TREATMENT OF TOPOLOGY OPTIMIZATION OF A TWO-DIMENSIONAL FIELD GOVERNED BY LAPLACE'S EQUATION UNDER NONLINEAR BOUNDARY CONDITION

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ABSTRACT
This paper presents a treatment of the topology optimization problem for two-dimensional fields governed by Laplace's equation. The study considers various boundary conditions, including Dirichlet, Neumann, Robin, and nonlinear radiation boundary conditions. Additionally, the topological derivative for a general objective functional comprising solely of boundary quantities is derived, with a special focus on the case of a radiation boundary condition in a black body. The accuracy of the derived adjoint problem and topological derivative is validated through several boundary element method calculations.

Keywords: topology optimization, nonlinear boundary condition, Laplace's equation, adjoint problem, topological derivative.

1 INTRODUCTION

The level-set based methods [1], [2] have been proposed as one of the robust topology optimization techniques and have been widely and successfully applied in various engineering applications. The primary successful application of topology optimization has been in the minimization of mean compliance for linear elastostatic bodies under linear boundary conditions, such as fixed and traction boundaries.

However, there are some structural optimization problems involving linear governing equations but under nonlinear boundary conditions, such as those encountered in galvanic cathodic protection [3], [4] and thermal radiation problems [5]–[8].

The level-set based methods utilize a continuous scalar function called the 'level set function', defined for points within a fixed design domain, to determine the region in which the material exists. This function assigns a positive value to points inside the material region, a zero value to boundary points, and a negative value to points outside the material region. Consequently, the actual material distribution can be extracted from the iso-surface of the level set function corresponding to the value zero.

The distribution of the level-set function is updated by solving an evolution equation [2], which is a type of reaction-diffusion equation. This evolution equation is accompanied by a source term corresponding to the 'topological derivative'. The topological derivative represents the sensitivity of the objective functional when an infinitesimal region is removed from the material region. The calculation of the topological derivative requires solving both the original problem and an adjoint problem.

In this study, we consider a structural optimization problem for boundary value problems governed by Laplace's equation with both linear and nonlinear boundary conditions. The variation of the objective functional, defined by a boundary integral of the potential and flux, is derived by appropriately defining an adjoint problem. Some numerical demonstrations are presented for both general and radiation-type nonlinear boundary conditions.

WIT Transactions on Engineering Sciences, Vol 135, © 2023 WIT Press
www.witpress.com, ISSN 1743-3533 (on-line)
doi:10.2495/BE460041

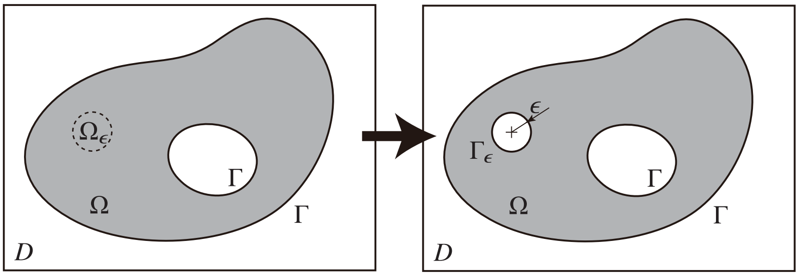

Figure 1: An infinitesimal circular region Ω_ϵ is removed from the material region Ω.

2 STATEMENT OF TOPOLOGY OPTIMIZATION

2.1 Level set method

We assume that a scalar function $\phi(\boldsymbol{x})$ is defined in $D \subset \mathbb{R}^2$. The material region Ω and its boundary Γ are assumed to be included in D, as follows:

$$\Omega = \{\boldsymbol{x} \mid 0 < \phi(\boldsymbol{x}) \leq 1\}, \tag{1}$$
$$\Gamma = \{\boldsymbol{x} \mid \phi(\boldsymbol{x}) = 0\}, \tag{2}$$
$$D \setminus \overline{\Omega} = \{\boldsymbol{x} \mid -1 \leq \phi(\boldsymbol{x}) < 0\}. \tag{3}$$

From this definition of $\phi(\boldsymbol{x})$, the material region can be described based on the distribution of $\phi(\boldsymbol{x})$. By changing the distribution of $\phi(\boldsymbol{x})$, the material distribution also changes accordingly. The level-set method of [2] assumes that the distribution of $\phi(\boldsymbol{x})$ is the solution of the following initial and boundary value problem:

$$\frac{\partial \phi(\boldsymbol{x}, t)}{\partial t} = -K J_T(\boldsymbol{x}, t) + \tau \nabla^2 \phi(\boldsymbol{x}, t) \quad \boldsymbol{x} \in D, \ \ t > 0, \tag{4}$$
$$\phi(\boldsymbol{x}, 0) = \phi_0 \quad \boldsymbol{x} \in D, \tag{5}$$
$$\phi(\boldsymbol{x}, t) = c \quad \boldsymbol{x} \in \partial D_+, \ \ t > 0, \tag{6}$$
$$\phi(\boldsymbol{x}, t) = -c \quad \boldsymbol{x} \in \partial D_-, \ \ t > 0, \tag{7}$$

where t is a fictitious time, K is a positive constant, τ is also a positive constant regularization parameter for blurring the distribution of ϕ, and ϕ_0 is the initial distribution of ϕ corresponding to the initial geometry of the material region Ω at $t = 0$. ∂D_+ and ∂D_- are parts of the boundary ∂D of the domain D, where $\partial D_+ \cup \partial D_- = \partial D$. The Dirichlet condition (6) with a positive constant $c \in (0, 1]$ restricts that the material always exists on ∂D_+, while (7) with a negative constant $-c \in [-1, 0)$ allows the material removal in D in the neighborhood of ∂D_-. J_T denotes the topological derivative, defined by

$$\delta J(\boldsymbol{x}) = J_T(\boldsymbol{x}) b(\epsilon) + o(b(\epsilon)), \tag{8}$$

where $\delta J(\boldsymbol{x})$ is the variation of J when an infinitesimal circular region of radius ϵ centered at $\boldsymbol{x}$ is removed as shown in Fig. 1, and $b(\epsilon)$ is a constant that vanishes as $\epsilon \to 0$. After $\delta J(\boldsymbol{x})$ is expanded analytically in the above form, the topological derivative is obtained by

$$J_T(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{\delta J(\boldsymbol{x})}{b(\epsilon)}. \tag{9}$$

2.2 Boundary value problem with a nonlinear boundary condition

Let us consider a boundary value problem with a general nonlinear boundary condition as follows:

$$\nabla \cdot (-k\nabla u) = 0 \quad \text{in} \quad \Omega, \tag{10}$$

$$u = \bar{u} \quad \text{on} \quad \Gamma_u, \tag{11}$$

$$q := -k\nabla u \cdot \boldsymbol{n} = \bar{q} \quad \text{on} \quad \Gamma_q, \tag{12}$$

$$q := -k\nabla u \cdot \boldsymbol{n} = \nu(u) \quad \text{on} \quad \Gamma_n, \tag{13}$$

where u is potential, k is a positive constant, both $\bar{u}$ and $\bar{q}$ are known functions, $\boldsymbol{n}$ is the unit outward normal vector to the boundary, and $\nu(u)$ is a nonlinear function of u. For thermal radiation problems of a black body, $\nu(u)$ can be given as

$$\nu(u) = \sigma u^4, \tag{14}$$

where σ is the Stefan–Boltzmann constant. In eqn (13), the interactions of the radiations are not taken into account for simplicity.

2.3 Objective functional and topological derivative

We consider the following augmented objective functional to minimize:

$$\begin{aligned} J &= \int_\Gamma f(u,q)\, d\Gamma + \int_\Omega \mu \nabla \cdot (-k\nabla u)\, d\Omega \\ &= \int_\Gamma f(u,q)\, d\Gamma + \int_\Gamma \mu q\, d\Gamma - \int_\Omega \nabla\mu \cdot (-k\nabla u)\, d\Omega, \end{aligned} \tag{15}$$

where μ is an adjoint variable.

By removing an infinitesimal circular region Ω_ϵ from Ω, the objective functional may change from J to $J + \delta J$, and after some lengthy operations, we obtain

$$\begin{aligned} J + \delta J = &\int_{\Gamma_u \cup \Gamma_q \cup \Gamma_n} (f + \delta f)\; d\Gamma + \int_{\Gamma_\epsilon} (f + \delta f)\, d\Gamma \\ &+ \int_{\Gamma_u \cup \Gamma_q \cup \Gamma_n} \mu(q + \delta q)\, d\Gamma + \int_{\Gamma_\epsilon} \mu(q + \delta q)\, d\Gamma \\ &- \int_{\Omega \setminus \Omega_\epsilon} (-k\nabla\mu) \cdot \nabla(u + \delta u)\, d\Omega. \end{aligned} \tag{16}$$

Subtracting both sides of eqn (15) from (16) results in

$$\begin{aligned} \delta J = &\int_{\Gamma_u} \left(\mu + \frac{\partial f}{\partial q}\right) \delta q\, d\Gamma - \int_{\Gamma_q} \left(\eta - \frac{\partial f}{\partial u}\right) \delta u\, d\Gamma \\ &- \int_{\Gamma_n} \left[\left(\eta - \frac{\partial f}{\partial u}\right) \delta u - \left(\mu + \frac{\partial f}{\partial q}\right) \delta q\right] d\Gamma + \int_{\Gamma_\epsilon} (f + \delta f)\, d\Gamma \\ &+ \int_{\Gamma_\epsilon} \mu(q + \delta q)\, d\Gamma - \int_{\Gamma_\epsilon} \eta\, \delta u\, d\Gamma \\ &+ \int_{\Omega \setminus \Omega_\epsilon} \nabla \cdot (-k\nabla\mu)\, \delta u\, d\Omega + \int_{\Omega_\epsilon} (-k\nabla\mu) \cdot \nabla u\, d\Omega, \end{aligned} \tag{17}$$

where $\eta = -k\boldsymbol{\nabla}\mu \cdot \boldsymbol{n}$.

On Γ_n, we find that $\delta q = \frac{\partial \nu(u)}{\partial u}\delta u$, thus, eqn (18) can be rearranged as

$$\begin{aligned}\delta J = &\int_{\Gamma_u}\left(\mu + \frac{\partial f}{\partial q}\right)\delta q\, d\Gamma - \int_{\Gamma_q}\left(\eta - \frac{\partial f}{\partial u}\right)\delta u\, d\Gamma \\ &- \int_{\Gamma_n}\left[\eta - \frac{\partial f}{\partial u} - \left(\mu + \frac{\partial f}{\partial q}\right)\frac{\partial \nu}{\partial u}\right]\delta u\, d\Gamma + \int_{\Gamma_\epsilon}(f + \delta f)\, d\Gamma \\ &+ \int_{\Gamma_\epsilon}\mu(q + \delta q)\, d\Gamma - \int_{\Gamma_\epsilon}\eta\, \delta u\, d\Gamma \\ &+ \int_{\Omega\setminus\Omega_\epsilon}\nabla\cdot(-k\nabla\mu)\,\delta u\, d\Omega + \int_{\Omega_\epsilon}(-k\nabla\mu)\cdot\nabla u\, d\Omega. \end{aligned} \tag{18}$$

Now, we employ μ, which is the solution of the following adjoint boundary value problem.

$$\nabla\cdot(-k\nabla\mu) = 0 \quad \text{in} \quad \Omega, \tag{19}$$

$$\mu = -\frac{\partial f}{\partial q} \quad \text{on} \quad \Gamma_u, \tag{20}$$

$$\eta := -k\boldsymbol{\nabla}\mu\cdot\boldsymbol{n} = \frac{\partial f}{\partial u} \quad \text{on} \quad \Gamma_q, \tag{21}$$

$$\eta := -k\boldsymbol{\nabla}\mu\cdot\boldsymbol{n} = \frac{\partial \nu}{\partial u}\mu + \frac{\partial f}{\partial u} + \frac{\partial \nu}{\partial u}\frac{\partial f}{\partial q} \quad \text{on} \quad \Gamma_n. \tag{22}$$

In the above boundary value problem, we find that the boundary condition for Γ_n is no longer a nonlinear boundary condition but a type of Robin boundary condition. By using the solution of the above boundary value problem, the variation of the objective functional can be obtained as

$$\begin{aligned}\delta J = &\int_{\Gamma_\epsilon}(f + \delta f)\, d\Gamma + \int_{\Gamma_\epsilon}\mu(q + \delta q)\, d\Gamma \\ &- \int_{\Gamma_\epsilon}\eta\,\delta u\, d\Gamma + \int_{\Omega_\epsilon}(-k\nabla\mu)\cdot\nabla u\, d\Omega, \end{aligned} \tag{23}$$

where the boundary condition for Γ_ϵ is also assumed to be a nonlinear one, i.e.,

$$q + \delta q = \nu(u + \delta u) \approx \nu(u) + \frac{\partial \nu}{\partial u}\delta u. \tag{24}$$

We consider the asymptotic behaviors of δu and δq, and the Taylor series expansions of u and q about the center of Ω_ϵ. After some lengthy operations, we find

$$\delta u\Big|_{\Gamma_\epsilon} \approx \epsilon\left(\frac{\partial u}{\partial x}(\boldsymbol{x}_0)\cos\theta + \frac{\partial u}{\partial y}(\boldsymbol{x}_0)\sin\theta\right), \tag{25}$$

and

$$q + \delta q\Big|_{\Gamma_\epsilon} \approx \nu(\boldsymbol{x}_0) + 2\epsilon\frac{\partial \nu}{\partial u}(\boldsymbol{x}_0)\left(\frac{\partial u}{\partial x}(\boldsymbol{x}_0)\cos\theta + \frac{\partial u}{\partial y}(\boldsymbol{x}_0)\sin\theta\right), \tag{26}$$

where $\boldsymbol{x}_0$ is the center of the circular region Ω_ϵ.

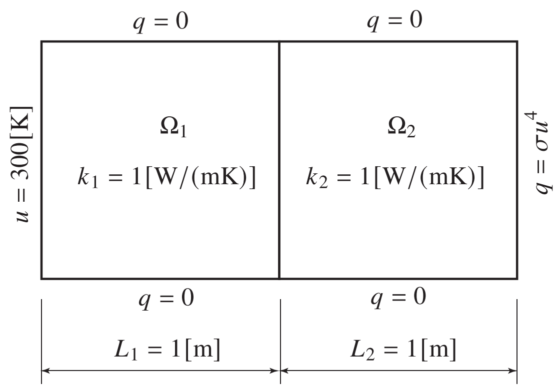

Figure 2: A domain consisting of two squares Ω_1 and Ω_2 subjected to a radiation boundary condition.

After evaluating the integrals for Γ_ϵ and Ω_ϵ, we obtain the variation of J as follows:

$$\delta J(\boldsymbol{x}_0) = \int_{\Gamma_\epsilon} (f + \delta f)\ d\Gamma + 2\pi\epsilon\mu(\boldsymbol{x}_0)\nu(\boldsymbol{x}_0) - 2\pi\epsilon^2 k\nabla\mu(\boldsymbol{x}_0)\cdot\nabla u(\boldsymbol{x}_0) + o(\epsilon^2). \quad (27)$$

Therefore, the topological derivative at a point $\boldsymbol{x}$ in Ω is obtained as

$$J_T(x) = \lim_{\epsilon\to 0}\frac{\delta J}{2\pi\epsilon} = \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} (f + \delta f)\ d\Gamma + \mu(\boldsymbol{x})\nu(\boldsymbol{x}). \quad (28)$$

The actual expression of the first term integral of J_T depends of the definition of the boundary objective functional f defined on the newly generated boundaries, but it should be chosen so that this limit can exist.

3 NUMERICAL EXAMPLE

We consider a steady-state heat conduction problem for a domain consisting of two squares with a side length of 1 m as shown in Fig. 2. The thermal conductivities of both domains are assumed to be the same, with $k_1 = k_2 = 1$ (W/(mK)). On the left-side boundary, the temperature $u = 300$ (K) is given, and on the right-side, the radiation condition $q = \sigma u^4$ is applied. The top and bottom boundaries are assumed to be adiabatic. On the interface boundary between Ω_1 and Ω_2, the temperature is continuous, and the heat fluxes are balanced. We discretized each edge with 10 constant boundary elements and calculated the temperatures and the adjoint temperatures on the boundaries and at twenty uniformly arranged internal points along the center line in the x direction. In Fig. 3, we show the temperature distributions calculated by using the boundary element method at the internal points and on the boundaries along the center lines in the x-direction, comparing them with the exact solutions. Both the BEM and exact solutions agree very well. The boundary element method required 48 iterations based on the fixed-point iteration approach until the temperature on the radiation boundary converged.

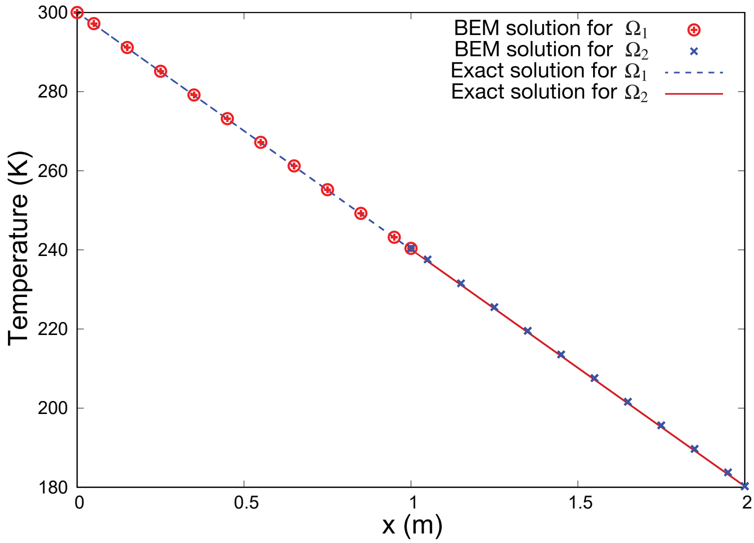

Figure 3: An infinitesimal circular region Ω_ϵ is removed from the material region Ω.

Next, we calculated the topological derivatives by using the expression of eqn (28). The boundary objective functional is defined as $f(u, q) = (u - \tilde{u})^2$, where $\tilde{u} = 300$ (K). Then, the adjoint boundary value problem becomes as follows:

$$\nabla \cdot (-k\nabla\mu) = 0 \quad \text{in} \quad \Omega, \tag{29}$$

$$\mu = 0 \quad \text{on} \quad \Gamma_u, \tag{30}$$

$$\eta = 2(u - \tilde{u}) \quad \text{on} \quad \Gamma_q, \tag{31}$$

$$\eta = 3\sigma u^3 \mu + 2(u - \tilde{u}) \quad \text{on} \quad \Gamma_n. \tag{32}$$

No objective functional is assumed on Γ_ϵ for simplicity; hence, the topological derivative becomes as follows:

$$J_T(x) = \mu(\boldsymbol{x})\nu(\boldsymbol{x}) = 3\sigma u(\boldsymbol{x})^3 \mu(\boldsymbol{x}). \tag{33}$$

In order to validate the topological derivative, we compared the values calculated by eqn (33) at 9×9 points uniformly arranged in each square region with those obtained by finite difference approximations of the objective functional for the original domain and the domain from which a small circular region is removed, i.e.,

$$J_T(\boldsymbol{x}) \approx \frac{J_\Omega - J_{\Omega \setminus \Omega_\epsilon}(\boldsymbol{x})}{2\pi\epsilon}, \tag{34}$$

where the radius of Ω_ϵ was set as $\epsilon = 0.01$ (m).

The boundary of the circular cavity Γ_ϵ was divided into 10 constant boundary elements, and the objective functional $\int_\Gamma (u - \tilde{u})^2 \, d\Gamma$ was calculated using the temperature values obtained by the BEM. In Fig. 4, we show the topological derivative values calculated using the analytical expression of eqn (33) and the approximate topological derivative values calculated using eqn (34). Both results show very good agreement, confirming the correctness of the representation of the newly derived topological derivative.

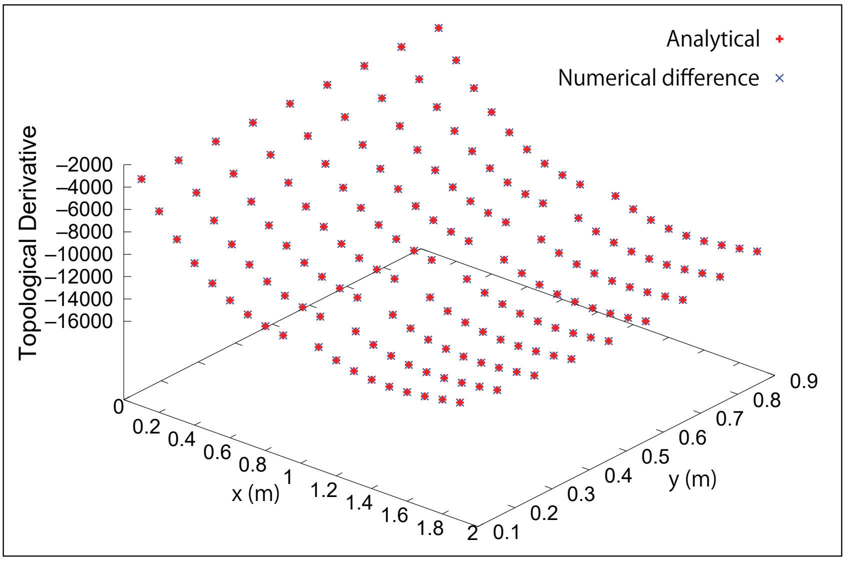

Figure 4: An infinitesimal circular region Ω_ϵ is removed from the material region Ω.

4 CONCLUSION

We considered topology optimization problems for a field governed by boundary value problems of Laplace's equation with nonlinear boundary conditions and derived the analytical expression of the topological derivative. Several numerical examples were demonstrated to validate the boundary element method and the topological derivative expression, particularly when a thermal radiation boundary condition was specified on some part of the boundary.

The topological derivative plays a crucial role in level-set based topology optimization, serving as the source term in the reaction-diffusion equation of the level set function. An extension of the current topological derivative expression is necessary to include the radiation condition that accounts for the thermal radiation from other boundaries.

ACKNOWLEDGEMENT

This work was supported by JSPS KAKENHI Grant Number JP19H00740.

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