Stability issues in 3D BEM formulations for transient elastodynamics

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Abstract

Time-domain boundary element method formulations (TD-BEM) are quite versatile in reproducing the transient response of finite and semi-infinite solid media and offer a number of advantages compared to transformed domain (Fourier or Laplace) approaches. They are known, however, to be prone to numerical instability, especially after a very large number of time steps. This seems to be more of a problem with 1D as compared to 3D formulations, although it is present in the latter ones. A careful investigation of the problem reveals that if the conventional BEM formulation using displacement and traction variables is replaced with one using velocity and traction pairs, the problem is ameliorated to a large extent and much better accuracy results.

Keywords: elastodynamics, stability, time-marching schemes, reciprocal theorems, convolution integrals.

1 Introduction

Although the importance of TD-BEM formulations in studying wave motion problems in elastic continua is indisputable, the conventional formulation in terms of the displacement and traction vectors (DBEM) has a latent problem when a large number of time steps are required. More specifically, unstable behaviour may occur that is known as ''intermittent instability'' [1]. Various schemes have been proposed in recent years, many of them drawn from finite element methodologies (such as the Wilson-theta integration algorithm) to eliminating or reduce this unstable behaviour of the time convolution integrals associated with the transient BEM. All such methods basically seek to modify

the underlying boundary integral equations so as to optimize the stability behaviour. Although stability may be enforced, it is usually accompanied by a reduction in accuracy, which is the consequence of the introduction of artificial damping and of natural period elongation [2]. Moreover, these algorithmic techniques are somewhat cumbersome, operating at different levels of complexity, and difficult to implement in existing BEM computer software codes. An alternative approach, as outlined in a recently by the present authors [3], is to introduce a new formulation based on the velocity reciprocal theorem (VBEM). This formulation has been successfully tested on simple problems, namely the single degree of freedom system and one dimensional wave propagation, with superior performance regarding stability as compared to formulations using the displacement reciprocal theorem. In this work, the fully 3D formulation is implemented following work outlined in ref. [4] and tested against benchmark problems. Again, overall superior performance of the VBEM compared to the DBEM is again established.

2   Velocity reciprocal theorem for 3D elastodynamics

Time domain BEM formulations use Graffi's reciprocal theorem [5], which employs displacement and traction fields and essentially is a work statement defined for a regular region V with boundary S (see Fig. 1) and material properties p, c_1, c_2 (mass density, pressure and shear wave speeds). In this work, we adopt an alternative strategy using a power statement that employs velocities in lieu of displacements. Specifically, consider two distinct elastodynamic states $[u_i, t_i, b_i]$ and $[u_i', t_i', b_i']$ comprising displacements, tractions and body forces in V with appropriate initial conditions. Next, define the velocity as the time derivative of displacement. For time $t \geq 0$ and by assuming (for simplicity) zero initial conditions, the following reciprocal theorem holds true

$$\int_S t_i * v'_i \, dS + \int_V \rho b_i * v'_i \, dV = \int_S t'_i * v_i \, dS + \int_V \rho b'_i * v_i \, dV \qquad (1)$$

where operation $*$ denotes Riemann's convolution. Utilizing this theorem with one state identified with the solution sought and the other with the fundamental singular solution pair, a velocity-based integral identity is:

$$c_{ij}(\xi) v_i(\xi, t) = \int_S \left(\dot{G}_{ij} * t_i(x) - F_{ij} * v_i(x) \right) dS(x) + p \int_V \dot{G}_{ij} * b_i(x) dV(x) \qquad (2)$$

The fundamental velocity solution $\dot{G}_{ij}(x,t; \xi, \tau = 0)$ is computed by noting that

$$\int_{c_1^{-1}}^{c_2^{-1}} \delta(t - \lambda r) \lambda \, d\lambda = \frac{t}{r^2} \left[H(t - \frac{r}{c_1}) - H(t - \frac{r}{c_2}) \right] \qquad (3)$$

and is given as:

$$\dot{G}_{ij} = (4\pi\rho)^{-1}\left\{a_{ij}(r)\dot{\delta}(t-\frac{r}{c_1})+b_{ij}(r)\dot{\delta}(t-\frac{r}{c_2})\right.$$
$$\left.+c_{ij}(r)\frac{t}{r^2}\left[\delta(t-\frac{r}{c_1})-\delta(t-\frac{r}{c_2})\right]+c_{ij}(r)\frac{1}{r^2}\left[H(t-\frac{r}{c_1})-H(t-\frac{r}{c_2})\right]\right\}$$

(4)

where spatial coefficients a_{ij}, b_{ij}, c_{ij} are the same as those appearing the conventional displacement BEM formulation [6] (i.e., the DBEM). This new formulation can easily be extended to incompressible materials [7], and becomes independent of the dilatational wave speed.
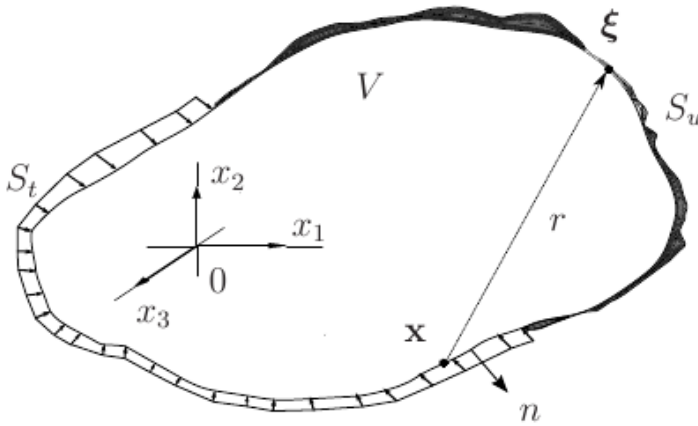

Figure 1:   Elastic solid body and BEM notation.

3   BEM formulations for displacements and for velocities

In order to proceed with formulating a BEM algorithm for elastodynamics, we need to establish spatial as well as temporal discretizations of the field variables. First, to integrate the time convolutions analytically, we proceed by dividing the time interval *[0,t]* into N equal time steps of length h and approximating any of the tractions, displacements (DBEM) or velocities (VBEM) in terms of basis functions. To study accuracy and stability, we consider a linear evolution of any time variable f (t) across a time-step as

$$f(t) = \sum_{n=1}^{N}\phi_n(t)\left(\frac{t_n-t}{h}f_{n-1}+\frac{t-t_{n-1}}{h}f_n\right)$$

(5)

where

$$\phi_n(t) = H(t-t_{n-1})-H(t-t_n)$$

(6)

and $t_n = nh$, $H(t)$ is the Heaviside function while f_{n-1}, f_n are values at discrete times t_{n-1}, t_n, respectively.

Regarding spatial discretization, a family of 2D surface isoparametric elements is used, namely the four, eight and nine node quadrilaterals (see Fig. 2). Spatial variation of any field variable f (x) is expressed in terms of the same shape functions M_a used for the geometry, i.e.,

$$f(x) = \sum_{a=1}^{q} M_a(x) f_a \tag{7}$$

In the above, q is the number of element's nodes, while f_a is a nodal value in space. After replacement in the BEM statement, space-time interpolation products arise, which in compact form read as follows:

$$f(x,t) = \sum_{n=1}^{N} \sum_{a=1}^{q} N_{1,n}(t) M_a(x) f_{a,n-1} + N_{2,n}(t) M_a(x) f_{a,n} \tag{8}$$

In the above,

$$N_{1,n(t)} = \phi_n(t) \frac{t_n - t}{h} \quad , \quad N_{2,n(t)} = \phi_n(t) \frac{t - t_{n-1}}{h} \tag{9}$$

Analytical spatial integrations of the resulting BEM system matrix coefficients are not in general possible and therefore numerical quadrature is used. Fundamental solution-shape function products are approximated by application of the Gauss-Legendre quadrature formula. There are two basic cases of integrand behaviour, namely non-singular and singular. Regarding the latter, weak singularities in both DBEM and VBEM are done by element subdivisions,
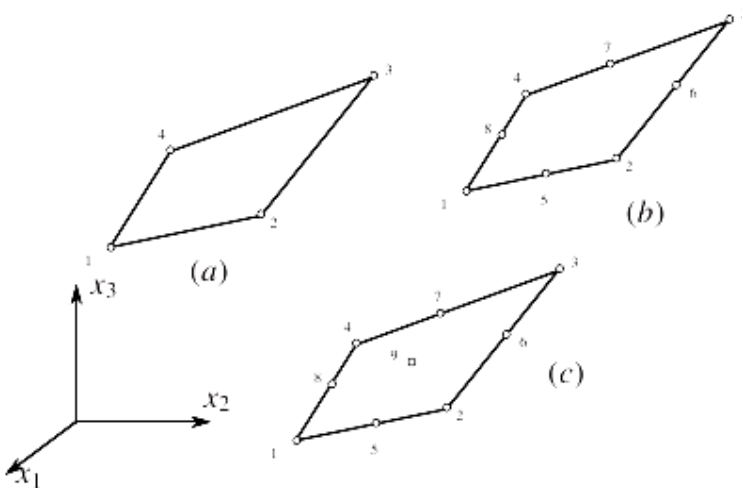

Figure 2:   Surface boundary elements: (a) four, (b) eight and (c) nine nodes.

followed by a series of transformations that reduces the order of singularity in the kernels by one and transforms them to regular ones. In the case of strongly singular traction kernels, the diagonal sub-matrices are computed utilizing indirect schemes based on the rigid body motion concept [8].

An important point for accuracy reasons in such formulations is identification of active zones of the integration along an element as it is sweep by the P and S waves. This is accomplished by invoking the causality principle [9] in the numerical integration scheme and employing element subdivision.

4   Stability analysis and BEM performance

We introduce a formal definition of stability as "Stability means that any 'initial' conditions at time t given by errors in the response, which may be due to round-off in the computer, do not grow in the integration". According to this definition [10] we distinguish between accuracy of a numerical method and its stability and focus on the latter. As example, we select a solid cube (see Fig. 3) with side length equal to L_{tot} . We discretize each side into ne × ne square boundary elements of length $L = L_{tot}/ne$. Next, we introduce the dimensionless Courant-Fredricks-Lewy (CFL) parameter as $\beta = cdt/L_W$, where c is either the P or S wave speed and L_W a representative boundary element length, such as the minimum distance between two nodes. We adopt the S wave speed in the CFL measure, since it has the advantage of being applicable to the incompressible material case. Then we assign fixed kinematic (displacements and velocities) boundary conditions on the negative x_1 face of the cube, zero tractions boundary conditions on the remaining faces and introduce unit initial 'error' for all boundary nodal values so as to trace its time evolution.
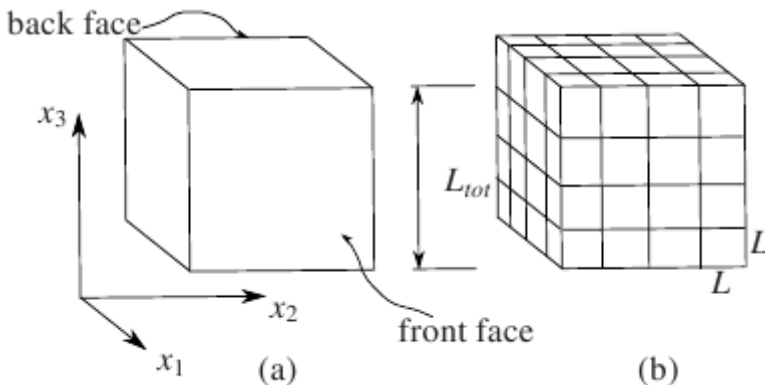

Figure 3:   (a) Geometry of solid cube used to study stability (b) BEM discretization.

4.1 Stability aspects of the DBEM formulation

The DBEM stability regions in terms on CFL parameter β_2 can be broken down in two parts: For $\beta_2 \leq 0.6$, the time-stepping algorithm shows instability that becomes more noticeable for diminishing β_2. For values of $\beta_2 > 0.6$, we observe a stable performance since the initial 'error' is damped as time evolves. Meanwhile, if the 'error' evolution is tracked over for a long time, *sudden instability* occurs as shown in Fig. 4. This has been reported elsewhere [11], and if we further increase β_2, we find a small stability region between *1.1≤ β_2≤ 1.2*, since for a large number of time steps the initial 'error' decreases. For values
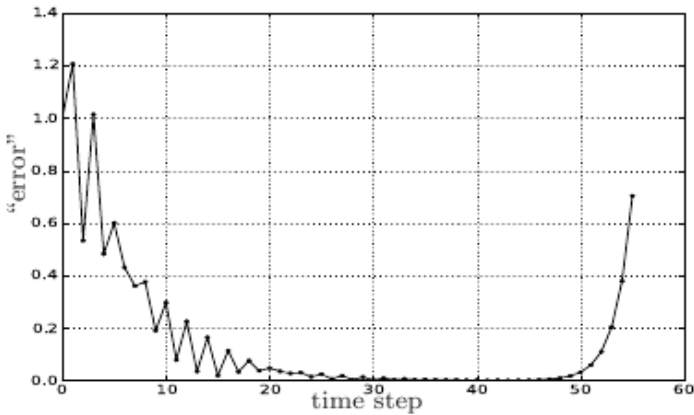
(a)
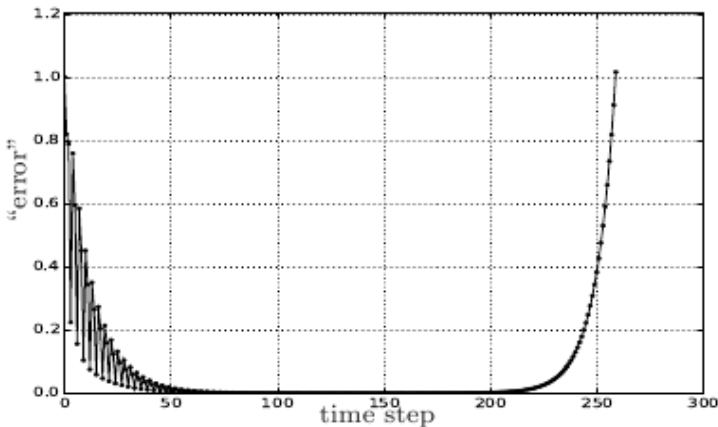
(b)

Figure 4:   Time evolution of DBEM displacement 'error' (a) β_2 = 0.8
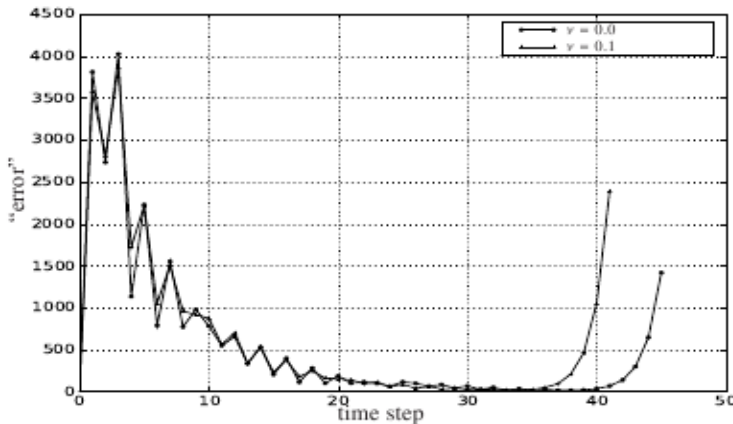(b) $\beta_2 = 0.9$.

Figure 5:   Time evolution of BDEM traction 'error' $\beta_2 = 0.75$ (a) $v = 0$
(b) $v = 0.1$.

greater than $\beta_2 > 1.2$, instability occurs once again. This kind of behavior, which is referred to in Ref. [12] as *intermittent instability*, is unacceptable as one cannot provide coherent guidelines on an appropriate choice of the time step.

Also, for Poisson's ratio values greater than zero, the DBEM present pronounced instability problems. This is observed in Fig. 5, where the 'error' evolution for tractions it is shown at a fixed β_2 value, but for two distinct values of Poisson's ratio v. For non-zero v, instability is manifested even for a small number of time steps. Furthermore, for large values of v it was impossible to define any stability region regarding the β_2 parameter, in terms of long time evolution, since *sudden instability* occurred.

4.2 Stability aspects of the VBEM formulation

The VBEM stability performance is shown here to be smoother and more predictable. Most important, there exists a lower limit for β_2 beyond which the time-stepping algorithm is found to be stable. Specifically, for Poisson's ratio range $0 \leq v \leq 0.5$, the stability region was identified as $0.7 \leq \beta_2 \leq 0.9$. These types of simulations are shown in Fig. 6. Corresponding values for the β_1 parameter may easily be retrieved, and we note that for large values of v, correspondingly very large values of β_1 result which produce substantial numerical damping. This unwelcome effect, however, may be reduced using with finer boundary element meshes.

In sum, the superior VBEM performance is probably due to the structure of the velocity fundamental solution, and specifically to the presence of the Dirac delta function that has a more 'local' space-time effect. In contrast, the displacement fundamental solution of the BDEM involves use of the Heaviside function.
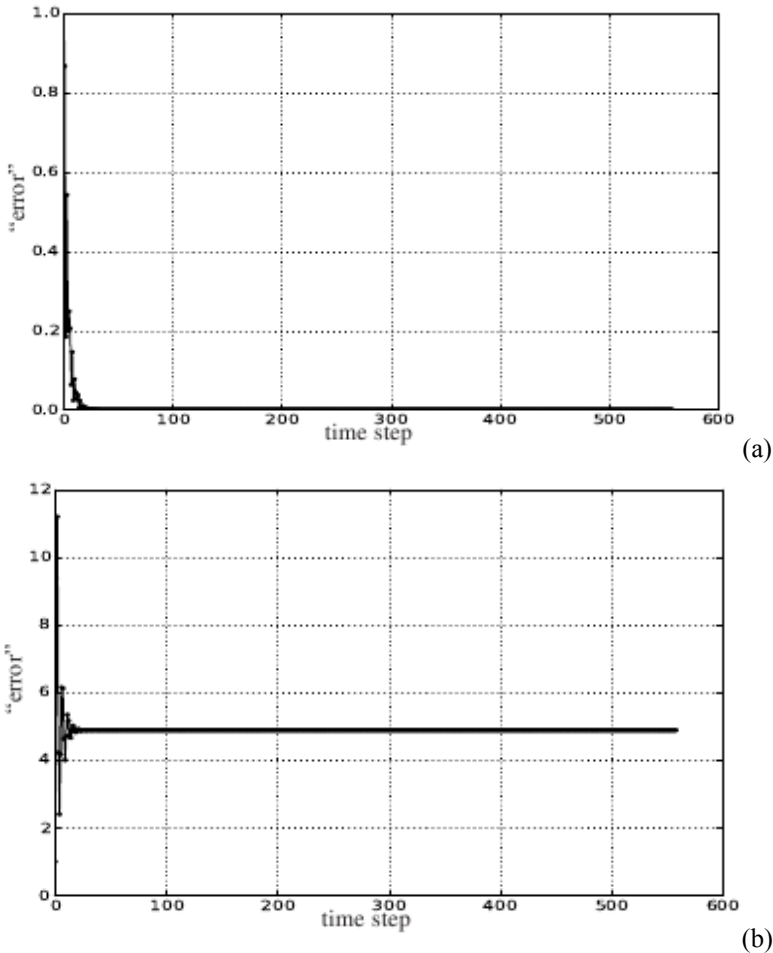
(a)

(b)

Figure 6:   Evolution of VBEM 'error' for $v = 0.0$ and $\beta_2 = 0.8$ (a) velocities (b) tractions.

5   Conclusions

A TD-BEM formulation for 3D elastodynamic problems has been proposed and tested. This new formulation is established in terms of velocities and tractions instead of the conventional one, and is based on a power-type reciprocal theorem. This formulation is found to have a better performance regarding stability and its incorporation into existing BEM codes is simple. It only requires the replacement of the displacement fundamental solution by its respective velocity solution as the kernel function. Furthermore, temporal integration of this new kernel is similar to that of the traction kernel, while spatial integrations are

identical to those of the displacement kernel. This way, we avoid here ''averaging'' methods for decreasing instability problems that have hitherto been used, especially in conjunction with 1D formulations. It is also interesting to note that the 2D elastodynamic case can also be formulated in terms of velocity-based quantities and better stability performance is to be expected in this case.

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