FINITE LINE METHOD FOR SOLVING CONVECTION–DIFFUSION EQUATIONS

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ABSTRACT
In this paper, a creative collocation-type numerical method, the Finite Line Method (FLM), is proposed for solving general convection–diffusion equations. The method is based on the use of a finite number of lines crossing each collocation point, and the Lagrange polynomial interpolation formulation to construct the shape functions over each line. The directional derivative technique is proposed to derive the first-order partial derivatives of any physical variables with respect to the global coordinates for the high-dimensional problems from the lines' ones and the high-order derivatives are evaluated from a recurrence formulation. The derived spatial partial derivatives are directly substituted into the governing partial differential equations and related boundary conditions of the convection–diffusion equations to set up the system of equations. The finite number of lines crossing each collocation point is called the line set. To evaluate the convection and diffusion terms accurately, two different line sets are used for these two terms, which are called the convection line set and central line set, respectively. The former is formed according to the velocity direction and is used for performing the upwind scheme in the computation of the convection term, and the latter is formed by the crossed lines including the collocation point at the center. A numerical example will be given to verify the correctness and stability of the proposed method.
Keywords: finite line method, cross-line method, free element method, mesh free method, convection–diffusion equation.

1 INTRODUCTION

Convection–diffusion equations (CDEs) appear in a variety of physical phenomena [1], such as in the heat transfer and fluid mechanics [2]. Numerical solution of CDEs has become an important means of analysis. Robust numerical methods for solving CDEs are the Boundary Element Method (BEM) [3], [4], the Finite Volume Method (FVM) [5], [6], the Finite Difference Method (FDM) [7], [8], the Finite Element Method (FEM) [9]–[11], and the Mesh Free Method (MFM) [12], [13]. Although these methods can be used to solve a wide range of engineering problems in solid mechanics, fluid mechanics and heat transfer, they also have some drawbacks. BEM leads to dense coefficient matrices and, as a result, the memory requirement grows quadratically with respect to the number of degrees of freedom; in FVM, it is difficult to set up high-order control volumes, so that the accuracy of fluxes, variables related to the gradient of basic physical variables, is not high. FDM may result in poor accuracy for irregular geometry problems and it is very tedious to derive the formulations for computing high-order partial derivatives using Taylor's series expansion [14], since, for different-order derivatives, there is need to derive different formulations, which is not convenient for setting up a unified scheme suitable for different problems; FEM usually requires a variational principle or energy equation to set up the solution algorithm, which is not convenient for solving problems governed by high-order partial differential equations (PDEs); and in MFM, it is difficult to impose boundary conditions and anisotropic material properties.

To overcome these drawbacks in stability, accuracy and suitability of irregular geometries, Gao et al. [15] proposed a new type of collocation method, called the free element method (FrEM), which absorbs the advantages of FEM in stability, FDM in easy use and

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MFM in suitability for complicated geometries. FrEM has been successfully used to solve heat conduction [16], piezoelectric [17], solid mechanics [18] and fluid mechanics [19] problems. However, when using high-order free elements in FrEM, a lot of element nodes are required. To overcome this weakness, Gao et al. recently constructed the cross-line elements (CLEs) for FrEM [20], which can use extremely few nodes to discretize problems' geometries and thus can achieve a very high computational efficiency. However, only the second-order CLEs were constructed in Gao et al. [20] and it is very difficult to construct high-order CLEs based on the standard shape function construction procedure. To solve this issue, a completely new numerical method, called the cross-line method (CLM) [21] (also called the finite line method (FLM) [22]), is proposed by Gao based on an innovative idea of using directional derivative technique.

In this paper, FLM is applied to solve convection–diffusion equations for the first time. In FLM, only two or three lines crossing a collocation point, which are called the line set, are required to derive the spatial partial derivatives used for collocating the governing differential equations and related Neumann boundary conditions. In solving CDEs, the upwind scheme is implemented to evaluate the convection term, while the diffusion term is computed using the central scheme which can give more accurate results. Thus, two different line sets are used for a same collocation point to evaluate the convection and diffusion terms appearing in the CDEs. The most distinct advantage of the proposed FLM is that arbitrary high-order schemes can be easily established by using the Lagrange polynomial interpolation formulation over each line. A numerical example will be given to demonstrate the correctness and efficiency of the proposed method.

2 FINITE LINE METHOD FOR EVALUATING SPATIAL PARTIAL DERIVATIVES

The most engineering problems are governed by the second or higher order partial derivatives with respect to the global coordinates. In the presented FLM, the formulations of evaluating spatial partial derivatives are derived based on a finite number of lines, called line set.

2.1 Line sets for different collocation points

In FLM, the computational domain is discretized into a certain number of collocation points (CPs) and at each CP, two lines or three lines crossing the CP are defined for 2D ($d = 2$) or 3D ($d = 3$) problems. Different line patterns are defined for different positions of CP, as shown in Fig. 1 for 2D problems using 5 nodes along each line and Fig. 2 for 3D problems using 3 nodes along each line.

It is noted that for the obvious view, the crossed lines shown in Figs 1 and 2 are straight lines. Nevertheless, curved lines can also be used, since the number of nodes defined over each line can be more than two for high order function approximations. It is also mentioned that although the same number of nodes are illustrated over each line of a line set shown in Figs 1 and 2, different numbers of nodes can also be utilized for different crossed lines in a line set.

2.2 Formulations for different order partial derivatives derived by a directional derivative technique

Assuming that there are m nodes defined over a line shown in Figs 1 and 2, the global coordinates denoted by $x = \{x_1, x_2\} = \{x, y\}$ for 2D and $x = \{x_1, x_2, x_3\} = \{x, y, z\}$ for 3D

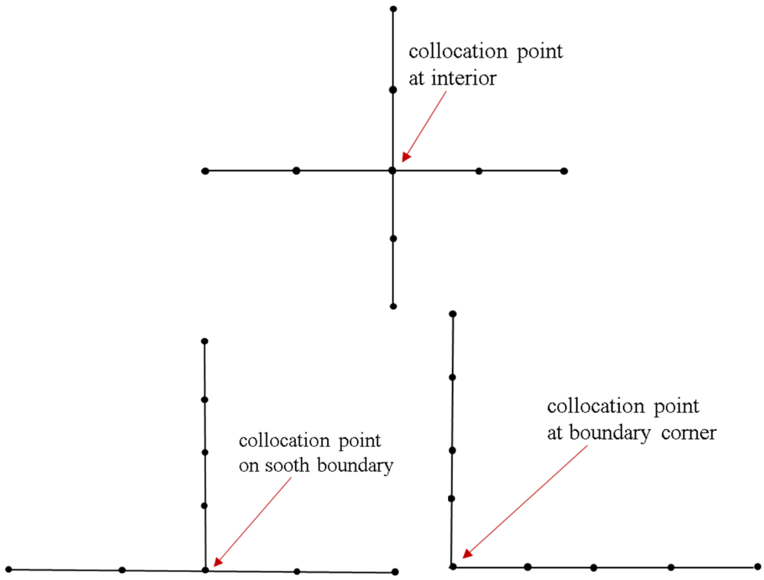


Figure 1: Line sets for different collocation point positions in 2D ($d = 2$) problems.

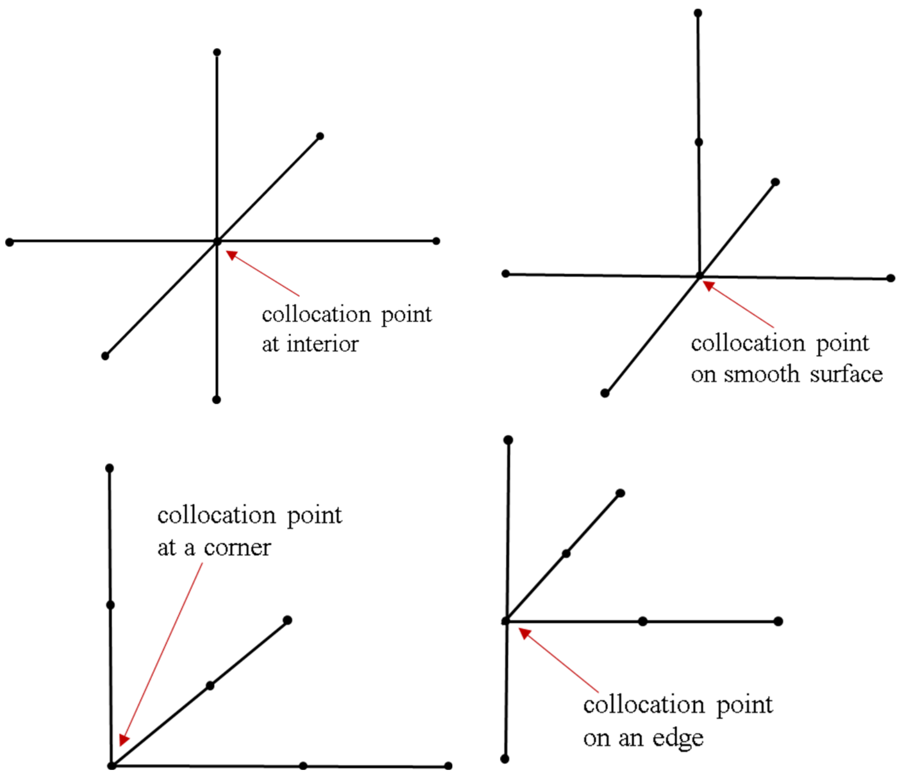


Figure 2: Line sets for different collocation point positions in 3D ($d = 3$) problems.

cases, and a physical variable $u(x)$ can be expressed in terms of the Lagrange polynomial interpolation formulation [15] as follows.

$$x_i = \sum_{\alpha=1}^{m} L_\alpha(l) x_i^\alpha , \tag{1}$$

$$u(\boldsymbol{x}) = \sum_{\alpha=1}^{m} L_\alpha(l) u^\alpha , \tag{2}$$

where l is the length of the current line, measured from node 1 to the current position $\boldsymbol{x}$ over the line, and x_i^α, $i = 1, ..., d$, and u^α are the values of coordinates and variable u at node α, and $L_\alpha(l)$ is the Lagrange polynomial formulation, which can be expressed [15], [21] as

$$L_\alpha(l) = \prod_{\beta=1,\beta\neq\alpha}^{m} \frac{l - l_\beta}{l_\alpha - l_\beta}, \tag{3}$$

where l_β is the value of l at the β-th node of the considered line.

In the following, the 2D problem is taken as the background of derivation. Let us denote the first-order partial derivatives of the function $u(\boldsymbol{x})$ with respect to the global coordinates $\boldsymbol{x}$ using $\partial u/\partial x_i$, and thus the directional derivative of $u(\boldsymbol{x})$ along the considered line can be expresses as

$$\frac{\partial u}{\partial l} = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial l} = [\frac{\partial x_1}{\partial l}, \frac{\partial x_2}{\partial l}] \begin{Bmatrix} \dfrac{\partial u}{\partial x_1} \\ \dfrac{\partial u}{\partial x_2} \end{Bmatrix}. \tag{4}$$

We can write up an equation like eqn (4) for each line crossing the considered collocation point, and thus after considering all lines of the line set, the following matrix equation can be written:

$$[J]\left\{\frac{\partial u}{\partial x}\right\} = \left\{\frac{\partial u}{\partial l}\right\} \quad \text{or} \quad \left\{\frac{\partial u}{\partial x}\right\} = [J]^{-1}\left\{\frac{\partial u}{\partial l}\right\}, \tag{5}$$

where

$$[J] = \begin{bmatrix} \dfrac{\partial x_1}{\partial l_1}, \dfrac{\partial x_2}{\partial l_1} \\ \dfrac{\partial x_1}{\partial l_2}, \dfrac{\partial x_2}{\partial l_2}, \end{bmatrix}, \tag{6}$$

$$\left\{\frac{\partial u}{\partial x}\right\}=\left\{\begin{matrix}\dfrac{\partial u}{\partial x_1}\\ \dfrac{\partial u}{\partial x_2}\end{matrix}\right\},\quad \left\{\frac{\partial u}{\partial l}\right\}=\left\{\begin{matrix}\dfrac{\partial u}{\partial l_1}\\ \dfrac{\partial u}{\partial l_2}\end{matrix}\right\}. \tag{7}$$

For the easy use, eqn (5) can be expressed as the following component form:

$$\frac{\partial u(\boldsymbol{x}^c)}{\partial x_i}=\sum_{\alpha=1}^{M_1} N_i'^{\alpha}(\boldsymbol{x}^c)u^{\alpha}\ , \tag{8}$$

where M_1 is the number of nodes defined over all lines of the considered line set for the 1st order partial derivative, and

$$N_i'^{\alpha}(\boldsymbol{x}^c)=\sum_{I=1}^{d}[J]_{iI}^{-1}\frac{\partial L_{\alpha}^{I}(l^c)}{\partial l_I}\,, \tag{9}$$

where the index I represents the line number and c denotes the value at the collocation point.

It is noted that since the size of matrices $[J]$ shown in eqn (6) is 2 for 2D and 3 for 3D problems, its inverse matrix can be easily obtained analytically.

For simpler use, eqn (8) can also be expressed as

$$\frac{\partial u(\boldsymbol{x}^c)}{\partial x_i}=d_i^{c\alpha}u^{\alpha}\quad \text{with}\quad d_i^{c\alpha}=N_i'^{\alpha}(\boldsymbol{x}^c)\,, \tag{10}$$

where the repeated index α represents the summation over all nodes of the considered line set.

The second-order partial derivatives can be easily evaluated by using eqn (10) twice, i.e.,

$$\frac{\partial^2 u(\boldsymbol{x}^c)}{\partial x_i\,\partial x_j}=d_{ij}^{c\alpha}u^{\alpha}\quad \text{with}\quad d_{ij}^{c\alpha}=d_j^{c\beta}d_i^{\beta\alpha}\,. \tag{11}$$

If needed, the formulations for more high-order derivatives can be established in a similar recursive manner. In eqns (10) and (11), $d_i^{c\alpha}$ and $d_{ij}^{c\alpha}$ are called the first order and second order derivative operators, respectively.

Eqns (10) and (11) can be directly substituted into the governing equations and related boundary conditions of a specific engineering problem to set up the discretized system of equations.

3 FINITE LINE METHOD FOR SOLVING CONVECTION–DIFFUSION EQUATIONS

The convection–diffusion equation for time-independent problems [1], [3] can be expressed as

$$\frac{\partial cv_i(\boldsymbol{x})u(\boldsymbol{x})}{\partial x_i}+\frac{\partial}{\partial x_i}\left(\lambda_{ij}(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_j}\right)+s(\boldsymbol{x})=0\,, \tag{12}$$

with the following boundary conditions:

$$u(\boldsymbol{x}) = \overline{u}, \qquad \boldsymbol{x} \in \partial\Omega_1, \tag{13}$$

$$-\lambda_{ij}(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_j} n_i(\boldsymbol{x}) = \overline{q}, \qquad \boldsymbol{x} \in \partial\Omega_2, \tag{14}$$

where c is a coefficient, v_i is the velocity, λ_{ij} the diffusion coefficient, and s the source term. The repeated index i and j represent the summation over d components. The typical engineering problem represented by eqn (12) is the energy equation in fluid mechanics [2], [19].

In eqn (12), the first term is the convection term, and the second one is the diffusion term. When we discretize eqn (12) in FLM, we need to employ two different line sets for these two terms, the upwind line set and the central line set (the central line set also being called the diffusion line set), respectively. The former embodies the influence of the incoming flow on the collocation point, while the latter embodies the influence of all surrounding nodes on the collocation point. Fig. 3 is an illumination of a 2D mesh connected by all line sets formed for all points, in which the nodes over the upwind and diffusion line sets for a particular collocation point are denoted by circles. In Fig. 3, the upwind line set is represented by nodes marked using black colored circles, and the central line set (diffusion line set) is marked using the red colored circles.

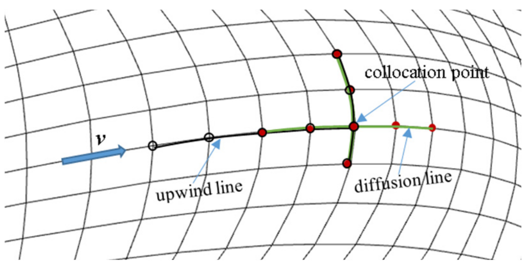


Figure 3: The upwind and diffusion line sets for a collocation point.

Thus, by using two different line sets, eqn (12) can be discretized into the following form for an internal collocation point $\boldsymbol{x}^c$:

$$\tilde{d}_i^{c\alpha} c^\alpha v_i^\alpha u^\alpha + d_i^{c\beta} \lambda_{ij}^\beta d_j^{\beta\alpha} u^\alpha + s(\boldsymbol{x}^c) = 0, \tag{15}$$

and the Neumann boundary condition eqn (14) can also be discretized as follows

$$-\lambda_{ij}(\boldsymbol{x}^c) n_i(\boldsymbol{x}^c) d_j^{c\alpha} u^\alpha = \overline{q}, \quad \boldsymbol{x}^c \in \partial\Omega_2. \tag{16}$$

In eqns (15) and (16), $\tilde{d}_i^{c\alpha}$ and $d_i^{c\beta}$ are the derivative operators computed based on the upwind and diffusion line sets, respectively, using eqns (10) and (11).

Collocating $\boldsymbol{x}^c$ at all points, the following matrix equation can be established by eqn (15), (16) or (13).

$$[C]\{u\} = \{f\}, \tag{17}$$

where $\{u\}$ is a vector consisting of all points u, and $\{f\}$ is a known vector formed by eqn (13) or (16) and the source term s in eqn (15).

By solving eqn (17), we can obtain the values of u at all points. It is noted that for a collocation point, only nodes over the crossed lines of the point have contributions, therefore the coefficient matrix $[C]$ in eqn (17) is extremely sparse and this makes eqn (17) can be quickly solved by a sparse equation solver.

4 NUMERICAL EXAMPLES

To verify the correctness and stability of the proposed FLM, a one-dimensional (1D) convection–diffusion equation, which has the exact solution, is considered in the following. The governing equation considered is as follows

$$\frac{\partial cvu}{\partial x} + \frac{\partial}{\partial x}\left(\Gamma \frac{\partial u}{\partial x}\right) = 0,$$

with the boundary conditions:

$$x=0, u = u_0; \quad x=L, u = u_L,$$

where $c = 1$, $\Gamma = 100$, $L = 100$, $u_0 = 300$, $uL = 800$, and two constant velocities are computed using $v = 3$ and $v = 5$, respectively.

The exact solution of the problem can be expressed as

$$\frac{u - u_0}{u_L - u_0} = \frac{\exp(cvx/\Gamma) - 1}{\exp(cvL/\Gamma) - 1}.$$

By using eqns (10) and (11), the discretized form of above equation can be expressed as

$$cv\tilde{d}_1^{c\alpha} u^\alpha + \Gamma d_{11}^{c\alpha} u^\alpha = 0$$

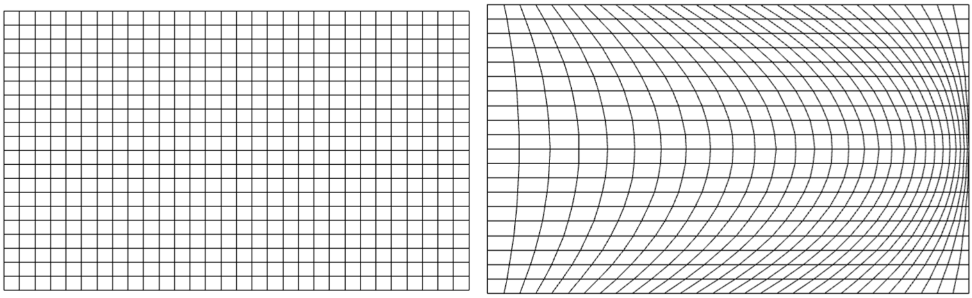
This example is a 1D problem, but the 2D algorithm is used to simulate it in the FLM simulation by using points of 100×60. To examine the performance of FLM on grids quality, two meshes are used, regular and irregular ones, as shown in Fig. 4 which are formed by connecting all lines established for all collocation points. In all computations, 3 and 5 nodes were examined over each line in both upwind and central schemes, and it was found that almost no differences exist in the computational results from these two cases. Fig. 5 shows the contour plots of the computed u based on the regular and irregular meshes, and Fig. 6 shows the distribution of computed u along x coordinate using $v = 3$ and $v = 5$, respectively.

From Fig. 5, it can be seen that the contour lines are very straight even using irregular meshes. This indicates that the proposed method in the paper is stable. And Fig. 6 shows that the computed results using regular and irregular meshes are in good agreement with the exact solutions. This indicates that the proposed FLM is correct and has a high accuracy.

5 CONCLUDING REMARKS

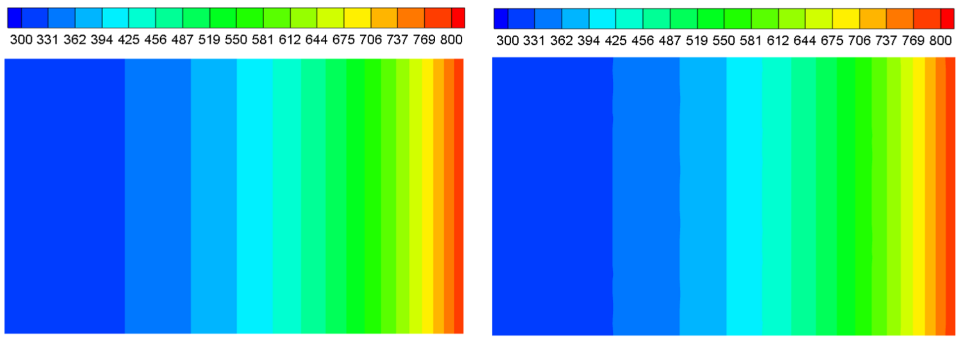
A new numerical method, finite line method (FLM), has been presented in the paper for solving convection–diffusion equations. The following concluding remarks can be drawn:

(1) The presented FLM only needs two or three lines to establish the formulations of computing different orders of spatial partial derivatives, which can be directly substituted into the differential equations to set up the system of equations.



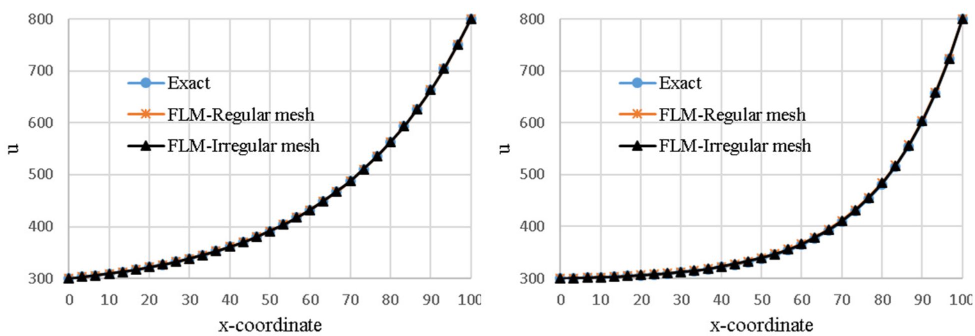
(a) (b)

Figure 4: Computational meshes connected by lines used for all points. (a) Regular mesh; and (b) Irregular mesh.



(a) (b)

Figure 5: Contour plots of computed *u* using two meshes. (a) From regular mesh; and (b) From irregular mesh.



(a) (b)

Figure 6: Distribution of computed u along x coordinate using two different velocities. (a) $v = 3$; and (b) $v = 5$.

(2) The upwind and central line sets should be used for simulating the convection and diffusion terms, respectively.
(3) The use of FLM is similar to FDM, having advantages of easy use and excellent flexibility; and, high-order function approximations can be easily formed in FLM.
(4) Irregular meshes can also give accurate results in FLM, and this is a distinct advantage of FLM over FDM.
(5) Attributed to the feature that FLM works based on lines, it is easy to prepare data for problems with complex geometries and complicated physical phenomena.

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