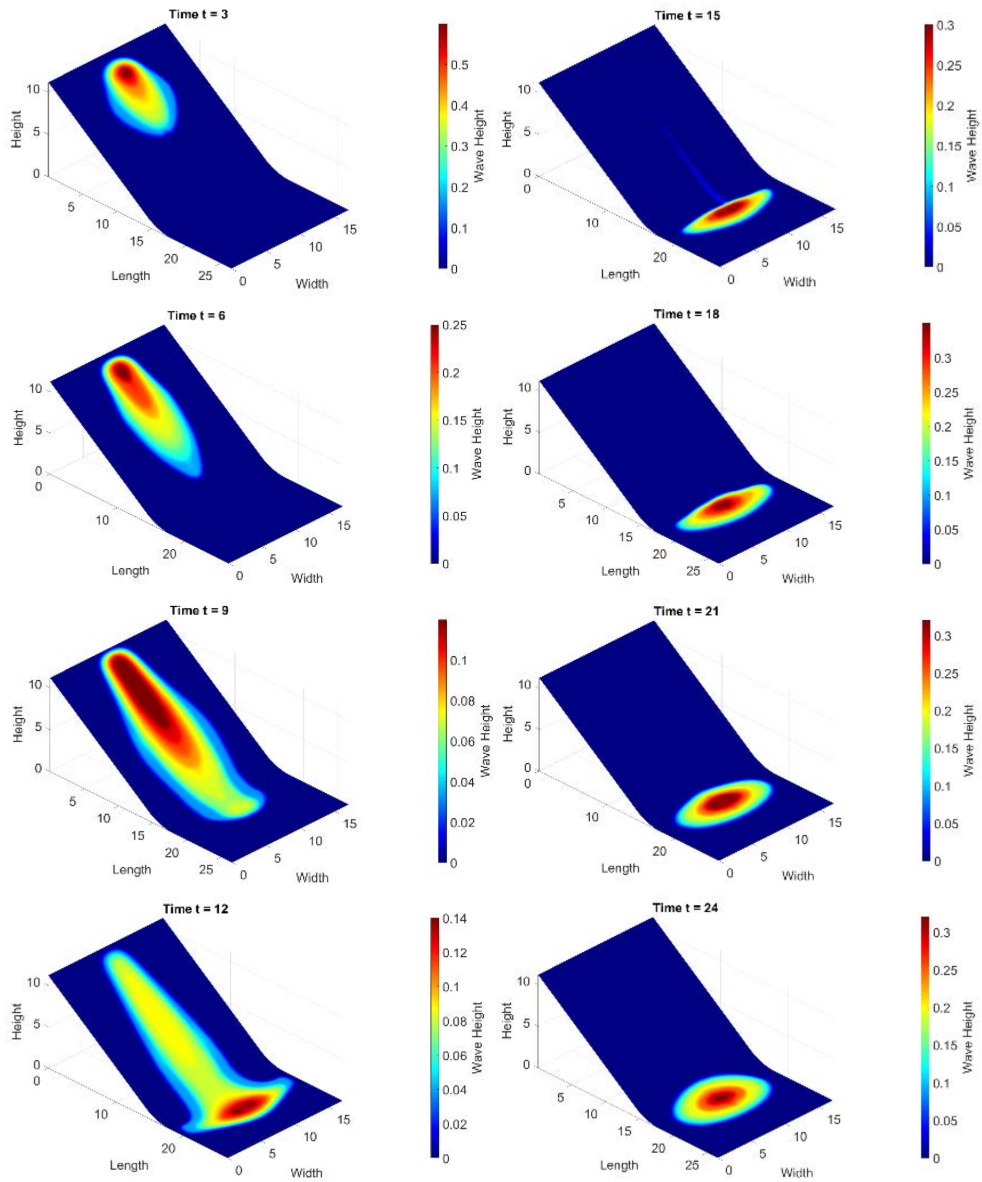

Figure 5: Regular grid, contours of depth.

Key observations include:

1. Rapid deformation and elongation of the initial hemispherical mass as it accelerates down the steep upper slope.
2. Development of a pronounced flow front, characterised by a sharp increase in flow depth.

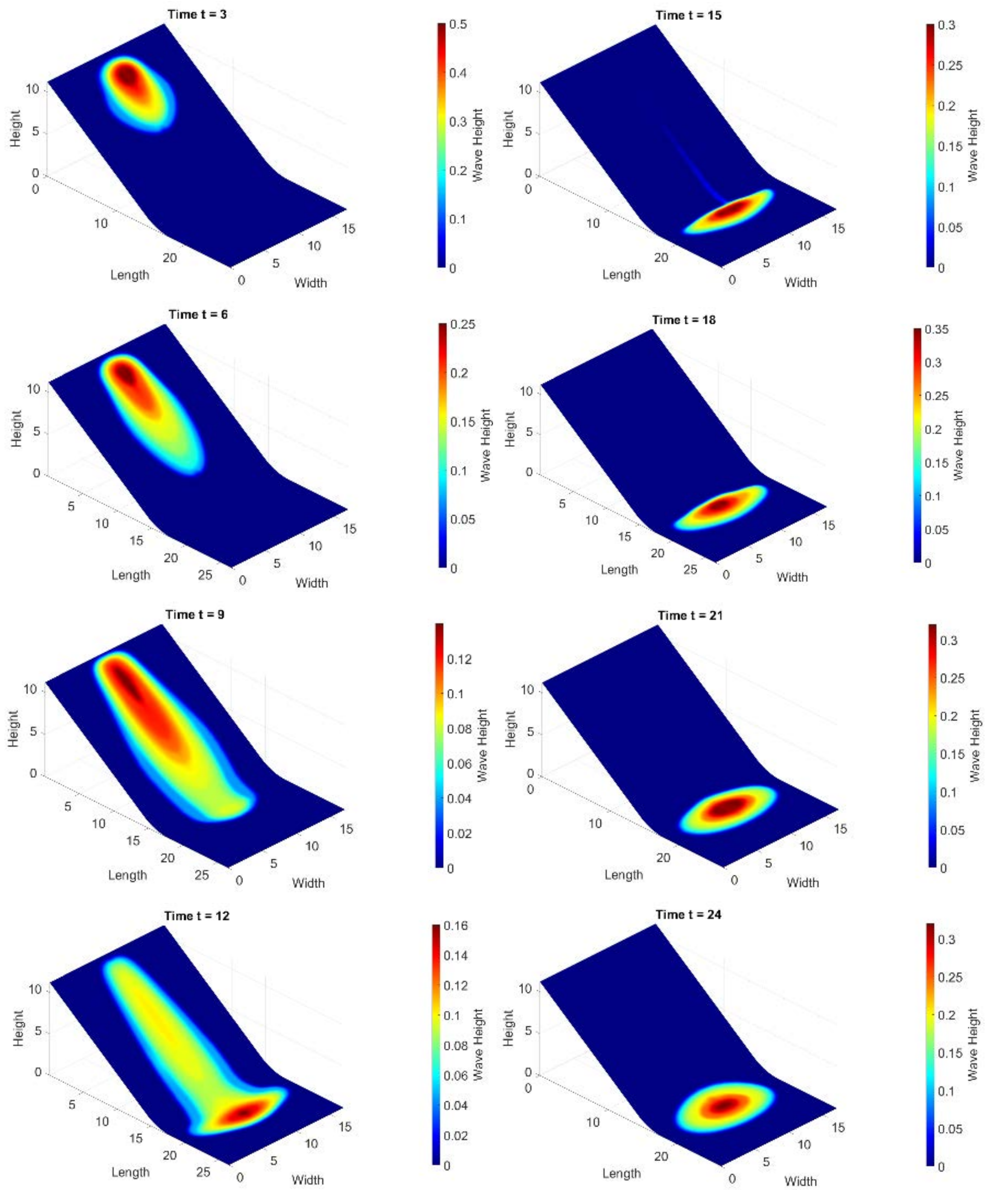

Figure 6: Irregular grid, contours of depth.

3. Deceleration and lateral spreading as the material enters the transition zone.
4. Continued spreading and gradual cessation of motion on the horizontal runout zone.

To assess the conservation properties of our numerical scheme (Fig. 7), we compute the relative error in total material volume over time:

$$\epsilon_V(t) = \frac{|V(t) - V(0)|}{V(0)} \tag{27}$$

where $V(t)$ is the total volume at time t.

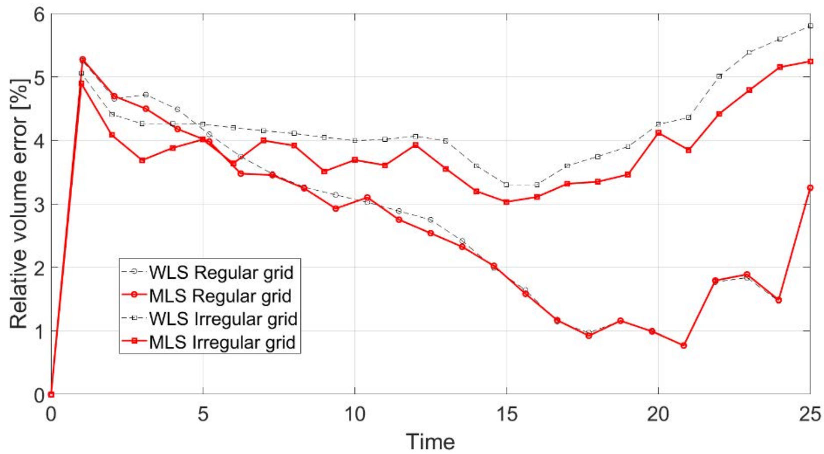

Figure 7: Course of the relative volume error over time. Comparison of MLS and WLS for regular and irregular grid.

5 CONCLUSIONS AND FUTURE WORK

This study has presented a robust numerical framework for modelling debris flows using the LMLS method. Our approach combines the extended Savage–Hutter theory for granular flows with a meshless spatial discretisation technique, offering several key advantages in simulating these complex geophysical phenomena.

The LMLS method, coupled with an RK3 time integration scheme, demonstrates excellent capability in capturing the nonlinear dynamics of debris flows. The numerical case study of dry cohesionless soil flow on an inclined slope showcases the method's ability to accurately model flow initiation, propagation, and deposition phases. Our approach exhibits strong conservation properties. The LMLS method's flexibility in handling regular and irregular point distributions is a significant advantage. This feature allows for adaptive resolution in areas of complex topography or high-flow gradients without complex remeshing algorithms. The proposed solution stabilisation process effectively mitigates numerical oscillations while preserving sharp flow features, which is essential for accurately representing the shock-like phenomena often observed in debris flows. The method's computational efficiency is demonstrated by its ability to simulate complex flows over extended periods on standard desktop hardware, making it accessible for research and practical engineering applications.

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