# CERTAIN RELATIONS BETWEEN THE MAIN MATRIX CONDITION NUMBER AND MULTIQUADRIC SHAPE PARAMETER IN THE NON-SYMMETRIC KANSA METHOD

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ABSTRACT
The Kansa method is one of the most popular meshless methods today. Its ease of implementation, high order of interpolation and ease of application to problems with complex geometry constitute its advantage over many other methods for solving partial differential equation-based problems. However, the Kansa method has a significant disadvantage – the need to find the shape parameter value despite these undeniable advantages. There are dozens of algorithms for finding a good shape parameter value, but none of them is proven to be optimal. Therefore, there is still a great scientific need to research new algorithms and improve those already known. In this work, an algorithm based on the study of the oscillation of certain shape parameter functions concerning the problems of two-dimensional heat flow in a material with spatially variable thermophysical parameters was investigated. It has been shown that algorithms of this type allow this class of problems to achieve solutions with high accuracy. At the same time, it was indicated that this direction of development of algorithms for searching for a good value of the shape parameter is auspicious. It is because this algorithm can be extended to a wide range of functions whose oscillation is studied and, consequently, its application to a broader range of problems.
*Keywords: Kansa method, multiquadric shape parameter, spatially variable thermophysical parameters, condition number.*

## 1 INTRODUCTION

The Kansa method's usefulness and accuracy strongly depend on the proper determination of the RBF shape parameter value. This issue is a scientific problem of significant interest among various research fields. Wang and Liu [1] studied the effect of shape parameters on the numerical accuracy of the radial point interpolation meshless method. A range of suitable shape parameters is obtained from the analysis of the condition number of the main system matrix, error of energy and irregularity of node distribution. However, it is observed that the widely used shape parameters for multiquadric and reciprocal multiquadric radial basis functions are not close to their optimums obtained in performed numerical experiments. Trahan and Wyatt [2] said that a good RBF shape parameter represents the best possible compromise between the inherent ill-conditioning of the RBF coefficient matrix and the smoothest possible interpolate. In the application presented in their paper, the famous Rippa technique may be unsuitable for choosing the good shape parameter, especially for time-dependent, where the quasi-optimal shape parameter value should be determined at every time step. Larsson and Fornberg [3] explored theoretically and numerically the behaviour of the interpolants in the limit of nearly flat radial basis functions. The approaches that determine the optimal RBF shape parameter value are explained through approximate expansions of the interpolation error. Further, Fornberg and Zuev [4] proposed the approach in which the shape parameter varies spatially, improving accuracy and numerical conditioning. Ferreira et al. [5] used the cross-validation technique to optimize the shape parameter for the radial basis functions in the mixed RBF-pseudospectral mode. Bayona et

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al. [6] also used the RBF-finite difference method's mixed mode. The local approximation error of RBF-finite difference formulas serves as the tool for estimating the optimal shape parameter that minimizes the solution error. Huang et al. [7] applied the increasingly flat radial basis function. They determined the optimal shape parameter for the elliptic partial differential equations solution using the root-mean-square error measure calculated using the exact solutions of presented examples. Boyd and Gildersleeve [8] used numerical experiments to examine the condition numbers of the interpolation matrix for many species of radial basis functions. Davydov and Oanh [9] formulated a new multilevel algorithm that effectively finds a near-optimal shape parameter, which helps to reduce the solution error significantly. Sarra [10] applied the extended precision floating-point arithmetic to improve the accuracy of RBF methods efficiently. Among other methods of determining the good value of the RBF shape parameter are the method based on the convergence analysis [11], the strategy with trigonometric, exponential and random variable shape parameter [12], the local optimization algorithm [13], the extended Rippa algorithm [14], the global genetic algorithm optimization method for the single [15] and variable [16] shape parameter value approach, the application of the principle of a minimum of the total potential energy [17] and the particle swarm optimization method [18]. As was shown above, the stability of the RBF method is often related to the ill-conditioning of the main matrix [19]. However, it is rare to find the method of determining the good value of the shape parameter using the measure of ill-conditioning. Haq and Hussain [20] solved the time-fractional Black–Scholes equations using the Kansa method. They formulated an algorithm for determining the quasi-optimal shape parameter value based on Zhang et al.'s paper [21]. This paper proposed a case-independent shape parameter selection strategy. In the proposed strategy, for a finite precision computation, the upper limit of the condition number is predetermined. Therefore, the shape parameter can be chosen where the condition number oscillates in the early stage. The authors solved equations modelled the transmission line model, the long straight cable channel model, and the grounding metal box model.

In the presented work, the authors expand the mentioned algorithm based on the behaviour of the condition number, especially in applications related to heat conduction problems in non-homogeneous bodies for double-precision floating-point calculations and low computational budget regime.

## 2 GOVERNING EQUATIONS

The paper considers a two-dimensional steady heat flow in a material with spatially variable thermophysical parameters. The fundamental equation solved in the paper is the two-dimensional heat equation, the form of which in the Cartesian coordinate system is as follows:

$$\frac{\partial \kappa}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \kappa}{\partial y}\frac{\partial u}{\partial y} + \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \qquad (1)$$

where $u = u(x, y)$ is temperature field function and $\kappa = \kappa(x, y)$ is the thermal conductivity function. Three types of boundary conditions were used:

1. Dirichlet boundary condition

$$u(\boldsymbol{x}_b) = u_b(\boldsymbol{x}_b), \qquad (2)$$

where $u_b$ is boundary temperature function, $\boldsymbol{x}_b$ is boundary point,

2. Neumann boundary condition

$$-\kappa(\boldsymbol{x}_b)\frac{\partial u(x_b)}{\partial n} = q_b(\boldsymbol{x}_b), \tag{3}$$

where $q_b$ is boundary heat flux function in the normal direction and $\partial/\partial n$ is the derivative in the normal direction,

3. Robin boundary condition

$$-\kappa(\boldsymbol{x}_b)\frac{\partial u(x_b)}{\partial n} = \alpha\big(u_\infty - u(\boldsymbol{x}_b)\big), \tag{4}$$

where $\alpha$ is heat transfer coefficient and $u_\infty$ is ambient temperature.

Eqn (1) was solved using the Kansa method, which is based on multiquadric radial basis function:

$$\varphi(\boldsymbol{x}_i, \boldsymbol{x}_j) = \sqrt{\left(\boldsymbol{x}_i - \boldsymbol{x}_j\right)^2 + \varepsilon^2}, \tag{5}$$

where $\varepsilon$ is the shape parameter. In the Kansa method, it is assumed that the value of an unknown variable $u$ at a given point $\boldsymbol{x}_i$ can be expressed as a linear combination of radial basis functions:

$$u(\boldsymbol{x}_i) = \sum_{j=1}^{n} \varphi(\boldsymbol{x}_i, \boldsymbol{x}_j)\, c_j, \tag{6}$$

where $c_j$ is a linear combination coefficient and $n$ is the number of collocation points. Writing eqn (6) for all collocation points leads to a $n \times n$ system of linear equations, which can be written in a matrix form:

$$\boldsymbol{u} = \boldsymbol{\phi}\boldsymbol{c}. \tag{7}$$

The vectors of spatial derivatives can be obtained by differentiating eqn (7) over corresponding variables:

$$\boldsymbol{u}_{,x} = \boldsymbol{\phi}_{,x}\boldsymbol{c}, \tag{8}$$

$$\boldsymbol{u}_{,y} = \boldsymbol{\phi}_{,y}\boldsymbol{c}, \tag{9}$$

$$\boldsymbol{u}_{,xx} = \boldsymbol{\phi}_{,xx}\boldsymbol{c}, \tag{10}$$

$$\boldsymbol{u}_{,yy} = \boldsymbol{\phi}_{,yy}\boldsymbol{c}. \tag{11}$$

Writing eqn (1) in matrix form for all collocation points and substituting eqns (8)–(11) leads to:

$$\Big(\boldsymbol{\kappa}_{,x}\boldsymbol{\phi}_{,x} + \boldsymbol{\kappa}_{,y}\boldsymbol{\phi}_{,y} + \boldsymbol{\kappa}\big(\boldsymbol{\phi}_{,xx} + \boldsymbol{\phi}_{,yy}\big)\Big)\,\boldsymbol{c} = \boldsymbol{0}. \tag{12}$$

Eqn (11) is the fundamental equation solved in the work. But, of course, eqn (1) alone is not sufficient to solve the problem because no boundary conditions have been imposed. Instead, they are imposed by replacing the equations corresponding to the boundary points in the system of eqn (12) with the corresponding equations describing the boundary conditions according to eqns (2)–(4). Eqn (12) then takes the following form:

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{B}, \tag{13}$$

which allows calculating coefficient vector $\boldsymbol{c}$.

As already mentioned in the introduction, the article aims to examine the idea of finding a good value for the shape parameter originally proposed by Zhang et al. concerning the heat conduction problems. The foundation of this algorithm is to examine the dependence of the interpolation matrix condition number on the value of the shape parameter. The condition number of a matrix is defined as the ratio of the largest singular value of that matrix to its smallest singular value. For the considered problem, this relationship, written as a function, is as follows:

$$\mathcal{L}(\varepsilon) = cond\left(\boldsymbol{\kappa}_{,x}\boldsymbol{\phi}_{,x} + \boldsymbol{\kappa}_{,y}\boldsymbol{\phi}_{,y} + \boldsymbol{\kappa}\left(\boldsymbol{\phi}_{,xx} + \boldsymbol{\phi}_{,yy}\right)\right). \tag{14}$$

It turns out that for the problem under consideration, the graph of function $\mathcal{L}(\varepsilon)$ Usually has a similar shape which is shown in Fig. 1.

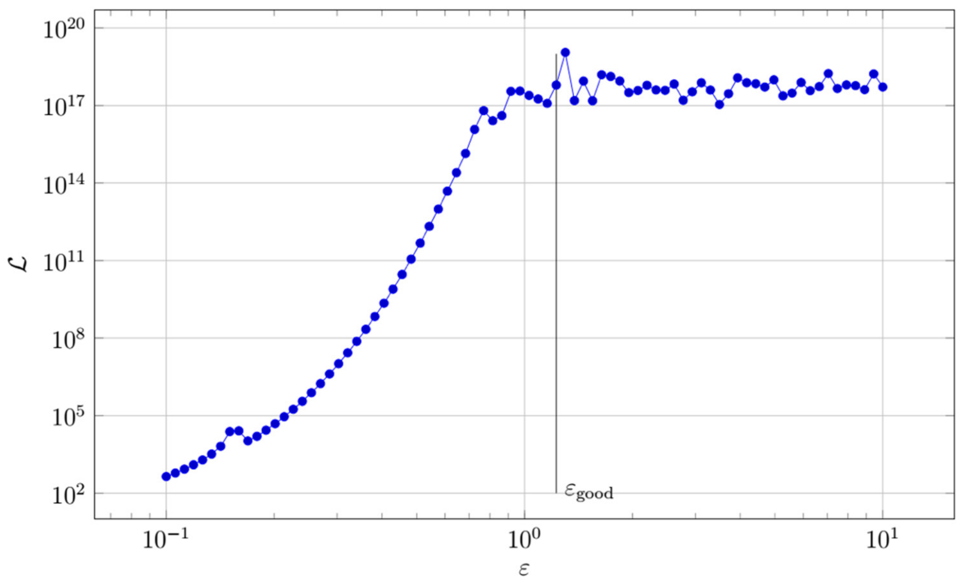


Figure 1: An example of the graph of the function $\mathcal{L}(\varepsilon)$.

In this algorithm, a good value of the shape parameter is one for which there is a transition from smooth to oscillating. This value is also marked in Fig. 1. This algorithm will be referred to as the condition algorithm at work.

## 3 NUMERICAL SETUP, REFERENCE SOLUTION AND ERROR MEASURE

Numerical simulations were performed in the computational domain, the scheme of which is shown in Fig. 2.

The domain with the dimension $L \times L = 1m \times 1m$ consisted of $25 \times 25 = 625$ equally distributed collocation points. The following harmonic function gives the distribution of thermal conductivity:

$$\kappa = \kappa(x) = \frac{\kappa_{max}+\kappa_{min}}{2} + \frac{\kappa_{max}-\kappa_{min}}{2}\cos\left(\omega\pi\frac{x}{L}\right), \tag{15}$$

where $\omega = 2$, $\kappa_{max} = 10\, W \cdot m^{-1} \cdot K^{-1}$, $\kappa_{min} = 1\, W \cdot m^{-1} \cdot K^{-1}$. As for the boundary conditions, the adiabatic boundary conditions $q_b = 0$ were imposed on the bottom and top boundary. Robin boundary condition with ambient temperature $u_\infty = 300\, K$ and heat transfer coefficient $\alpha = 10\, W \cdot m^{-2} \cdot K^{-1}$ was imposed on the left boundary. Dirichlet boundary condition with temperature $u_b = 400\, K$ was imposed on the right boundary. The problem formulated above has an analytical solution given by the equation:

$$u(x) = C_1 \int \frac{dx}{\kappa} + C_2, \tag{16}$$

where $C_1$, $C_2$ are the constants of integration which can be calculated using the boundary conditions. The integral in eqn (16) for distribution given by eqn (15) has the following solution:

$$\int \frac{dx}{\kappa} = \frac{2L}{\omega\pi\sqrt{\kappa_{max}\kappa_{min}}} \tan^{-1}\left(\tan\left(\omega \frac{\pi}{2} \frac{x}{L}\right) \sqrt{\frac{\kappa_{min}}{\kappa_{max}}}\right), \tag{17}$$

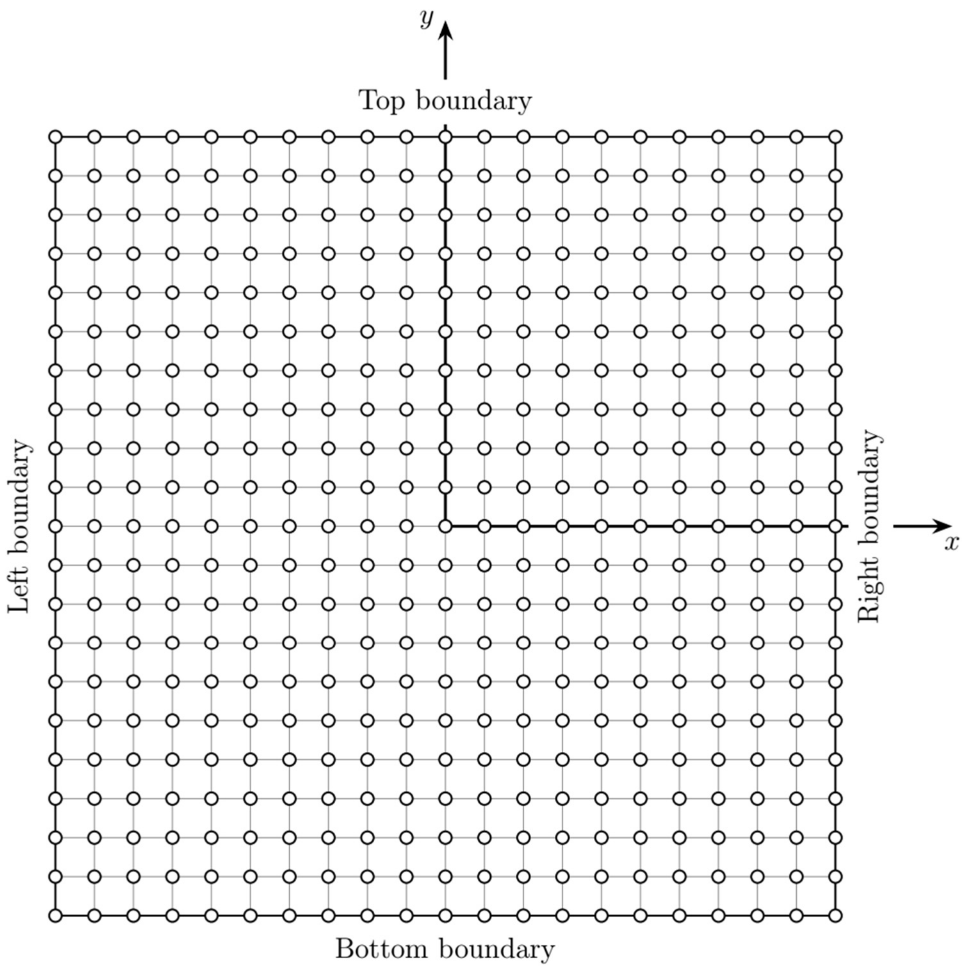


Figure 2: An example of the graph of the function $\mathcal{L}(\varepsilon)$.

The following error measure was used to test the accuracy of the solution:

$$\xi = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{u_k(x_i)-u_a(x_i)}{u_a(x_i)}\right|, \tag{18}$$

where $u_k$ is Kansa method solution while $u_a$ is an analytical solution.

## 4 RESULTS AND DISCUSSION

As part of the research, simulations of the presented problem were performed for 200 different values of shape parameter $\varepsilon$ logarithmically distributed in the range $10^{-1} - 10^{1}$. For each value, the values of the $\mathcal{L}$ function and the error measure $\xi$ were calculated. Fig. 3 shows the graph of function $\mathcal{L}(\varepsilon)$ and $\xi(\varepsilon)$ along with the marked transition point into the oscillatory state, which indicates a good shape parameter value according to the condition algorithm.

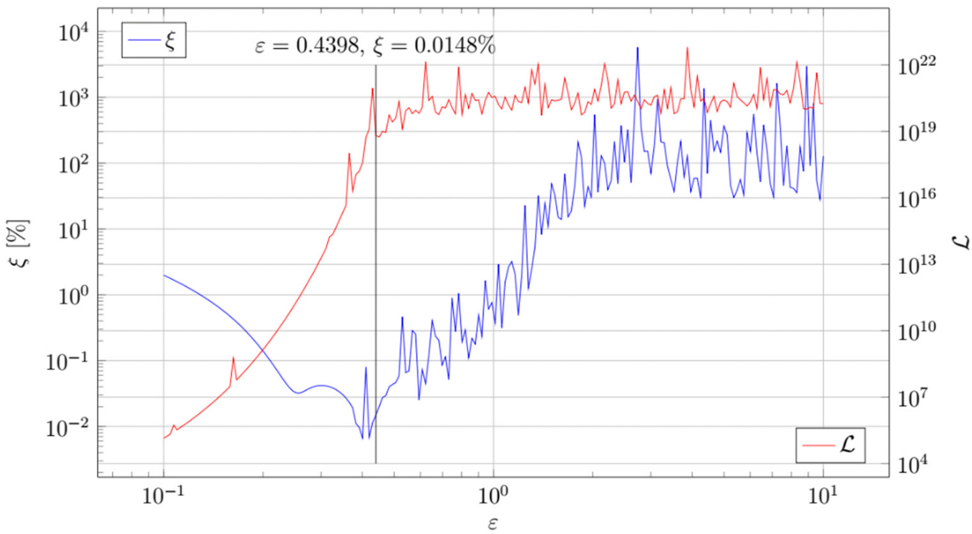


Figure 3: $\mathcal{L}$ function and error measure $\xi$ versus shape parameter $\varepsilon$.

Fig. 3 shows that the value of shape parameter obtained using condition algorithm leads to the solution for which the error measure $\xi$ was $0.0148\%$ which is a low value. This shows that the condition algorithm is suitable for finding a good shape parameter value for this problem class. However, it is worth comparing the results of this algorithm with the results of the currently most popular algorithm for finding a good value of the shape parameter – the Fasshauer [22] algorithm, which is based on finding the minimum of the $E_\infty(\varepsilon)$ function where $E_\infty$ is the infinity norm of the Rippa error function [23]. Also, searching for the minimum of this function, the place where this function goes into the oscillatory state (as in the condi algorithm) will be searched. Fig. 4 shows the graph of the function $E_\infty(\varepsilon)$ and $\xi(\varepsilon)$ along with the marked the minimum of $E_\infty$ function as well as the transition place of the function $E_\infty$ into the oscillatory state.

As can be seen in Fig. 4, the Fasshauer algorithm made it possible to obtain a solution with a low value of the error measure $\xi = 0.0357\%$, but slightly higher than in the case of the condition algorithm. As for the modified Fasshauer algorithm, it gave even better results than even the condition algorithm $\xi = 0.0068\%$. As the results show, the idea of studying the transition of a certain function to the oscillatory state does not have to be limited to the $\mathcal{L}$ function; it has been shown that also good results can be obtained by studying the $E_\infty$ function in this way. Therefore, naturally, there is an idea to investigate other functions as well. Accordingly, two other functions were investigated in this way: $\|c\|^2$ and $\|Ac - B\|^2$. Figs 5 and 6 show the graphs of functions $\|Ac - B\|^2(\varepsilon)$, $\|c\|^2(\varepsilon)$, and $\xi(\varepsilon)$ along with the marked place of the transition of the function $\|Ac - B\|^2$ into the oscillatory state.

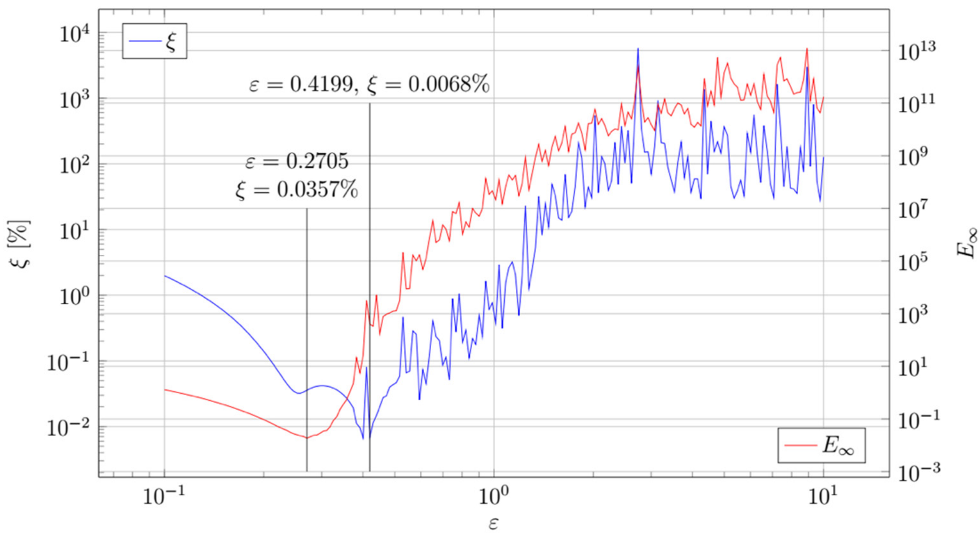


Figure 4: $E_\infty$ function and error measure $\xi$ versus shape parameter $\varepsilon$.

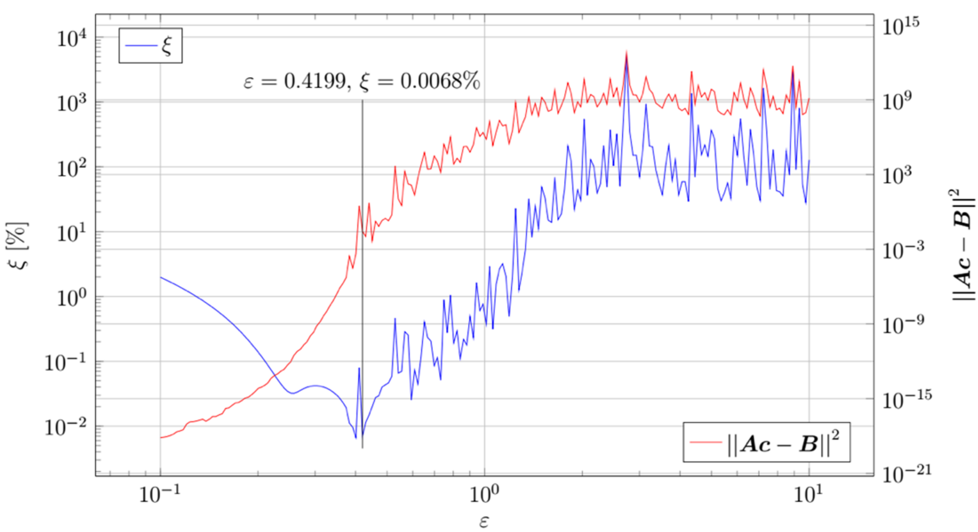


Figure 5: $\|Ac - B\|^2$ function and error measure $\xi$ versus shape parameter $\varepsilon$.

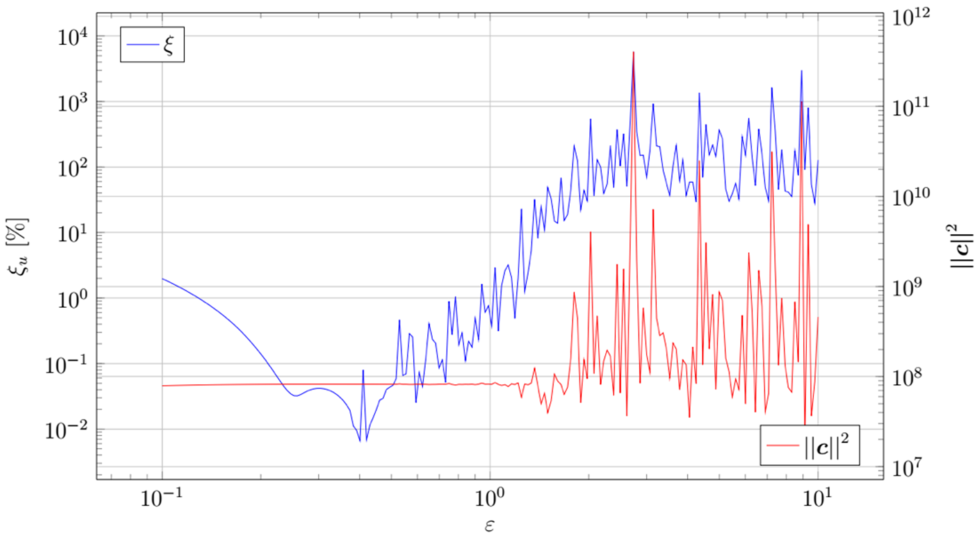


Figure 6: $||c||^2$ function and error measure $\xi$ versus shape parameter $\varepsilon$.

As can be seen, the function $||Ac - B||^2$ gave an epsilon value leading to an error with the value $\xi = 0.0068\%$ – the same as in the case of the modified Fasshaur algorithm. As for the $||c||^2$ function, the point where its oscillation begins falls within the range of the already significant error. Therefore, it can be stated that for this class problem, searching for a good shape parameter value using the condition algorithm is effective for the $||Ac - B||^2$ function, but not for the $||c||^2$ function.

## 5 CONCLUSIONS

In summary, it could be noted that the condition algorithm turned out to be extremely promising. The considered class of problem for as many as three functions led to obtaining results with higher accuracy than the results obtained when using the Fasshauer algorithm, which is today a fundamental and well-known algorithm for finding a good shape parameter value. It opens an auspicious direction for the further development of algorithms for searching for a good value of shape parameter based on the idea behind the condition algorithm. Furthermore, applying this approach to other functions may extend this algorithm's applicability to another class's problems.

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