HIGH-ACCURATE APPROACH FOR REPEATED INTEGRALS CALCULATION FROM DOUBLE LAYER POTENTIAL WITH PIECEWISE-CONSTANT DENSITY OVER TRIANGULAR PANELS

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ABSTRACT

The algorithm of vortex particle methods of computational hydrodynamics for three-dimensional incompressible flows simulation includes the solution of a boundary integral equation (BIE) that describes vorticity generation on the body surface. Instead of the 'traditional' approach, which leads to hyper-singular integral equation, an alternative way allows for considering the BIE with weak singularity, that can be solved with acceptable accuracy by using the Galerkin approach with piecewise-constant solution representation. The kernel of the BIE is the gradient of fundamental solution of the Laplace equation. For the panels with common edge or common vertices high-accurate algorithm is developed, that is based on singularity additive exclusion and its analytical integration. The resulting formulae are written down in such a way to avoid accumulation of roundoff errors. The special cases are considered for which ambiguities arise: limit values of the corresponding terms are calculated analytically and linear expansions are presented in neighbourhood of 'ambiguous cases' with respect to small parameters which use allows for achieving high accuracy of repeated integrals computation.

Keywords: vortex particle method, boundary integral equation, Galerkin approach, repeated integral, singularity exclusion, common edge, common vertex, limit value, linear expansion.

1 INTRODUCTION

The boundary element method [1]–[3] is a powerful tool for solving a wide class of engineering problems. The main object here is boundary integral equations (BIEs), which solution makes it possible to find solutions of boundary value problems for partial differential equations of elliptic type. In this paper, the problem is considered that arises when implementing vortex particle methods [4], [5] of computational fluid dynamics when modelling the spatial flow of an incompressible medium around bodies.

Discrete analogues of BIEs are systems of linear algebraic equations with dense matrices. Despite the significant amount of research carried out by various authors concerning the construction of discrete analogues of BIEs, this issue continues to attract attention; in particular, it is of interest to construct numerical schemes based on the Galerkin approach. As numerical experiments show, schemes of such type turn out to be very accurate even with a piecewise constant representation of the solution and the use of low-quality surface meshes [6]. The main difficulty in this case is the need to calculate repeated integrals of the BIE kernel function over parts (panels) of the surface: integrals of unbounded functions that can be improper, singular or hyper-singular in cases where the panels have common points. To calculate such integrals, one can use the Taylor–Duffy approach [7], which is quite general, but the corresponding computational subroutine turns out to be quite time-consuming. The same applies to the methodology proposed in Dodig et al. [8].

The aim of this paper is to develop an algorithm for calculating with high accuracy the integrals that arise at construction of a discrete analogue of the BIE with a piecewise constant representation of the solution and a triangulated representation of the domain boundary. The

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obtained results can be applicable for the problems, where there is a need of solving the Laplace equation, and solution of the Dirichlet/Neumann problem is represented through a double/single layer potential, respectively. Similar integrals arise when a vortex sheet is considered on the body surface and its influence is calculated [9].

2 INTEGRAL CALCULATION OVER TRIANGULAR PANEL

Under assumption of piecewise-constant representation of the solution (single, double or vortex layer density) over triangulated body surface, the problem of discretisation of the governing boundary integral equation using the simplest numerical scheme – the collocation method – can be formulated as a problem of integration of the gradient of fundamental solution of the Laplace equation over triangular panel.

Such integral can be calculated exactly for arbitrary triangle K_j and arbitrary observation point M_i which position is $\mathbf{r}_i$:

$$\mathbf{J}(M_i, K_j) = \int_{K_j} \frac{\mathbf{r}_i - \xi}{4\pi |\mathbf{r}_i - \xi|^3} dS_\xi = \frac{1}{4\pi}\Big(\Theta(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c)\mathbf{n}_j + \mathbf{\Psi}(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c) \times \mathbf{n}_j\Big). \tag{1}$$

Here

$$\Theta(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c) = 2\arctan\big(\mathbf{v}'_a\mathbf{v}'_b\mathbf{v}'_c,\ 1 + (\mathbf{v}'_a \cdot \mathbf{v}'_b) + (\mathbf{v}'_b \cdot \mathbf{v}'_c) + (\mathbf{v}'_c \cdot \mathbf{v}'_a)\big),$$

$$\mathbf{\Psi}(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c) = \ln\left(\frac{|\mathbf{v}_a|}{|\mathbf{v}_b|}\frac{1+\cos\varphi_c^a}{1-\cos\varphi_c^b}\right)\boldsymbol{\tau}_c + \ln\left(\frac{|\mathbf{v}_b|}{|\mathbf{v}_c|}\frac{1+\cos\varphi_a^b}{1-\cos\varphi_a^c}\right)\boldsymbol{\tau}_a + \ln\left(\frac{|\mathbf{v}_c|}{|\mathbf{v}_a|}\frac{1+\cos\varphi_b^c}{1-\cos\varphi_b^a}\right)\boldsymbol{\tau}_b,$$

the function $\arctan(y, x)$ has values in range $(-\pi, \pi]$ and means the principal value of the argument of complex number $z = x + iy$.

The quantities included in eqn (1) have a clear geometric meaning (Fig. 1): the primes indicate the unit vectors of the corresponding vectors; $\mathbf{v}_0 = 1/3(\mathbf{v}_a + \mathbf{v}_b + \mathbf{v}_c)$; vector $\mathbf{n}_j$ is the unit vector of the normal to the plane of the triangle, that should be oriented in such a way that when observing the triangle against the direction of this vector, the sides having lengths L_a, L_b and L_c are traversed counterclockwise; $\Theta(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c)$ is a value equal in modulus to the solid angle at which the triangle is visible from the observation point, the expression for calculating the module of the solid angle is borrowed from van Oosterom and Strackee [10], while $\Theta(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c) > 0$ if the observation point lies from the panel in the direction of the normal vector $\mathbf{n}_j$, i.e., at $\mathbf{v}_0 \cdot \mathbf{n}_j > 0$; $\boldsymbol{\tau}_a$, $\boldsymbol{\tau}_b$ and $\boldsymbol{\tau}_c$ are unit vectors that correspond to the vectors directed along the sides of the triangle K_j.

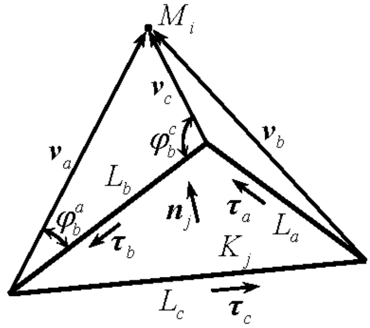

Figure 1: Observation point M_i, triangular panel K_j and all necessary angles and vectors.

Let us consider special cases when eqn (1) cannot be applied straightforwardly.

1. If the observation point M_i lies inside the influencing panel K_j, which in practical calculations, depending on the situation, can either be postulated explicitly or require checking the condition that the distance from the point to the panel is small compared to the linear size of the panel, then the expression (1) remains valid if one takes $\Theta(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c) = 0$. The limit values of the integral on both sides of the panel $\mathbf{J}^{\lim}(M_i, K_j)$ are obtained by taking $\Theta(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c) = \pm 2\pi$; the sign is determined by the side from which the observation point approaches the panel.
2. When the observation point M_i lies in the plane of the triangular panel on the continuation of one of the sides of the triangle in the direction opposite to the corresponding vector $\boldsymbol{\tau}$, i.e., when one of the following conditions is satisfied,

$$\cos\varphi_c^a = -1, \quad \cos\varphi_a^b = -1, \quad \cos\varphi_b^c = -1,$$

an ambiguity arises in the corresponding term in the expression for $\boldsymbol{\Psi}(\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c)$. In such situations, the logarithm that contains ambiguity should be replaced with its limit value:

$$\lim_{\cos\varphi_c^a \to -1} \ln\left(\frac{|\mathbf{v}_a|}{|\mathbf{v}_b|}\frac{1+\cos\varphi_c^a}{1-\cos\varphi_c^b}\right) = -\ln\frac{|\mathbf{v}_a|}{|\mathbf{v}_b|}, \quad \lim_{\cos\varphi_a^b \to -1} \ln\left(\frac{|\mathbf{v}_b|}{|\mathbf{v}_c|}\frac{1+\cos\varphi_a^b}{1-\cos\varphi_a^c}\right) = -\ln\frac{|\mathbf{v}_b|}{|\mathbf{v}_c|},$$

$$\lim_{\cos\varphi_b^c \to -1} \ln\left(\frac{|\mathbf{v}_c|}{|\mathbf{v}_a|}\frac{1+\cos\varphi_b^c}{1-\cos\varphi_b^a}\right) = -\ln\frac{|\mathbf{v}_c|}{|\mathbf{v}_a|}.$$

Note that practical calculations are inevitably accompanied by errors that are caused, at least, by the finite precision of the representation of real numbers, so equalities of the type $\cos\varphi_c^a = -1$ will not be satisfied exactly in the mathematical sense (with very rare exceptions). At the same time, operations with a value of $\cos\varphi_c^a$ close to (-1) will lead to a rapid accumulation of errors due to the subtraction of close numbers. Thus, in practice it is necessary to enter a small value of 'angular tolerance' $\varepsilon > 0$ and check the conditions

$$1 + \cos\varphi_c^a < \frac{\varepsilon^2}{2}, \qquad 1 + \cos\varphi_a^b < \frac{\varepsilon^2}{2}, \qquad 1 + \cos\varphi_b^c < \frac{\varepsilon^2}{2},$$

meaning that the angles φ_c^a, φ_a^b and φ_b^c differ from the flat angle by less than ε. For small values of ε the limit values differ from the exact ones by $O(\varepsilon^2)$.

The only case when the integral $\mathbf{J}(M_i, K_j)$ does not converge arises if the observation point M_i lies on sides of the panel K_j.

3 REPEATED INTEGRAL CALCULATION OVER TWO TRIANGULAR PANELS

First of all, let us note, that it is impossible to obtain exact analytical expressions in closed form for repeated integrals form the gradient of Newtonian potential over two triangular panels. Numerical integration over the control panel K_i using Gaussian (or some other) quadrature rules [11] can be done only for the cases when the panels K_i and K_j do not have common points: the results of calculating the internal integral eqn (1) is now a bounded function together with their derivatives with respect to spatial coordinates, therefore, high accuracy of calculating the external integral normally can be achieved when using 'standart' quadrature rules, even with a relatively small number of nodes.

If the panels K_i and K_j have a common edge or vertex, the internal integral, i.e., the function $\mathbf{J}(M_i, K_j)$, turns out to be unbounded when approaching the common points, and the external integral becomes improper. To calculate it with high accuracy, it is suggested to perform singularity extraction in the integrand, i.e., to write it down as a sum

$$\mathbf{J}(M_i, K_j) = \mathbf{J}^{\text{reg}}(M_i, K_j) + \mathbf{J}^{\text{sing}}(M_i, K_j),$$

where the singular part includes the singular parts of both functions Θ and $\mathbf{\Psi}$:

$$\mathbf{J}^{\text{sing}}(M_i, K_j) = \frac{1}{4\pi}\left(\Theta^{\text{sing}}(M_i, K_j)\mathbf{n}_j + \mathbf{\Psi}^{\text{sing}}(M_i, K_j) \times \mathbf{n}_j\right).$$

Note, that $\Theta(M_i, K_j)$ is always bounded, but has unbounded first derivations which, in fact, are 'isolated' in singular part $\Theta^{\text{sing}}(M_i, K_j)$. Then the regular part (dependencies on the position of observation point M_i and panel K_j are omitted)

$$\mathbf{J}^{\text{reg}} = \frac{1}{4\pi}\left((\Theta - \Theta^{\text{sing}})\mathbf{n}_j + (\mathbf{\Psi} - \mathbf{\Psi}^{\text{sing}}) \times \mathbf{n}_j\right),$$

and also its spatial derivatives turn out to be smooth bounded functions, so their integration can be performed numerically, if no one of the nodes of the used quadrature rule coincide with the common points of the considered panels. It introduces a non-burdensome restriction for the quadrature rule.

For integrals from singular parts of the corresponding functions

$$\int_{K_i} \Theta^{\text{sing}} dS_r, \qquad \int_{K_i} \mathbf{\Psi}^{\text{sing}} dS_r$$

it is possible to derive exact expressions in closed form, which differ significantly for the cases of panels with a common edge and a common vertex.

Note that if the control and influencing panels coincide ($i = j$), then the repeated integral becomes singular; its principal value in the Cauchy sense is equal to zero.

3.1 Panels with common edge

Let us consider the control panel K_i and the influencing panel K_j with common edge, which directing unit vector is $\boldsymbol{\tau}_c$ (Fig. 2).

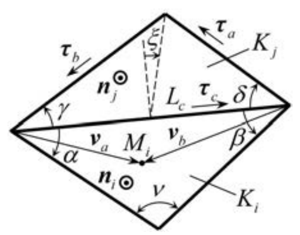

Figure 2: Two panels with common edge.

The expressions for singular parts of the inner integral $\mathbf{J}(M_i, K_j)$ are the following:

$$\Theta^{\text{sing}}(M_i, K_j) = 2\Big(\arctan\big(\mathbf{v}'_a\boldsymbol{\tau}_b\boldsymbol{\tau}_c,\ (\boldsymbol{\tau}_b - \boldsymbol{\tau}_c)\cdot(\boldsymbol{\tau}_b + \mathbf{v}'_a)\big) - \arctan\big(\mathbf{v}'_b\boldsymbol{\tau}_a\boldsymbol{\tau}_c,\ (\boldsymbol{\tau}_a - \boldsymbol{\tau}_c)\cdot(\boldsymbol{\tau}_a - \mathbf{v}'_b)\big)\Big),$$

$$\mathbf{\Psi}^{\text{sing}}(M_i, K_j) = \boldsymbol{\tau}_c \ln\left(\frac{|\mathbf{v}_b|}{|\mathbf{v}_a|}\frac{\boldsymbol{\tau}_c \cdot (\boldsymbol{\tau}_c - \mathbf{v}'_b)}{\boldsymbol{\tau}_c \cdot (\boldsymbol{\tau}_c - \mathbf{v}'_a)}\right) - \boldsymbol{\tau}_b \ln\left(\frac{|\mathbf{v}_a|}{L_c}\boldsymbol{\tau}_b \cdot (\boldsymbol{\tau}_b + \mathbf{v}'_a)\right)$$
$$-\boldsymbol{\tau}_a \ln\left(\frac{|v_b|}{L_c}\boldsymbol{\tau}_a \cdot (\boldsymbol{\tau}_a - \mathbf{v}'_b)\right).$$

The expression for Θ^{sing} as well as scalar multipliers for unit vectors $\boldsymbol{\tau}_p$ in expression $\boldsymbol{\Psi}^{\text{sing}}$ can be integrated analytically in closed form over the control panel K_i:

$$\int_{K_i} \Theta^{\text{sing}}(M_i, K_j) dS_r = S_i(q_\Theta(\xi,\alpha,\beta,\gamma,\mu,\lambda) + q_\Theta(\xi,\beta,\alpha,\delta,\sigma,\theta)), \quad (2)$$

$$\int_{K_i} \boldsymbol{\Psi}^{\text{sing}}(M_i, K_j) dS_r = S_i(q_\Psi(\xi,\alpha,\beta,\mu,\gamma,\lambda)\boldsymbol{\tau}_b$$
$$+q_\Psi(\xi,\beta,\alpha,\sigma,\delta,\theta)\boldsymbol{\tau}_a - q_{\alpha\beta}(\alpha,\beta)\boldsymbol{\tau}_c). \quad (3)$$

Hereinafter α and β mean the angles of the control panel K_i adjuscent to the common edge; $\nu = \pi - \alpha - \beta$ is the opposite angle; γ and δ mean the angles of the influencing panel K_j, that are adjuscent to the angles α and β, respectively; S_i is the area of the panel K_i; ξ is the angle between the planes of the panels K_i and K_j, that can have both positive and negative values (it is measured as shown in Fig. 2):

$$\xi = \arctan(\mathbf{n}_i\mathbf{n}_j\boldsymbol{\tau}_c, \mathbf{n}_i \cdot \mathbf{n}_j),$$

the rest angles are defined as following:

$$\sigma = \pi - \arccos(\cos\alpha\cos\delta + \cos\xi\sin\alpha\sin\delta),$$
$$\mu = \pi - \arccos(\cos\beta\cos\gamma + \cos\xi\sin\beta\sin\gamma),$$
$$\lambda = \pi - \arccos(\cos\alpha\cos\gamma - \cos\xi\sin\alpha\sin\gamma),$$
$$\theta = \pi - \arccos(\cos\beta\cos\delta - \cos\xi\sin\beta\sin\delta),$$

which can be interpreted as a formulation of the cosine theorem for trihedral angles with the corresponding plane and dihedral angles.

The auxiliary functions q_Θ, q_Ψ and $q_{\alpha\beta}$, that depend only on the above mentioned angles, have the following expressions:

$$q_\Theta(\xi,\alpha,\beta,\gamma,\mu,\lambda) = \phi(\xi,\alpha,\gamma,\lambda) +$$
$$+\frac{\sin\gamma\sin\nu}{\sin\alpha(1-\cos^2\mu)}\Big((\cos\beta\sin\gamma - \cos\xi\sin\beta\cos\gamma)\phi(\xi,\pi-\alpha,\pi-\gamma,\lambda) +$$
$$+\frac{1}{2}\sin\xi\sin\beta\left((1+\cos\mu)\ln\frac{1+\cos\beta}{1-\cos\nu} + (1-\cos\mu)\ln\frac{1-\cos\beta}{1+\cos\nu} + 2\ln\frac{1+\cos\lambda}{1-\cos\gamma}\right)\Big), \quad (4)$$

$$q_\Psi(\xi,\alpha,\beta,\gamma,\mu,\lambda) = \frac{3}{2} - \frac{1}{\sin\alpha(1-\cos^2\mu)}\Big(\sin\beta(\cos\nu + \cos\mu\cos\lambda)\ln(1+\cos\lambda) +$$
$$+\sin\nu(\cos\beta + \cos\mu\cos\gamma)\ln(1-\cos\gamma) + \sin\beta(1-\cos\mu)(\cos\nu - \cos\lambda)\ln\frac{\sin\beta}{\sin\nu} +$$
$$+\sin\nu\sin\beta(\sin\beta\cos\gamma - \cos\xi\sin\gamma\cos\beta)\,ln\frac{1-\cos\nu}{1+\cos\beta} +$$
$$+\phi(\xi,\pi-\alpha,\pi-\gamma,\lambda)\sin\xi\sin\gamma\sin\nu\sin\beta\Big),$$

$$q_{\alpha\beta}(\alpha,\beta) = \frac{\sin\nu}{\sin\beta}\ln\left(\frac{\alpha}{2}\frac{\nu}{2}\right) + \frac{\sin\nu}{\sin\alpha}\ln\left(\frac{\beta}{2}\frac{\nu}{2}\right) + \ln\left(\frac{\alpha}{2}\frac{\beta}{2}\right), \quad (5)$$

where ϕ is equal to

$$\phi(\xi, \alpha, \gamma, \lambda) = 2 \arctan(\sin\xi \sin\alpha \sin\gamma, 1 - \cos\alpha + \cos\gamma + \cos\lambda).$$

The above formulae are applicable for arbitrary mutual arrangement of triangular panels K_i and K_j, except some special case: if they lie in the same plane ($\xi = 0$) and one of the conditions $\beta = \gamma$ or $\alpha = \delta$ is satisfied, in the above formulae for q_Θ and q_Ψ an ambiguity of the form [0/0] arises. Note that the same ambiguity also arises at $|\xi| \approx \pi$, however, let us do not consider this case, since it would correspond to an infinitely thin surface.

This problem can be solved by replacing the terms with ambiguities with their limit values, however, such approach in the general case does not allow for high accuracy achieving in calculating integrals (if one has in mind computations with double precision, typical for engineering practice, then it seems consistent to consider 'high accuracy' mean a relative error at the level not higher than 10^{-8}). The latter is due to the fact that in practice, instead of the conditions $\xi = \gamma - \beta = 0$ or $\xi = \delta - \alpha = 0$, it is necessary to check 'weaker' conditions

$$|\xi| < \varepsilon \;\wedge\; |\gamma - \beta| < \varepsilon \quad \text{or} \quad |\xi| < \varepsilon \;\wedge\; |\delta - \alpha| < \varepsilon .$$

As a small parameter ε, it is impossible to take a value that is too small (for example, close to machine epsilon, about 10^{-14} or 10^{-15}): in this case, even if the above 'criterion' for compliance with a special case is not met, in calculations being performed according to general formulae, eqns (2)–(5), problems of error accumulation will arise when subtracting close numbers. A numerical experiment shows that normally suitable are values no less than $\varepsilon = 10^{-6}$. Thus, the error of replacing of true values with the limit ones will be $O(\varepsilon)$, which, taking into account the above considerations, does not guarantee high accuracy in calculating the corresponding integrals.

This issue can be overcome by replacing of the expressions containing ambiguities not with their limit values [12], but with linear expansions, so that the resulting error is $O(\varepsilon^2)$. Expanding the expressions (4) and (5) into series with respect to small parameters up to linear terms and calculating the limit values of the coefficients, one can finally obtain

$$q_\Theta(\alpha, \beta, \mu, \gamma, \lambda) = \frac{\xi}{4}\left(5 + \cos 2\beta - 2\frac{\sin\frac{(\alpha-\beta)}{2}}{\cos\frac{\nu}{2}} - 2\frac{\sin^2\beta \sin\nu}{\sin\alpha} \ln\left(\tan\frac{\nu}{2} \tan\frac{\beta}{2}\right) - \sin 2\beta \tan\frac{\alpha}{2}\right),$$

$$q_\Psi(\alpha, \beta, \mu, \gamma, \lambda) = \frac{1}{2} - \ln 2 - 2\ln\left(\sin\frac{\beta}{2}\right) + \frac{\sin\beta - \sin\nu}{\sin\alpha} + \frac{\cos\nu \sin\beta}{\sin\alpha} \ln\left(\tan\frac{\beta}{2} \tan\frac{\nu}{2}\right) +$$
$$+ \frac{\gamma-\beta}{24 \sin\alpha \sin\beta}\left(\sin\nu\left((3\cos 2\beta - 1)\ln(1 + \cos\nu) + 4\ln\frac{\sin\beta}{\sin\nu}\right) - 24\cos^2\frac{\beta}{2} \cos\nu \tan\frac{\nu}{2} + \right.$$
$$\left. + \sin\nu\left(4\ln\frac{1-\cos\nu}{1+\cos\beta} - (3\cos 2\beta - 1)\ln\left(2\tan^2\frac{\beta}{2}\sin^2\frac{\nu}{2}\right)\right)\right).$$

3.2 Panels with common vertex

Now let us consider the control panel K_i and influencing panel K_j with common vertex (Fig. 3). All auxiliary vectors and angles are also introduced in Fig. 3.

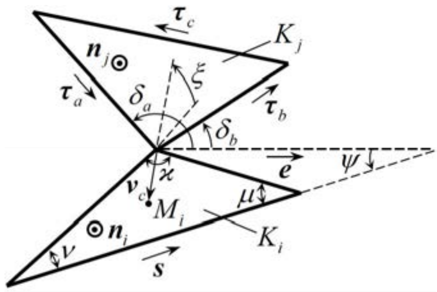

Figure 3: Two panels with common edge.

The unit direction vector for the line of intersection of the planes of the panels, denoted as $\boldsymbol{e}$, is chosen codirected to the vector product of the normal vectors to the panels $\mathbf{n}_i \times \mathbf{n}_j$. If the panels K_i and K_j lie in the same plane, i.e., at $\mathbf{n}_i \times \mathbf{n}_j = \mathbf{0}$ (in practice, for $|\mathbf{n}_i \times \mathbf{n}_j| < \varepsilon^2$), the direction of the unit vector $\mathbf{e}$ can be chosen arbitrarily; for clarity, let us take $\mathbf{e} = \boldsymbol{\tau}_b$. After introducing the unit vector $\mathbf{e}$, the orientation of the panel K_j is fully determined by the angles δ_a and δ_b between its sides and the vector $\mathbf{e}$ (Fig. 3):

$$\delta_a = \arctan(-\mathbf{e}\boldsymbol{\tau}_a\mathbf{n}_j, -\mathbf{e}\cdot\boldsymbol{\tau}_a), \qquad \delta_b = \arctan(\mathbf{e}\boldsymbol{\tau}_b\mathbf{n}_j, \mathbf{e}\cdot\boldsymbol{\tau}_b).$$

If at least one of the angles δ_a or δ_b turns out to be close to $\pm\pi$, that in practice means checking the condition

$$\pi - |\delta_a| < \varepsilon \quad \vee \quad \pi - |\delta_b| < \varepsilon,$$

or, what is the same,

$$1 + \cos\delta_a < \frac{\varepsilon^2}{2} \quad \vee \quad 1 + \cos\delta_b < \frac{\varepsilon^2}{2},$$

it is convenient to change the direction of the unit vector $\mathbf{e}$ to the opposite, in this case the angle will become close to zero; the value of the second angle should be recalculated. The same operation should be performed when the following condition is met:

$$\delta_a \cdot \delta_b < 0 \quad \wedge \quad |\delta_a - \delta_b| > \pi.$$

After the direction of the unit vector $\boldsymbol{e}$ is finally determined, the angle ξ between the planes of triangular panels K_i and K_j can be calculated:

$$\xi = \arctan(\mathbf{n}_i\mathbf{n}_j\mathbf{e},\ \mathbf{n}_i \cdot \mathbf{n}_j).$$

Positive or negative sign of the angle ξ determines mutual orientation of the panels.

Then the expressions for the singular parts of the integral $\mathbf{J}(M_i, K_j)$ can be written in the following form:

$$\Theta^{\text{sing}}(M_i, K_j) = 2\Big(\arctan\big(\mathbf{v}'_c\boldsymbol{\tau}_a\mathbf{e},\ (\mathbf{e}-\mathbf{v}'_c)\cdot(\mathbf{e}-\boldsymbol{\tau}_a)\big) + \arctan\big(\mathbf{v}'_c\boldsymbol{\tau}_b\mathbf{e},\ (\mathbf{e}-\mathbf{v}'_c)\cdot(\mathbf{e}+\boldsymbol{\tau}_b)\big)\Big),$$

$$\boldsymbol{\Psi}^{\text{sing}}(M_i, K_j) = -\left(\boldsymbol{\tau}_a \ln\left(\frac{|\mathbf{v}_c|}{\sqrt{S_i}}(1+\boldsymbol{\tau}_a\cdot\mathbf{v}'_c)\right) + \boldsymbol{\tau}_b \ln\left(\frac{|\mathbf{v}_c|}{\sqrt{S_i}}(1-\boldsymbol{\tau}_b\cdot\mathbf{v}'_c)\right)\right).$$

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Here, as earlier, S_i means the area of the panel K_i; the prime means the unit vector of the corresponding vector.

As a result, the singular part Θ^{sing} and the multipliers for the vectors $\boldsymbol{\tau}_a$, $\boldsymbol{\tau}_b$ in the expression for $\boldsymbol{\Psi}^{\text{sing}}$ can be integrated analytically over the control panel K_i:

$$\int_{K_i} \Theta^{\text{sing}}(M_i, K_j) dS_r = S_i(q^{\Theta}(\delta_a) - q^{\Theta}(\delta_b) + 4\pi p), \tag{6}$$

$$\int_{K_i} \boldsymbol{\Psi}^{\text{sing}}(M_i, K_j) dS_r = S_i(q^{\Psi}(\delta_a)\boldsymbol{\tau}_a + q^{\Psi}(\delta_b)\boldsymbol{\tau}_b), \tag{7}$$

where p is an integer, the meaning of which will be explained below; expressions for the functions q^{Θ} and q^{Ψ} are the following:

$$q^{\Theta}(\delta) = \frac{2}{\sin\text{ù}\ \sin\psi}\Bigg(A_\nu \sin\mu \sin(\nu+\psi) - A_\mu \sin\nu \sin(\mu-\psi) - \\ -\frac{1}{D}\sin\mu \sin\nu \sin\delta \left(W\cos\eta + \frac{\sin\psi \sin\xi}{2}(\Lambda_1 - \Lambda_2\cos\sigma)\right)\Bigg), \tag{8}$$

$$q^{\Psi}(\delta) = \frac{3-\ln 2}{2} + \frac{\sin\mu \sin\nu}{\sin\text{ù}}\Bigg(\ln\frac{1+\cos\lambda}{1+\cos\theta}\cot\psi + \\ + \frac{1}{D}\left(\Lambda_1 \frac{\sin\delta\cos\eta}{\sin\psi} + \Lambda_2\cos\chi - 2W\sin\delta\sin\xi - G\ln\frac{\sin\nu}{\sin\mu}\right)\Bigg) - \\ -\frac{1}{\sin\text{ù}}\left(\sin\mu\cos\nu\ln\frac{1+\cos\theta}{\sin\nu} + \cos\mu\sin\nu\ln\frac{1+\cos\lambda}{\sin\mu}\right) - \frac{1}{2}\ln\frac{\sin\mu\sin\nu}{\sin\text{ù}}. \tag{9}$$

Additional notations introduced for simplicity are the following:

$$A_\mu = \arctan\left(\tan\frac{\delta}{2}\cos\frac{\mu-\psi}{2}\sin\xi,\ \tan\frac{\delta}{2}\cos\frac{\mu-\psi}{2}\cos\xi + \sin\frac{\mu-\psi}{2}\right),$$

$$A_\nu = \arctan\left(\tan\frac{\delta}{2}\sin\frac{\nu+\psi}{2}\sin\xi,\ \tan\frac{\delta}{2}\sin\frac{\nu+\psi}{2}\cos\xi + \cos\frac{\nu+\psi}{2}\right),$$

$$W = \arctan\Bigg(\sin\delta\sin\frac{\text{ù}}{2}\sin\xi,\ \cos\frac{\text{ù}}{2} + \sin\delta\cos\left(\frac{\mu-\psi}{2} - \frac{\nu+\psi}{2}\right)\cos\xi + \\ + \cos\delta\sin\left(\frac{\mu-\psi}{2} - \frac{\nu+\psi}{2}\right)\Bigg),$$

$$D = \sin^2(\delta-\psi) + \sin\delta\sin\psi(1-\cos\xi)\big(\cos(\delta-\psi) + \cos\sigma\big),$$

$$G = \cos\psi\left(\sin\delta\cos\sigma\cos\xi + \cos\delta\cos\chi - \frac{\sin^2\delta}{\sin\psi}\right),$$

$$\Lambda_1 = \ln\left(\frac{1+\cos\lambda}{1+\cos\theta}\frac{\sin\nu}{\sin\mu}\right), \quad \Lambda_2 = \ln\left(\tan\frac{\nu}{2}\tan\frac{\mu}{2}\right).$$

Let us explain the meaning of the quantities in the presented formulae. The angle ù is the angle of the control panel K_i at the vertex common with the influencing panel K_j; μ and ν are the other angles of the panel K_i (Fig. 3); ψ is the angle between the vector **e** and the directive vector **s** of the side of the panel K_i, which lies opposite the common vertex:

$$\psi = \arctan(\mathbf{e}\mathbf{s}\mathbf{n}_i,\ \mathbf{e}\cdot\mathbf{s});$$

for $\cos\sigma$, $\cos\chi$, $\cos\eta$, $\cos\theta$ and $\cos\lambda$ the following relations are valid:

$$\cos\sigma = \sin\delta\sin\psi\cos\xi + \cos\delta\cos\psi,$$

$$\cos\chi = \sin\delta\cos\psi\cos\xi - \cos\delta\sin\psi,$$

$$\cos\eta = \cos\delta\sin\psi\cos\xi - \sin\delta\cos\psi,$$

$$\cos\theta = \sin\delta\sin(\nu+\psi)\cos\xi + \cos\delta\cos(\nu+\psi),$$

$$\cos\lambda = \sin\delta\sin(\mu-\psi)\cos\xi - \cos\delta\cos(\mu-\psi).$$

The function $\Theta^{\text{sing}}(M_i, K_j)$ is given by the sum of two arctangents, so it is necessary to monitor the continuity of their branches by appropriate choice the value of the integer parameter p, however, the corresponding conditions are very cumbersome and inconvenient to check in practice. Instead, one can take into account the geometric sense of the function $\Theta(M_i, K_j)$: its absolute value is equal to the value of the solid angle at which the influencing panel K_j is visible from the observation point M_i. Thus, its magnitude cannot exceed 2π. Guided by this, after calculating the sum of two integrals

$$\int_{K_i}\Theta(M_i, K_j)dS_r = \int_{K_i}\Theta^{\text{sing}}dS_r + \int_{K_i}(\Theta - \Theta^{\text{sing}})dS_r,$$

where the first term is calculated using the presented analytical formulae, and the second numerically (since the integrand is a smooth function), it is necessary to choose a value of the parameter p for which

$$-2\pi S_i \text{ „ } \int_{K_i}\Theta(M_i, K_j)dS_r \text{ „ } 2\pi S_i.$$

If the control and influencing panels do not have common internal points, this can be done in the only way.

The presented formulae for the integrals of $\Theta^{\text{sing}}(M_i, K_j)$and $\mathbf{\Psi}^{\text{sing}}(M_i, K_j)$ are valid for arbitrary relative positions of panels with a common vertex, except several special cases that arise for zero or close to zero values of $\sin\psi$, $\sin\delta$, $\sin\xi$ and some other quantities included in these expressions. According to the choice of the direction of the vector **e**, excluding the possibility of $\delta \approx \pm\pi$, one obtains that the condition $\sin\delta \approx 0$ is equivalent to the condition $\delta \approx 0$, while $\sin\psi$ and $\sin\xi$ vanish both at zero angles and at cases when the angles are equal to $\pm\pi$.

In the cases considered below, when using the given 'general' formulae, ambiguities of the type [0/0] or $[\infty - \infty]$ arise. Let us present expressions for calculating the corresponding quantities, valid in the vicinity of these special cases and written in the form of expansions with respect to small parameter(s) with taking into account constant and linear terms. The upper asterisk index '*' indicates the limit value of the corresponding parameter.

1. Let the consider the case when $|\sin\xi| < \varepsilon$, $1 - |\cos\sigma| < \varepsilon^2/2$, and at the same time $|\sin\psi| \geq \varepsilon$. The small quantities here are $(\xi - \xi^*)$ and $(\delta - \cos(\xi^*)\psi - \Delta^*)$, where ξ^* and Δ^* can take values 0 or $\pm\pi$. Calculating all the necessary limits, one obtains:

$$q^{\Theta}(\delta) = \frac{\pi}{2}(1-\cos\sigma)\left(1-\operatorname{sign}(\delta^*)\cos\xi^*\right)+$$

$$+(\xi-\xi^*)\left(1-\cos\sigma^*\left(\cos\psi+\frac{1}{2}\sin\psi(\cot\nu-\cot\mu)\right)+C\frac{\sin\psi}{2\sin\grave{u}}\right),$$

$$q^{\Psi}(\delta) = \frac{1}{2}\left(1 - ln\frac{2\sin\mu\sin\nu}{\sin\quad\varkappa}\right)+$$

$$+\frac{\cos\sigma^*}{\sin\grave{u}}\left(\left(1+\cos\nu\ln\left(\tan\frac{\nu}{2}\right)\right)\sin\mu-\left(1+\cos\mu\ln\left(\tan\frac{\mu}{2}\right)\right)\sin\nu\right)+$$

$$+\frac{1}{2}\left(\cos\xi^*(\delta-\Delta^*)-\psi\right)\left(\cot\nu-\cot\mu-C\frac{\cos\sigma^*}{\sin\grave{u}}\right),$$

where $C = \cot\mu\sin\nu + \cot\nu\sin\mu - \sin\mu\sin\nu\ln\left(\tan\frac{\mu}{2}\tan\frac{\nu}{2}\right)$.

2. If $|\sin\xi| < \varepsilon$, but at the same time $1 - |\cos\sigma| \geq \frac{\varepsilon^2}{2}$ and $|\sin\psi| \geq \varepsilon$, then one small parameter arises, namely $(\xi - \xi^*)$, where, as earlier, ξ^* can take the values 0 and $\pm\pi$. The expansion of the function q^{Θ}, which contains ambiguity, has the form

$$q^{\Theta}(\delta) = \frac{2}{\sin\grave{u}\ \sin\psi}\left(\arg\left(\operatorname{sign}(c_\varsigma s_\varsigma)\right)\frac{\sin\left((\cos\xi^*)\delta\right)\sin\mu\sin\nu}{\sin\varsigma}-\right.$$

$$\left.-\arg(\operatorname{sign}s_\varsigma)\sin\nu\sin(\mu-\psi)+\arg(\operatorname{sign}c_\varsigma)\sin\mu\sin(\nu+\psi)\right)+$$

$$+2(\xi-\xi^*)\frac{\cos\xi^*}{\sin\grave{u}\ \sin\psi}\sin\frac{\delta}{2}\left(\frac{\sin\mu\sin(\nu+\psi)}{c_\varsigma}\sin\frac{\nu+\psi}{2}-\frac{\sin\nu\sin(\mu-\psi)}{s_\varsigma}\cos\frac{\mu-\psi}{2}+\right.$$

$$\left.+\frac{\sin\mu\sin\nu}{\sin\varsigma}\cos\frac{\delta}{2}\left(\frac{\cos\xi^*\sin\delta}{c_\varsigma s_\varsigma}\cos\frac{\grave{u}}{2}-\frac{\sin\psi}{\sin\varsigma}\left(\ln\frac{s_\varsigma^2\sin\nu}{c_\varsigma^2\sin\mu}-\cos\varsigma\ln\left(\tan\frac{\mu}{2}\tan\frac{\nu}{2}\right)\right)\right)\right).$$

Here, for brevity, the notations are used:

$$\varsigma = (\cos\xi^*)\delta-\psi\ ,\quad c_\varsigma = \cos\frac{\varsigma-\nu}{2}\ ,\quad s_\varsigma = \sin\frac{\varsigma+\mu}{2}\ ;$$

the function $\arg(z) = \arctan(\operatorname{Im}z, \operatorname{Re}z)$ means the principal value of the argument of the corresponding complex number.
Note that the function $q^{\Psi}(\delta)$ in this case does not contain ambiguity and can be calculated using the general formula, eqn (9).

3. If $|\sin\psi| < \varepsilon$, but $|\sin\delta| \geq \varepsilon$, then the expansions with respect to the small parameter $(\psi - \psi^*)$, where $\psi^* = 0$ or $\psi^* = \pm\pi$, up to linear terms, have the form

$$q^{\Theta}(\delta) = -\frac{\pi}{2}(1-\cos\xi)\left(1-\operatorname{sign}\psi^*\right)+\frac{2}{\sin\grave{u}}\left(T_\nu\sin\mu\cos\nu+T_\mu\sin\nu\cos\mu+\right.$$

$$+\frac{\sin\mu\sin\nu}{2\sin\delta}\Big(2T\cos\delta\cos\xi+(L_f+L_p\cos\delta)\cos\psi^*\sin\xi\Big)\Bigg)+$$

$$+\left(\psi-\psi^*\right)\left(\frac{\sin\xi}{\sin\delta}\left(\cos\delta\frac{\sin\nu-\sin\mu}{\sin\grave{u}}+1\right)+\right.$$

$$\left.+\frac{\sin\mu\sin\nu}{\sin\grave{u}\,\sin^2\delta}\left(T\Big((1+\cos^2\delta)\cos 2\xi+\sin^2\delta\Big)+\cos\psi^*\sin 2\delta\left(L_f\cos\delta+L_p\frac{1+\cos^2\delta}{2}\right)\right)\right)\Bigg),$$

$$q^{\Psi}(\delta)=\frac{3}{2}-\frac{1}{2}\ln\frac{2\sin\mu\sin\nu}{\sin\grave{u}}-\frac{1}{\sin\grave{u}}\Bigg(\cos\nu\sin\mu\ln\frac{A}{\sin\nu}+\sin\nu\cos\mu\ln\frac{B}{\sin\mu}-$$

$$-\frac{\sin\mu\sin\nu}{\sin\delta}\Big((L_f\cos\delta+L_p)\cos\psi^*\cos\xi-2T\sin\xi\Big)\Bigg)+$$

$$+\left(\psi-\psi^*\right)\left(\frac{\cos\xi}{\sin\delta}\left(\cos\delta+\frac{\sin\nu-\sin\mu}{\sin\grave{u}}\right)+\right.$$

$$\left.+\frac{\sin\mu\sin\nu}{\sin\grave{u}\,\mathrm{ssin}^2\delta}\Big(L_f\,\big((1+\cos^2\delta)\cos 2\xi+\sin^2\delta\big)+\cos\delta\big(L_p\cos 2\xi\cos\psi^*-2T\sin 2\xi\big)\Big)\right).$$

The following notations are used here:

$$A=1+\cos\delta\cos\nu+\cos\psi^*\cos\xi\sin\delta\sin\nu,\quad B=1-\cos\delta\cos\mu+\cos\psi^*\cos\xi\sin\delta\sin\mu,$$

$$T_\nu=\arctan\left(\sin\xi\sin\nu\tan\frac{\delta}{2},\ (1+\cos\nu)\cos\psi^*+\cos\xi\sin\nu\tan\frac{\delta}{2}\right),$$

$$T_\mu=\arctan\left(\sin\xi\sin\mu\tan\frac{\delta}{2},\ (1-\cos\mu)\cos\psi^*+\cos\xi\sin\mu\tan\frac{\delta}{2}\right),$$

$$T=\arctan(\sin\grave{u}\ \sin\delta\sin\xi,\ A+B+\cos\grave{u}\ +1),$$

$$L_f=\ln\frac{A\sin\mu}{B\sin\nu},\quad L_p=\ln\left(\tan\frac{\mu}{2}\tan\frac{\nu}{2}\right).$$

4. When $|\sin\psi|<\varepsilon$ and $|\sin\delta|<\varepsilon$ the expression $q^{\Theta}(\delta)$ depends only on the small parameter $(\psi-\psi^*)$, where, as earlier, $\psi^*=0$ or $\psi^*=\pm\pi$:

$$q^{\Theta}(\delta)=\frac{\pi}{2}\left(1-\cos\psi^*\right)(1+\mathrm{sign}\,\psi^*)+\left(\psi-\psi^*\right)S\sin\xi.$$

In this case, the expansion of $q^{\Psi}(\delta)$ also contains a term proportional to the small parameter δ:

$$q^{\Psi}(\delta)=\frac{1}{2}\Bigg(\mathrm{sign}\left(\cos\psi^*\right)\left(\ln\frac{\tan\nu/2}{\tan\mu/2}+\frac{\sin(\mu-\nu)}{\sin\grave{u}}\ln\left(\tan\frac{\mu}{2}\tan\frac{\nu}{2}\right)-2\frac{\sin\nu-\sin\mu}{\sin\grave{u}}\right)+$$

$$+\ln(\cot\nu+\cot\mu)+1-\ln 2\Bigg)+\Big(\delta-\left(\psi-\psi^*\right)\cos\xi\Big)S,$$

where $S=\dfrac{\cos\psi^*}{2\sin\grave{u}}\left((\cos\mu+\cos\nu)\dfrac{\tan\nu/2}{\tan\mu/2}-\sin\mu\sin\nu\ln\left(\tan\dfrac{\mu}{2}\tan\dfrac{\nu}{2}\right)\right).$

4 CONCLUSION

In the present paper, analytical expressions convenient for use in practical computations are obtained for integrating the gradient of the fundamental solution of the Laplace equation over a triangular cell of a surface mesh. A semi-analytical technique for calculating repeated integrals of unbounded functions has been implemented, which made it possible to propose an approach for calculating repeated integrals of the gradient of the Newtonian potential over two cells. All possible cases of mutual arrangement of triangular cells are considered, including special cases when analytical formulas contain ambiguities. The source code of the implementation of the suggested algorithm of integration is freely available on github (https://github.com/vortexmethods/integrator).

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