A Kansa-RBF method for Poisson problems in annular domains

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Abstract

We employ a Kansa-radial basis function (RBF) method for Poisson boundary value problems in annular domains. This discretization leads, for any choice of RBF, to linear system matrices possessing block circulant structures. The linear systems can be solved efficiently using Matrix Decomposition Algorithms (MDAs). The feasibility of the proposed technique is illustrated by a numerical example.
Keywords: radial basis functions, Poisson equation, fast Fourier transforms, Kansa method.

1 Introduction

We consider the discretization of Poisson boundary value problems in annular domains using a Kansa radial basis function (RBF) method [5], see also [3]. For any choice of RBF, for an appropriate choice of collocation points, such discretizations lead to linear systems in which the coefficient matrices possess block circulant structures. These systems can be solved efficiently using Matrix Decomposition Algorithms (MDAs) [1] with Fast Fourier Transforms (FFTs). Such MDAs have been used in the past in various applications of the Method of Fundamental Solutions to boundary value problems in geometries possessing radial symmetry, see e.g., [6, 7], as well as RBF approximations and their derivatives in circular domains in [8], see also [4].


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2 The problem

We consider the Poisson equation

$$\Delta u = f \quad \text{in} \quad \Omega, \tag{1a}$$

subject to the Dirichlet boundary conditions

$$u = g_1 \quad \text{on} \quad \partial\Omega_1, \tag{1b}$$

$$u = g_2 \quad \text{on} \quad \partial\Omega_2, \tag{1c}$$

in the annulus

$$\Omega = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : \varrho_1 < |\boldsymbol{x}| < \varrho_2 \right\}.$$

The boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$ where

$$\partial\Omega_1 = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| = \varrho_1 \right\} \quad \text{and} \quad \partial\Omega_2 = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| = \varrho_2 \right\}.$$

3 Kansa's method

We define the M angles

$$\vartheta_m = \frac{2\pi(m-1)}{M}, \quad m = 1, \ldots, M,$$

and the N radii

$$\mathsf{r}_n = \varrho_1 + (\varrho_2 - \varrho_1)\,\frac{n-1}{N-1}, \quad n = 1, \ldots, N.$$

The collocation points $\{(x_{mn}, y_{mn})\}_{m=1,n=1}^{M,N}$ are given from

$$x_{mn} = \mathsf{r}_n \cos(\vartheta_m + \frac{2\pi\alpha_n}{N}), \quad y_{mn} = \mathsf{r}_n \sin(\vartheta_m + \frac{2\pi\alpha_n}{N}), \tag{2}$$

where the parameters $\{\alpha_n\}_{n=1}^{N} \in [-1/2, 1/2]$ correspond to rotations of the collocation points and may be used to produce more uniform distributions.

In the current application of Kansa's method, we take

$$u_{MN}(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} \phi_{mn}(x, y), \qquad (x, y) \in \bar{\Omega}, \tag{3}$$

where the MN coefficients $\{(a_{mn})\}_{m=1,n=1}^{M,N}$ are unknown. The RBFs $\phi_{mn}(x, y)$ can be expressed in the form

$$\phi_{mn}(x, y) = \Phi(r_{mn}), \quad \text{where} \quad r_{\mathsf{n}}^2 = (x - x_{mn})^2 + (y - y_{mn})^2.$$

These coefficients are determined by collocating the differential equation (1a) and the boundary conditions (1b)-(1c) in the following way:

$$\begin{aligned} \Delta u_{MN}(x_{mn}, y_{mn}) &= f(x_{mn}, y_{mn}), \quad m = 1, \ldots, M,\ n = 2, \ldots, N-1, \\ u_{MN}(x_{m1}, y_{m1}) &= g_1(x_{m1}, y_{m1}), \quad m = 1, \ldots, M, \\ u_{MN}(x_{mN}, y_{mN}) &= g_2(x_{mN}, y_{mN}), \quad m = 1, \ldots, M, \end{aligned} \tag{4}$$

yielding a total of MN equations.

The vectorization of the arrays of unknown coefficients and collocation points from

$$\mathsf{a}_{(n-1)M+m} = a_{mn}, \quad \mathsf{x}_{(n-1)M+m} = x_{mn},$$

$$\mathsf{y}_{(n-1)M+m} = y_{mn}, \quad m = 1, \ldots, M,\ n = 1, \ldots, N,$$

equations (4) yield an $MN \times MN$ system of the form

$$A\,\boldsymbol{a} = \begin{pmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,N} \\ A_{2,1} & A_{2,2} & \ldots & A_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,1} & A_{N,2} & \ldots & A_{N,N} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_N \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_N \end{pmatrix} = \boldsymbol{b}\,. \tag{5}$$

The $M \times M$ submatrices $A_{n_1,n_2},\ n_1, n_2 = 1, \ldots, N$ are defined as follows:

$$\begin{aligned} \left(A_{n_1,n_2}\right)_{m_1,m_2} &= \Delta\phi_{m_2,n_2}(x_{m_1,n_1}, y_{m_1,n_1}), \\ m_1, m_2 = 1, \ldots, M,\ n_1 &= 2, \ldots, N-1,\ n_2 = 1, \ldots, N, \end{aligned}$$

$$\left(A_{1,n}\right)_{m_1,m_2} = \phi_{m_2,n}(x_{m_1,1}, y_{m_1,1}), \quad m_1, m_2 = 1, \ldots, M,\ n = 1, \ldots, N,$$

$$\left(A_{N,n}\right)_{m_1,m_2} = \phi_{m_2,n}(x_{m_1,N}, y_{m_1,N}), \quad m_1, m_2 = 1, \ldots, M,\ n = 1, \ldots, N,$$

while the $M \times 1$ vectors $\boldsymbol{a}_n, \boldsymbol{b}_n, n = 1, \ldots, N$ are defined as

$$\left(\boldsymbol{a}_n\right)_m = a_{mn},\ m = 1, \ldots, M,\ N = 1, \ldots, N,$$

$$\left(\boldsymbol{b}_n\right)_m = f(x_{mn}, y_{mn}), \quad m = 1, \ldots, M,\ n = 2, \ldots, N-1,$$

$$\left(\boldsymbol{b}_1\right)_m = g_1(x_{m1}, y_{m1}), \quad \left(\boldsymbol{b}_N\right)_m = g_2(x_{mN}, y_{mN}), \quad m = 1, \ldots, M.$$

It can be easily shown that each of the $M \times M$ submatrices A_{n_1,n_2}, $n_1, n_2 = 1, \ldots, N$ in the coefficient matrix in (5) is circulant [2]. Hence matrix A in system (5) is block circulant.

4 Matrix decomposition algorithm

First, we define the unitary $M \times M$ Fourier matrix

$$U_M = \frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \bar{\omega} & \bar{\omega}^2 & \cdots & \bar{\omega}^{M-1} \\ 1 & \bar{\omega}^2 & \bar{\omega}^4 & \cdots & \bar{\omega}^{2(M-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \bar{\omega}^{M-1} & \bar{\omega}^{2(M-1)} & \cdots & \bar{\omega}^{(M-1)(M-1)} \end{pmatrix}, \tag{7}$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/M}$, $\mathrm{i}^2 = -1$.

If I_N is the $N \times N$ identity matrix, pre–multiplication of system (5) by $I_N \otimes U_M$ yields

$$(I_N \otimes U_M)\, A\, (I_N \otimes U_M^*)\, (I_N \otimes U_M)\, \boldsymbol{a} = (I_N \otimes U_M)\, \boldsymbol{b}$$

or

$$\tilde{A}\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{b}}, \tag{8}$$

where

$$\tilde{A} = (I_N \otimes U_M)\, A\, (I_N \otimes U_M^*)$$

$$= \begin{pmatrix} U_M A_{1,1} U_M^* & U_M A_{1,2} U_M^* & \cdots & U_M A_{1,N} U_M^* \\ U_M A_{2,1} U_M^* & U_M A_{2,2} U_M^* & \cdots & U_M A_{2,N} U_M^* \\ \vdots & \vdots & & \vdots \\ U_M A_{N,1} U_M^* & U_M A_{N,2} U_M^* & \cdots & U_M A_{N,N} U_M^* \end{pmatrix}$$

$$= \begin{pmatrix} D_{1,1} & D_{1,2} & \cdots & D_{1,N} \\ D_{2,1} & D_{2,2} & \cdots & D_{2,N} \\ \vdots & \vdots & & \vdots \\ D_{N,1} & D_{N,2} & \cdots & D_{N,N} \end{pmatrix}, \tag{9}$$

and

$$\tilde{\boldsymbol{a}} = (I_N \otimes U_M)\, \boldsymbol{a} = \begin{pmatrix} U_M \boldsymbol{a}_1 \\ U_M \boldsymbol{a}_2 \\ \vdots \\ U_M \boldsymbol{a}_N \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{a}}_1 \\ \tilde{\boldsymbol{a}}_2 \\ \vdots \\ \tilde{\boldsymbol{a}}_N \end{pmatrix},$$

$$\tilde{\boldsymbol{f}} = (I_N \otimes U_M)\, \boldsymbol{b} = \begin{pmatrix} U_M \boldsymbol{b}_1 \\ U_M \boldsymbol{b}_2 \\ \vdots \\ U_M \boldsymbol{b}_N \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{b}}_1 \\ \tilde{\boldsymbol{b}}_2 \\ \vdots \\ \tilde{\boldsymbol{b}}_N \end{pmatrix}. \tag{10}$$

From the properties of circulant matrices [2], each of the $M \times M$ matrices $D_{n_1,n_2},\ n_1, n_2 = 1, \cdots, N$, is diagonal. If, in particular

$$D_{n_1,n_2} = \text{diag}\left(D_{n_1,n_{2_1}}, D_{n_1,n_{2_2}}, \ldots, D_{n_1,n_{2_M}}\right)$$

and

$$A_{n_1,n_2} = \text{circ}\left(A_{n_1,n_{2_1}}, A_{n_1,n_{2_2}} \ldots, A_{n_1,n_{2_M}}\right),$$

we have, for $n_1, n_2 = 1, \cdots, N$,

$$D_{n_1,n_{2_m}} = \sum_{k=1}^{M} A_{n_1,n_{2_k}} \omega^{(k-1)(m-1)},\ m = 1, \cdots, M. \tag{11}$$

Since the matrix $\tilde{A}$ consists of N^2 blocks of order M, each of which is diagonal, the solution of system (8) can be decomposed into solving the M independent systems of order N

$$E_m\, \boldsymbol{x}_m = \boldsymbol{y}_m, \quad m = 1, \cdots, M, \tag{12}$$

where

$$(E_m)_{n_1,n_2} = D_{n_1,n_{2_m}},\ n_1, n_2 = 1, \cdots, N,$$

and

$$(\boldsymbol{x}_m)_n = (\tilde{\boldsymbol{a}}_n)_m,\ (\boldsymbol{y}_m)_n = \left(\tilde{\boldsymbol{b}}_n\right)_m,\ n = 1, \cdots, N. \tag{13}$$

Having obtained the vectors $\boldsymbol{x}_m,\ m = 1, \cdots, M$, we can recover the vectors $\tilde{\boldsymbol{a}}_n,\ n = 1, \cdots, N$ and, subsequently, the vector $\boldsymbol{a}$ from

$$\boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_N \end{pmatrix} = (I_N \otimes U_M^*)\, \tilde{\boldsymbol{a}} = \begin{pmatrix} U_M^* \tilde{\boldsymbol{a}}_1 \\ U_M^* \tilde{\boldsymbol{a}}_2 \\ \vdots \\ U_M^* \tilde{\boldsymbol{a}}_N \end{pmatrix}. \tag{14}$$

Clearly, FFTs are used in the calculation of $\tilde{\boldsymbol{f}}$ in (10) with a cost of $O(NM \log M)$. Also, FFTs are used in the calculation of the diagonal elements in (11) with a cost of $O(N^2 M \log M)$. Finally, FFTs are used in the recovery of the coefficients $\boldsymbol{a}$ in (14) at a cost of $O(NM \log M)$. The most expensive part of the proposed algorithm is the solution of systems (12) with a cost of $O(MN^3)$. The FFTs are carried out using the MATLAB© [9] commands `fft` and `ifft`.

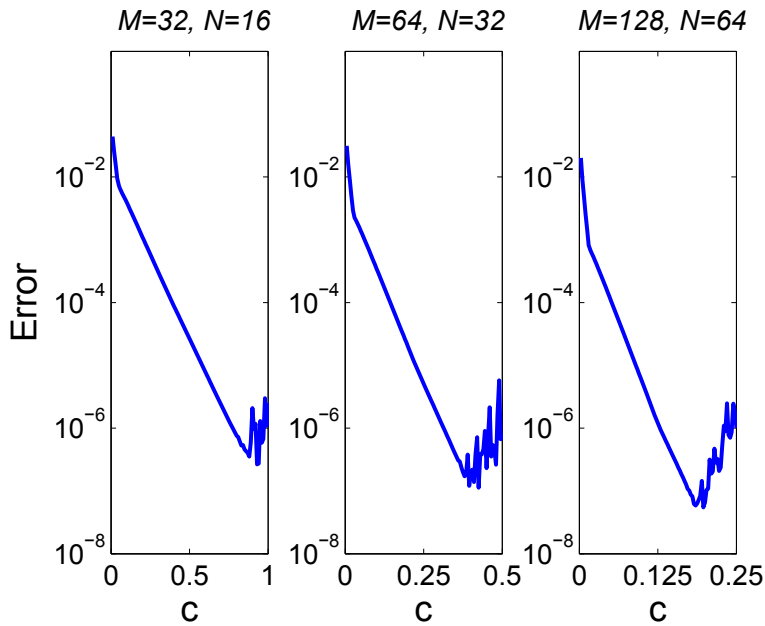

Figure 1: Example 1: Error versus c.

5 Numerical results

We took collocation points described by $\alpha_n = (-1)^n/4, n = 1, \ldots, N$ (cf. (2)), and calculated the maximum relative error E over $\mathcal{M}\mathcal{N}$ test points defined by

$$\mathrm{r}_n \left(\cos\vartheta_m, \sin\vartheta_m\right), \text{ where } \vartheta_m = \frac{2\pi(m-1)}{\mathcal{M}}, \ m = 1, \ldots, \mathcal{M},$$

$$\mathrm{r}_n = \varrho_1 + (\varrho_2 - \varrho_1)\,\frac{n-1}{N-1}, \ n = 1, \ldots, \mathcal{N}.$$

We chose $\mathcal{N} = 25, \mathcal{M} = 50$ so that the test points are different than the boundary collocation points. The maximum relative error E is defined as

$$E = \frac{||u - u_N||_{\infty,\partial\Omega}}{||u||_{\infty,\partial\Omega}}.$$

As RBFs in (3) we used the multiquadric basis functions

$$\phi_{mn}(x, y) = \sqrt{r_{mn}^2 + c^2}, \quad r_{mn}^2 = (x - x_{mn})^2 + (y - y_{mn})^2,$$

where c is the shape parameter.

We considered the Dirichlet boundary value problem (1) for the Poisson equation corresponding to the exact solution $u = \mathrm{e}^{x+y}$. In Figure 1 we present the

error in u for various numbers of degrees of freedom versus the shape parameter c. As expected, the accuracy of the approximation for each set of (M, N) improves with c up to a point and then deteriorates.

6 Conclusions

We have applied a Kansa-RBF method for the solution of Poisson boundary value problems in annular domains. With an appropriate choice of collocation points the discretization yields linear systems in which the coefficient matrices are block circulant. Thus these systems may be solved efficiently using MDAs. The major advantages of the proposed technique are its potential for solving large-scale problems efficiently and its applicability for solving a large class of partial differential equations. Several techniques for finding an appropriate shape parameter may be found in the literature of RBFs. The issue of finding a suitable shape parameter for multiquadric basis functions in the context of circulant matrices will be considered in a future investigation. The extension of this technique to problems governed by the inhomogeneous biharmonic equation and the Cauchy–Navier equations of elasticity is currently under investigation.

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