INTEGRAL EQUATIONS FOR MODELLING OF FRACTURE INITIATION AND DEVELOPMENT IN LAYERED POROELASTIC MEDIA

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ABSTRACT
In this study, we consider a special incorrectly posed boundary value problem of the theory of cracks, which arises when modelling the initiation and development of fracture on the interface between poroelastic materials. The main feature of the problem is the formulation of boundary conditions, which is different from the standard formulations. The problem is considered for a strip, where three conditions are set on one side of the strip and one on the other side, which makes it possible to classify this formulation as a semi-inverse one.
Keywords: layered poroelastic media, cracks, delamination, semi-inverse boundary value problem, integral equations.

1 INTRODUCTION

This study presents the development of mathematical foundations for ill-posed boundary value problems for layered elastic media. It is assumed that some experimental strain/displacement data can be measured on the stress-free part of the surface. This is the main difference from the classical formulations. The latter are well developed and do not present significant difficulties, for example, Ufljand [1]. Shevljakov [2] presents a general method capable of working with two-dimensional and three-dimensional layered structures, employing the Fourier transform and matrix algorithms, as well as some solutions for standard formulations of elastic boundary value problems (BVP). On the other hand, there are inverse formulations that can be reduced to the initial value Cauchy problem (see Schwab [3]), who relates such problems to the (**u**,**p**) problem, which indicates that the displacement, **u**, and stress, **p**, vectors are known on part of the boundary while no conditions are set on the other part of the boundary. This problem has a unique solution and is therefore conditionally ill-posed. There are other formulations that can be reduced to the (**u**,**p**) problem (see [4]–[7]).

The ill-posed problems mentioned above are of interest for applications, especially for the problem of delamination detection on the interfaces between piecewise inhomogeneous layers, for example, coatings or functionally graded materials, layered rocks in geomechanics, live tissues in biomechanics, and others. There are a number of standard methods that can be used to monitor surface displacement or strain, however, in some situations it may not be possible to measure all displacements on a body surface. For example, optical methods can easily measure the deflection of a stress-free surface, however, when measuring lateral strains by using strain gauges, the simultaneous deflection measurements can be complicated. Taking this into account, in Galybin [8] a new formulation has been proposed, in which three conditions are specified on the surface of the coating and one condition on the interface between the coating and the substrate. This formulation is referred to as a semi-inverse problem. In this study, a somewhat similar formulation for an elastic strip is analysed. As shown in Galybin and Rogerson [9], the case of a single strip provides information regarding the distribution of stresses and

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displacements on the interfaces between the layers and can be used for fracture identification. Further we consider a model for describing the initiation and development of interlayer hydraulic fractures, which follows from the poroelastic formulation of the problem under rapid loading.

2 BASIC RELATIONSHIPS OF POROELASTICITY

The linear theory of poroelasticity was proposed by Biot in 1941 [10]. For isotropic material the constitutive equations can be presented in the form [11]:

$$\varepsilon_{ij} = \tfrac{1}{2G}\sigma_{ij} - \left(\tfrac{1}{6G} - \tfrac{1}{9K}\right)\delta_{ij}\sigma_{kk} + \tfrac{1}{3H}\delta_{ij}p, \quad \theta = \tfrac{\sigma_{kk}}{3H} + \tfrac{p}{R}. \tag{1}$$

Here $\varepsilon_{ij}, \sigma_{ij}$ are the strain and stress components, respectively, p is the pore pressure, θ is the variation in liquid content, G is the shear modulus, K is the volume modulus, H and R are the poroelastic constants, $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ is the Kronecker delta. It can be seen that the deviatoric stresses $\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$ (hereafter summation with respect to the repeated index) do not depend on the volumetric strain and pore pressure, while the volumetric strains and the variation in liquid content depend as follows:

$$\varepsilon = \tfrac{P}{K} + \tfrac{p}{H}, \quad \theta = \tfrac{P}{H} + \tfrac{p}{R}, \tag{2}$$

where $P = \frac{1}{3}\sigma_{kk}$.

Two limiting states of a poroelastic body can be considered: drained response, $p = 0$, and undrained response, $\theta = 0$. In both cases, the ratios for volumetric strain remain the same, but with different bulk moduli, which, for undrained response takes the form:

$$K_u = K\left(1 + \tfrac{KR}{H^2 - KR}\right). \tag{3}$$

It is convenient to introduce the Biot constant $\alpha = \frac{K}{H}$ characterizing the compressibility and the constant $B = \frac{R}{H} = \frac{K_u - K}{\alpha K_u}$ (the Skempton coefficient). In this case, the relationships for volumetric strain take the form:

$$\varepsilon = \tfrac{P + \alpha p}{K}, \quad \theta = \tfrac{\alpha}{K}\left(P + \tfrac{p}{B}\right), \quad P = -\alpha M\theta + K_u\varepsilon, \quad p = M(\theta - \alpha\varepsilon), \tag{4}$$

where the notation $M = \frac{K_u - K}{\alpha^2} = \frac{H^2 R}{H^2 - KR}$ is introduced, which is often called the Biot modulus. It characterizes the increase in fluid volume as a result of an increase in pore pressure. When considering specific problems, for convenience, other poroelastic constants can also be introduced, see details in Detournay and Cheng [11]. We further use Poisson's ratio for undrained response, ν_u, instead of K_u, which is introduced by analogy with the standard theory of elasticity, i.e. as:

$$\nu_u = \frac{3K_u - 2G}{2(3K_u + G)}. \tag{5}$$

The constants B and M can be expressed in terms of ν_u if necessary.

For plane strain conditions, it should be assumed that $\sigma_{33} = \nu\sigma_{kk} - \alpha(1 - 2\nu)p$, and then eqn (1) takes the form:

$$2G\varepsilon_{ij} = \sigma_{ij} - \nu\sigma_{kk}\delta_{ij} + \alpha(1-2\nu)p\delta_{ij}$$
$$2G\left(\varepsilon_{ij} - \tfrac{1+\nu_u}{3}B\theta\delta_{ij}\right) = \sigma_{ij} - \nu\sigma_{kk}\delta_{ij} \tag{6}$$
$$2G\theta = \alpha(1-2\nu)\left(\sigma_{kk} + \tfrac{3}{B(1+\nu_u)}p\right).$$

The full system of differential equations of poroelasticity includes the equations of equilibrium (in the absence of body forces):

$$\frac{\partial\sigma_{ij}}{\partial x_j} = 0, \quad i,j = 1,2\,, \tag{7}$$

and the equation that follows from continuity and Darcy's law:

$$\frac{\partial\theta}{\partial t} - \chi\Delta p = 0\,, \tag{8}$$

where χ is the permeability coefficient. The latter equation can be converted to the Laplace equation for steady flow conditions, i.e. to the following form:

$$\Delta p = 0\,. \tag{9}$$

The above equations assume that the fluid is incompressible and the skeleton is connected, as well as a fraction of the volume occupied by the fluid, which allows it to flow under load. These equations, if necessary, can be reformulated in displacements (the Lame equations).

3 MODEL OF CRACK INITIATION ON THE INTERFACE

Further on, we consider a model in which the poroelasticity equations are formally independent of the unknown θ and, consequently, of the pore pressure. This is true for the stress tensor deviator, but, as can be seen from the above, it is not true for the volumetric strain. The reason why we exclude the fluid equation from consideration further on is as follows. With the rapid deformation of the medium, which occurs with a decrease in the volume of the fluid-saturated medium, the pore fluid does not have time to quickly flow out to remote parts where the pressure is lower, while the skeleton deforms instantly. This, actually, corresponds to the case $\theta = 0$. On the other hand, with very slow deformation, the pore pressure disappears, $p = 0$. Thus, we have limiting cases of either complete drainage or non-drainage. Both cases are described by the equations of the static theory of elasticity either with the usual bulk modulus or with the drained modulus determined by eqn (3). Let us consider a strip of poroelastic fluid-saturated material (undrained response) attached to the base made of a material of significantly lower permeability. After applying the load to the layer (Fig. 1(a)), the volume decreases sharply in the area of application of the load and the liquid is displaced in all directions. If the base has a weaker permeability, then the fluid accumulates at the interface, forming an analogue of a hydraulic fracture. When the layer is rapidly deformed in the transverse direction, for example, when the biological tissue is heated (Fig. 1(b)), the liquid is also displaced and a hydraulic fracture is formed. In both cases, the pore pressure in the fracture is unknown because we do not use all the information about fluid motion in the saturated medium. It should be noted that the shear stress on the interface can be neglected, since they are equal to zero in the area where the crack developed, and on the other part, the tangential stresses are either small, as in the case

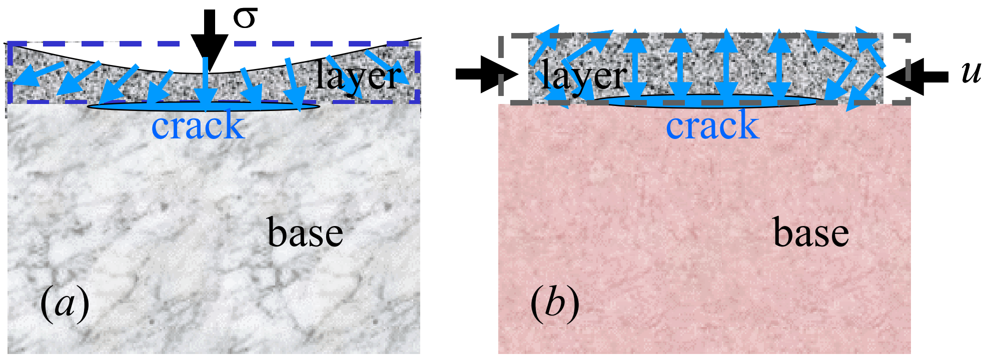

Figure 1: Scheme of the formation of an interface crack. (a) Under the application of a load; and (b) Under compression of the layer. In both cases, the liquid is rapidly forced out to the interface. The dashed line shows the position of the layer before deformation.

in Fig. 1(a), or, as in the case depicted in Fig. 1(b), the development of shear zones has occurred due to relative displacement of the layer and base edges. Thus, on the interface, the shear stresses can be assumed to be zero, while the normal distribution remains unknown. Accordingly, the problem on this part of the layer boundary remains underdetermined. At the upper boundary, the shear stresses are absent in both cases, and the normal stresses are known in the case of loading in Fig. 1(a), or they are also equal to zero if the layer is rapidly shortened. Since four scalar conditions are required for the solvability of the problem (in the classical case, two on each part of the boundary), it is necessary to introduce an additional condition on the upper boundary of the layer. Such a condition can be obtained by monitoring the normal component of the displacements on the stress-free part of the boundary, i.e. deflection. Instead of monitoring the deflection, it is natural to measure the slope of the tangent to the deflection, i.e. the derivative of the deflection with respect to the tangential coordinate, which is easy to acquire by optical methods. Accordingly, we arrive at the formulation when one condition is set on one part of the boundary of the strip (the absence of tangential stresses), and three conditions on the other part. In addition to the absence of shear stresses, the normal stresses and the derivative of the deflection in the tangential direction are known. Such a problem can be referred to as semi-inverse boundary value problem that was first formulated in Galybin [8] for the case of a shear zone. Further, a more general case is considered, which is a combined open-shear crack, and its open part is a hydraulic fracture. In the future, for the mathematical formulation of the problem, it will be necessary to use the equations of the plane theory of elasticity in complex coordinates, as well as the Fourier transform, which is given below.

3.1 Kolosov–Muskhelishvili formulas

The Kolosov–Muskhelishvili formulas for the stress and displacement components in the Cartesian coordinate system *Oxy* have the following form [12]:

$$
\begin{aligned}
&P \equiv \frac{\sigma_{xx}+\sigma_{yy}}{2} = \Phi(z)+\overline{\Phi(z)}, \\
&D \equiv \frac{\sigma_{yy}-\sigma_{xx}}{2} + i\sigma_{xy} = \overline{z}\Phi'(z)+\Psi(z), \\
&2GW \equiv 2G(u_x+iu_y) = \kappa\phi(z) - z\overline{\Phi(z)} - \overline{\psi(z)}.
\end{aligned} \tag{10}
$$

Here $\sigma_{xx},\sigma_{yy},\sigma_{xy}$ are the stress components, P is the mean stress, D is the complex stress deviator, $W = u_x + iu_y$ is the complex displacement vector with x and y components respectively, G is the shear modulus, κ = (3 – 4ν) for plane stress and κ = (3 – ν)/(1 + ν) for plane stress, $\nu = \nu_u$ is Poisson’s ratio for the undrained case, determined by eqn (5), $\Phi(z) = \phi'(z)$, $\Psi(z) = \psi'(z)$ are complex potentials (holomorphic functions of the complex variable $z = x + iy$).

For straight boundaries it is convenient to replace the potential Ψ(z) with a new unknown holomorphic function Ω(z) as follows:

$$\Omega(z) = z\Phi'(z) + \Phi(z) + \Psi(z)\,. \tag{11}$$

Then the expression for the complex stress deviator takes the form:

$$D(z,\overline{z}) = (\overline{z} - z)\Phi'(z) - \Phi(z) + \Omega(z)\,. \tag{12}$$

Differentiating the third formula in eqn (10) with respect to the variable x and using eqns (11) and (12), one can obtain the following expressions for the derivative of the displacement vector $W' = w = u + iv$, and its complex conjugate, $\overline{W}' = \overline{w} = u - iv$, in terms of holomorphic functions Φ(z) and Ω(z):

$$2Gw = \kappa\Phi(z) - \overline{\Omega(z)} + (\overline{z} - z)\overline{\Phi'(z)}, \quad 2G\overline{w} = \kappa\overline{\Phi(z)} - \Omega(z) - (\overline{z} - z)\Phi'(z)\,. \tag{13}$$

The sum of the first and second formulas in eqn (10) followed by complex conjugation leads to the following equivalent relations:

$$
\begin{aligned}
\sigma \equiv \sigma_{yy} + i\sigma_{xy} &= \overline{\Phi(z)} + \Omega(z) + (\overline{z} - z)\Phi'(z), \\
\overline{\sigma} = \sigma_{yy} - i\sigma_{xy} &= \Phi(z) + \overline{\Omega(z)} - (\overline{z} - z)\overline{\Phi'(z)}.
\end{aligned} \tag{14}
$$

The expressions for the complex potentials Φ(z) and Ω(z) in terms of the stress and displacement vectors are found from eqns (13) and (14) as follows:

$$(\kappa+1)\Phi(z) = \overline{\sigma} + 2Gw\,, \tag{15}$$

$$(\kappa+1)\Omega(z) = \kappa\sigma - 2G\overline{w} + (\kappa+1)(z - \overline{z})\Phi'(z)\,. \tag{16}$$

Summation of eqns (15) and (16) gives:

$$\Phi(z) + \Omega(z) - (z - \overline{z})\Phi'(z) = \tfrac{\kappa\sigma + \overline{\sigma} + 2G(w - \overline{w})}{\kappa+1}\,. \tag{17}$$

If $y = 0$, then eqn (17) takes the form:

$$\Phi(x,0) + \Omega(x,0) = \sigma_{yy}(x,0) + i\left[\tfrac{\kappa-1}{\kappa+1}\sigma_{xy}(x,0) + \tfrac{4G}{\kappa+1}v(x,0)\right]. \tag{18}$$

It follows from eqn (18) that if the right-hand side of eqn (18) is known, then we arrive at the initial Cauchy problem for finding the holomorphic function Φ(z)+Ω(z). This fact will be used later to solve the semi-inverse BVP.

3.2 Fourier transform

Fourier transform, **F** (.) (abbreviated further as FT) and its inverse, $\mathbf{F}^{-1}$ (.) are determined as follows [13]:

$$\mathrm{F}\big(Y(x)\big) \equiv \hat{Y}(s) = \int_{-\infty}^{\infty} Y(x) e^{-isx} dx, \qquad \mathrm{F}^{-1}\big(\hat{Y}(s)\big) \equiv Y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Y}(s) e^{isx} ds\,. \tag{19}$$

where s is real, Im $(s) = 0$.

The FT of the derivative is:

$$\mathrm{F}\big(Y'(x)\big) = is\hat{Y}(s)\,. \tag{20}$$

The FT for complex conjugate functions:

$$\mathrm{F}\big(\overline{Y(x)}\big) = \widehat{\overline{Y(s)}} = \int_{-\infty}^{\infty} \overline{Y(x)} e^{-isx} dx = \overline{\int_{-\infty}^{\infty} Y(s) e^{isx} ds} = \overline{\hat{Y}(-s)}\,. \tag{21}$$

If $Y_0(x)$ is real-valued, then $\overline{\hat{Y}_0(s)} = \hat{Y}_0(-s)$.

Applying FT with respect to the variable x to any holomorphic function H(z) = $H(x, y)$ leads (due to the Cauchy–Riemann equations) to the following relationships for transformations of the complex potentials:

$$\begin{pmatrix} \hat{\Phi}(s,y) \\ \hat{\Omega}(s,y) \end{pmatrix} = e^{-ys} \begin{pmatrix} \hat{\Phi}(s,0) \\ \hat{\Omega}(s,0) \end{pmatrix}. \tag{22}$$

Applying complex conjugation to eqn (22), from eqn (20) one finds that:

$$\begin{pmatrix} \overline{\hat{\Phi}(-s,y)} \\ \overline{\hat{\Omega}(-s,y)} \end{pmatrix} = e^{ys} \begin{pmatrix} \overline{\hat{\Phi}(-s,0)} \\ \overline{\hat{\Omega}(-s,0)} \end{pmatrix}. \tag{23}$$

Relationship eqns (22) and (23) are used below.

4 MATHEMATICAL FORMULATION

A strip of thickness $2h$ with elastic constants G and κ is considered. Let us introduce the Cartesian coordinate system Oxy such that the x-axis coincides with the lower boundary of the layer, Γ_1, and the y-axis is directed towards the upper boundary, Γ_2. Thus, the elastic band occupies the region $\{-\infty<x_n<\infty,\ 0<y<2h\}$.

Further on, it is assumed that three boundary conditions are set on Γ_1:

$$\sigma_{yy}(x,0) = \sigma_1(x), \quad \sigma_{xy}(x,0) = \tau_1(x), \quad v(x,0) = v_1(x)\,, \tag{24}$$

and just one condition on Γ_2:

$$\sigma_{xy}(x,2h) = \tau_2(x)\,. \tag{25}$$

Thus, we consider a boundary value problem for a strip, in which, for generality, the shear stresses are introduced, which will be set to zero from now on:

$$\tau_1(x) = \tau_2(x) = 0 \,. \tag{26}$$

If the boundary Γ_1 is loaded by concentrated forces acting only at points t_j (j = 1… J), then the complex stress vector on Γ_1 has the form:

$$\sigma_1(x) + i\tau_1(x) = \sum_{j=0}^{J} P_j \delta(x - t_j), \quad x, t_j \in \Gamma_1 \,, \tag{27}$$

where P_j are the complex intensities of the applied loads and δ(t) is the delta function.

The assumption made above ensures that the deflection of the lower boundary can be measured over the entire surface, and therefore the tangential derivative of the normal displacements is also known at the boundary, except for a countable number of points t_j, where it suffers discontinuities and tends to infinity. Monitoring of tangential displacements (or surface deformation) is also possible. However, this is more difficult than deflection monitoring, which can be done with optical methods. For this reason, we do not use any data regarding the measurements of the field $u_x(x,0)$. Instead, eqn (25) is introduced on the other side of the strip. Therefore, it is obvious that such formulation is different from the well-known correct formulations, as well as from the Cauchy problem considered in Galybin and Rogerson [9]. This, to some extent, is related to the problem of detecting the slip zone at the interface between the coating and the substrate, considered in Galybin [8]. The difference from the last mentioned problem is that no connection is introduced between the stress and displacement vectors at the upper boundary, as was used in Galybin [8], due to the continuity conditions at the interface, and not a slip zone is considered, but a crack of general type.

4.1 Solutions of semi-inverse BVP for an elastic strip in the Fourier domain

This subsection presents solutions to semi-inverse boundary value problems for an elastic strip with boundaries $\Gamma_1 = \{z = t : -\infty < t < \infty\}$ and $\Gamma_2 = \{z = t + 2ih : -\infty < t < \infty\}$. We present complete solutions in terms of the FT of the complex potentials that allow one to analyse the stress state at any point in the strip.

Firstly, the boundary conditions eqn (24) are used, from which it follows that the boundary value of the holomorphic function Φ(z) + Ω(z) is known on the boundary Γ_1 (y = 0). Let us apply FT to eqn (17) and use the properties of the FT to obtain:

$$(1 + 2sy)\hat{\Phi}(s, y) + \hat{\Omega}(s, y) = \hat{\sigma}_{yy}(s, y) + i\left[\tfrac{\kappa-1}{\kappa+1}\hat{\sigma}_{xy}(s, y) + \tfrac{4G}{\kappa+1}\hat{v}(s, y)\right]. \tag{28}$$

On the boundary Γ_1 (y = 0) one can represent eqn (28) in terms of the FT of eqn (24) as follows:

$$\hat{\Phi}(s, 0) + \hat{\Omega}(s, 0) = f(s) \,, \tag{29}$$

where it is denoted:

$$f(s) = \hat{\sigma}_1(s) + i\left[\tfrac{\kappa-1}{\kappa+1}\hat{\tau}_1(s) + \tfrac{4G}{\kappa+1}\hat{v}_1(s)\right]. \tag{30}$$

Using eqns (22) and (23), one can transform eqn (29) to the form:

$$\hat{\Omega}(s,y) = e^{-ys} f(s) - \hat{\Phi}(s,y). \tag{31}$$

Due to eqn (21), the complex conjugation of eqn (31) leads to:

$$\overline{\hat{\Omega}(s,y)} = e^{ys}\,\overline{f(-s)} - \overline{\hat{\Phi}(-s,y)} \tag{32}$$

Applying the FT to the stress and displacement vectors and eliminating the FT of Ω by employing eqns (31) and (32), we obtain:

$$\begin{aligned} \hat{\sigma}_{yy}(s,y) + i\hat{\sigma}_{xy}(s,y) &= e^{ys}\,\overline{\hat{\Phi}(-s,0)} - (1-2ys)e^{-ys}\hat{\Phi}(s,0) + e^{-ys} f(s), \\ \hat{\sigma}_{yy}(s,y) - i\hat{\sigma}_{xy}(s,y) &= e^{-ys}\hat{\Phi}(s,0) - (1+2ys)e^{ys}\,\overline{\hat{\Phi}(-s,0)} + e^{ys}\,\overline{f(-s)}, \end{aligned} \tag{33}$$

$$\begin{aligned} 2G\big(\hat{u}(s,y) + i\hat{v}(s,y)\big) &= \kappa e^{-ys}\hat{\Phi}(s,0) + (1+2ys)e^{ys}\,\overline{\hat{\Phi}(-s,0)} - e^{ys}\,\overline{f(-s)}, \\ 2G\big(\hat{u}(s,y) - i\hat{v}(s,y)\big) &= \kappa e^{ys}\,\overline{\hat{\Phi}(-s,0)} + (1-2ys)e^{-ys}\hat{\Phi}(s,0) - e^{-ys} f(s). \end{aligned} \tag{34}$$

Taking the difference of the equations in eqns (33) and (34) on Γ_1, one finds:

$$\begin{aligned} i\hat{\tau}_1(s) &= \overline{\hat{\Phi}(-s,0)} - \hat{\Phi}(s,0) + \tfrac{1}{2} f(s) - \tfrac{1}{2}\overline{f(-s)}, \\ 4iG\hat{v}_1(s,0) &= (\kappa-1)\hat{\Phi}(s,0) - (\kappa-1)\overline{\hat{\Phi}(-s,0)} + f(s) - \overline{f(-s)}. \end{aligned} \tag{35}$$

From eqn (35) the following equivalent relations are found:

$$\begin{aligned} \overline{\hat{\Phi}(-s,0)} &= \hat{\Phi}(s,0) + i\hat{\tau}_1(s) - \tfrac{1}{2}\Big[f(s) - \overline{f(-s)}\Big], \\ \overline{\hat{\Phi}(-s,0)} &= \hat{\Phi}(s,0) - \tfrac{4G}{\kappa-1} i\hat{v}_1(s) + \tfrac{1}{\kappa-1}\Big[f(s) - \overline{f(-s)}\Big]. \end{aligned} \tag{36}$$

Substituting any of the formulas in eqn (36) into eqns (33) and (34) allows one to express the stress and displacement vectors (and their conjugates) in terms of the only unknown $\hat{\Phi}(s,0)$. The latter is found from eqn (25) on Γ_2 as detailed below.

Consider the difference of the formulas in eqn (33):

$$i\hat{\sigma}_{xy}(s,y) = (1+ys)e^{ys}\,\overline{\hat{\Phi}(-s,0)} - (1-ys)e^{-ys}\hat{\Phi}(s,0) + \tfrac{e^{-ys} f(s) - e^{ys}\overline{f(-s)}}{2}. \tag{37}$$

Substituting the first formula from eqn (36) into eqn (37) leads to:

$$\begin{aligned} i\hat{\sigma}_{xy}(s,y) &= \big[(1+ys)e^{ys} - (1-ys)e^{-ys}\big]\hat{\Phi}(s,0) + \tfrac{e^{-ys} f(s) - e^{ys}\overline{f(-s)}}{2} + \\ &\quad + (1+ys)e^{ys}\Big(i\hat{\tau}_1(s) - \tfrac{f(s)-\overline{f(-s)}}{2}\Big) = \\ &= 2\big[\sinh(ys) + ys\cosh(ys)\big]\hat{\Phi}(s,0) - 2\sinh(ys) f(s) + (1+ys)e^{ys} i\hat{\tau}_1(s) - yse^{ys}\tfrac{f(s)-\overline{f(-s)}}{2}. \end{aligned} \tag{38}$$

Assuming $y = 2h$ and using the FT of $\tau_2(x)$ on the boundary Γ_2, one finds the solution for unknown FT of the function $\hat{\Phi}(s,0)$ in the form:

$$\hat{\Phi}(s,0) = \frac{i\big[\hat{\tau}_2(s) - (1+2hs)e^{2hs}\hat{\tau}_1(s)\big] + \sinh(2hs) f(s) + hse^{2hs}\big(f(s) - \overline{f(-s)}\big)}{2\sinh(2hs) + 4hs\cosh(2hs)}. \tag{39}$$

If we assume the absence of shear stresses on the boundaries, then eqn (39) takes the following form:

$$\hat{\Phi}(s,0)=\frac{2\sinh(2hs)f(s)+hse^{2hs}(f(s)-\overline{f(-s)})}{2\sinh(2hs)+4hs\cosh(2hs)},\quad f(s)=\hat{\sigma}_1(s)+i\,\tfrac{4G}{\kappa+1}\hat{v}_1(s). \tag{40}$$

From eqn (40) it is evident that the numerator must vanish when s tends to zero (to compensate for the first-order singularity in the denominator at $s = 0$). This should also ensure the global equilibrium of the entire strip. It is quite evident that this condition for eqn (34) is satisfied.

Let us find the distribution of the normal stresses on the boundary Γ_2, for which we sum up eqn (33) and use eqn (36):

$$\hat{\sigma}_{yy}(s,y)=-2ys\sinh(ys)\hat{\Phi}(s,0)+\tfrac{1}{2}\left\{e^{-ys}f(s)+e^{ys}\overline{f(-s)}+yse^{ys}\left[f(s)-\overline{f(-s)}\right]\right\}. \tag{41}$$

Next, we set $y = 2h$ and substitute the solution for $\hat{\Phi}(s,0)$, which leads to the expression:

$$\begin{aligned}\hat{\sigma}_{yy}(s,2h)=&-4hs\sinh(2hs)\frac{2\sinh(2hs)f(s)+hse^{2hs}(f(s)-\overline{f(-s)})}{2\sinh(2hs)+4hs\cosh(2hs)}+\\&+\tfrac{1}{2}\left\{e^{-2hs}f(s)+e^{2hs}\overline{f(-s)}+2hse^{2hs}\left[f(s)-\overline{f(-s)}\right]\right\}.\end{aligned} \tag{42}$$

The latter expression can be simplified, taking into account that the half-difference $f(s)-\overline{f(-s)}$ is $i\,\tfrac{4G}{\kappa+1}\hat{v}_1(s)$, but this does not facilitate the direct use of the FT inversion formula.

5 REDUCTION TO INTEGRAL EQUATIONS

From the structure of the solutions obtained in the previous section, it can be seen that $\hat{\Phi}(s,0)$ can be directly found by inverting FT, as well as $\hat{\Omega}(s,0)$, which can be seen from the following formula:

$$\begin{aligned}\begin{pmatrix}\Phi(x,0)\\ \Omega(x,0)\end{pmatrix}&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix}\hat{\Phi}(s,0)\\ \hat{\Omega}(s,0)\end{pmatrix}e^{isx}ds=\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix}1\\ -1\end{pmatrix}\hat{\Phi}(s,0)e^{isx}ds+\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix}0\\ 1\end{pmatrix}f(s)e^{isx}ds=\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix}1\\ -1\end{pmatrix}\hat{\Phi}(s,0)e^{isx}ds+\begin{pmatrix}0\\ 1\end{pmatrix}f(x).\end{aligned} \tag{43}$$

However, direct inversion of the solution $\hat{\Phi}(s,y)$ is impossible, since some integrals diverge due to the presence of the factor $\exp(-sy)$ in their integrands. Therefore, it is proposed, following the approaches outlined in Galybin [8] and Galybin and Rogerson [9], to reduce the inversion of FT to integral equations of the first kind by using the following manipulations.

We represent the exponent $\exp(-sy)$ as the difference of hyperbolic functions:

$$e^{-ys}=\cosh(ys)-\sinh(ys), \tag{44}$$

followed by substituting eqn (44) into eqn (22). Thus, one finds:

$$\begin{pmatrix} \hat{\Phi}(s,y) \\ \hat{\Omega}(s,y) \end{pmatrix} = \left(\cosh(ys) - \sinh(ys)\right) \begin{pmatrix} \hat{\Phi}(s,0) \\ \hat{\Omega}(s,0) \end{pmatrix} = \cosh(ys)\left(1 - \tan(ys)\right) \begin{pmatrix} \hat{\Phi}(s,0) \\ \hat{\Omega}(s,0) \end{pmatrix}. \quad (45)$$

One can further divide eqn (45) by $\cosh(ys) > 0$ and use eqn (43) to obtain:

$$\frac{1}{\cosh(ys)} \begin{pmatrix} \hat{\Phi}(s,y) \\ \hat{\Omega}(s,y) \end{pmatrix} = \left(1 - \tan(ys)\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \hat{\Phi}(s,0) - \left(1 - \tan(ys)\right) \begin{pmatrix} 0 \\ f(s) \end{pmatrix}. \quad (46)$$

Then we use the table integrals from Bateman and Erdelyi [13] to obtain the following inverse transformations

$$\mathrm{F}_{-1}\left(\cosh_{-1}(\pi s)\right) = \frac{1}{2\pi} \frac{1}{\cosh \frac{x}{2}}, \quad \mathrm{F}_{-1}\left(\tanh(\pi s)\right) = \frac{-1}{2\pi i} \frac{1}{\sinh \frac{x}{2}}. \quad (47)$$

Since the inversion of $\hat{\Omega}(s,y)$ is expressed linearly in terms of the inversion of $\hat{\Phi}(s,y)$, we can further consider only the first row of eqn (46). Assuming $y = 2h$ and applying the convolution theorem, one obtains the following integral equation for the determination of the unknown complex potential $\Phi(z)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(t,y)}{\cosh \frac{\pi}{2} \frac{t-x}{y}} dt = y\Phi(x,0) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(t,0)}{\sinh \frac{\pi}{2} \frac{t-x}{y}} dt, \quad 0 < y \le 2h \ . \quad (48)$$

Eqn (48) is the fundamental integral equation of the considered boundary value problem. It belongs to the class of Fredholm integral equations of the first kind. Therefore, its solution may be unstable. In this case, it is necessary to use special regularization methods, for example, Tikhonov's regularization [14] or truncation of the SVD expansion. It is worth noting that the right hand side of eqn (48) is subjected to measurement errors because $\Phi(x,0)$ contains deflection monitoring data, which emphasises the importance of regularization. It has been numerically shown that the SVD regularization provides reasonable results in a somewhat similar problem, see Galybin and Rogerson [9] for more details, where the considered integral equation is reduced to the inversion of the Stieltjes transform. Eqn (48) can also be reduced to the same problem by introducing the following substitutions:

$$x = -\tfrac{2}{\pi} y \ln \zeta, \quad t = -\tfrac{2}{\pi} y \ln \eta, \quad 0 < \zeta, \eta < \infty \ . \quad (49)$$

Further discussion of the numerical approach is out of scope of this paper.

As a result of solving eqn (48), the boundary values of both complex potentials are determined, and, consequently, the distribution of normal stresses at the interface, i.e. inversion of eqn (42). Analysis of this solution makes it possible to reveal the presence of a crack at the interface, which may occur in the area where these stresses are tensile.

6 CONCLUSIONS

This paper presented an approach to handle poroelastic boundary value problems for a layer attached to the base without modelling fluid transport. Instead we have introduced unknown pressure at the interface between the layer and the base and reduced the problem to the semi-inverse BVP, which necessitate monitoring the deflection of the layer. The problem has been further reduced to a system of Fredholm integral equations of the first kind.

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