Probabilistic fatigue crack growth using BEM and reliability algorithms

E. D. Leonel & W. S. Venturini
University of São Paulo. School of Engineering of São Carlos, Department of Structural Engineering, Brazil

Abstract

Fatigue and crack propagation are phenomena affected by high uncertainties, where deterministic methods fail to predict accurately the structural life. This paper aims at coupling reliability analysis with boundary element method (BEM) in modeling probabilistic fatigue crack growth. BEM has been recognized as an accurate and efficient numerical technique in modeling crack growth problems. The dual BEM approach was adopted to model crack growth. The couple BEM and reliability algorithms allows us to consider uncertainties during the crack propagation process. In addition, it calculates the probability of fatigue failure for complex structural geometry and loading. Two coupling procedures are considered: direct coupling of reliability and mechanical solver and indirect coupling by the response surface method. Numerical applications show the performance of the proposed models in lifetime assessment under uncertainties, where the direct method has shown faster convergence than response surface method.
Keywords: fatigue crack growth, BEM, structural reliability.

1 Introduction

Fatigue and crack propagation problems have been widely studied by the scientific community in recent years, because crack growth can explain the failure of structures. The accurate modelling of complex engineering structures, including complex geometries and boundary conditions, requires numerical techniques. Therefore, to model crack propagation problems, numerical models are required because the structural geometry and, consequently, the boundary conditions in these problems change at each crack length increment.

WIT Transactions on Modelling and Simulation, Vol 52,
www.witpress.com, ISSN 1743-355X (on-line)
doi:10.2495/BE110011

Particularly, the boundary element method (BEM) has already been recognized as an accurate and efficient numerical technique to deal properly with many problems in engineering, especially crack growth problems. Compared to finite element and other domain mesh methods, BEM is recommended to analyze crack propagation problems due to its efficiency of dimensionality reduction, as only the boundary is discretized. The BEM is even more efficient for mixed mode propagation, as the remeshing problems are avoided, whatever the structural complexity. For these reasons, BEM has been widely applied to deal with fatigue and crack propagation problems [1–4].

Fatigue crack growth is a slow process that includes large uncertainties and requires appropriate inspection plan in order to prevent the risk of failure. The fatigue crack growth is governed by several parameters as structural geometry, initial crack size and configuration, material properties and loading history. All these conditions are random and appropriate analysis method is required on the basis of probabilistic models. An appropriate analysis of fatigue phenomena is performed by considering the problem in a probabilistic manner. Numerous studies have been carried out on probability-based fatigue assessment as [5–8]. Although these works, and many others [9–11], have developed probability-based fatigue assessment approaches and applications, the model assumptions are often very restrictive, leading to inappropriate application to practical engineering structures.

This paper aims at developing a general procedure allowing to deal efficiently random fatigue crack growth for complex structures. This goal is achieved by coupling a reliability model with the mechanical model based on BEM, which one simulates accurately fatigue and arbitrary crack propagation. The BEM model is based on the dual BEM formulation in which singular (displacement integral equation) and hyper-singular (traction integral equation) integral equations are adopted. Displacement integral equations are used for collocation points along the crack surface, whereas traction integral equations are used for collocation points along the opposing crack surface. This technique avoids singularities in the resulting algebraic system of equations, despite the fact that two of the collocation points defined on the opposite crack surfaces have the same coordinates.

Two reliability procedures are considered: the direct method (DM) based on implicit limit state function and the response surface method (RSM) based on polynomial approximation of the mechanical behavior. The DM is based on the direct application of first order reliability method (FORM), where the derivatives of the mechanical response are calculated by finite difference technique applied to BEM model. In the RSM, the implicit mechanical response, given by BEM, is approximated by a polynomial function, known as response surface. In this approach, the implicit mechanical response is converted into an explicit one. The reliability procedure is then applied to this response surface instead of the BEM itself. Finally, the failure probability regarding fatigue can be defined and the most probable crack path determined.

The developed procedures are applied to random fatigue problems, where the DM has shown to give faster convergence, with respect to RSM.

2 BEM formulation

The BEM has shown to be an attractive approach in various engineering fields, such as contact problems, fatigue and fracture mechanics, due to its precision and robustness in modelling structures with high stress concentration. In two-dimensional elasticity, the boundary element formulation is obtained considering a homogeneous domain Ω, with a boundary Γ. The equilibrium equation can be written in terms of displacements as:

$$u_{i,jj} + \frac{1}{1-2\upsilon} u_{j,ji} + \frac{b_i}{\mu} = 0 \tag{1}$$

where μ is the shear modulus, υ is Poisson's ratio, u_i are the displacement components and b_i are the body forces. In this equation i=1,2 because it is 2D case. Using Betti's theorem, the singular integral for displacements is given, without body forces, by:

$$c_{il}(f,c)u_l(f) + \fint_\Gamma P_{il}^*(f,c)u_l(c)\,d\Gamma = \int_\Gamma P_l(c)u_{il}^*(f,c)\,d\Gamma \tag{2}$$

where P_l and u_l are respectively the tractions and displacements at the boundary, $\fint$ is the integral of Cauchy principal value and the term c_{il} is equal to $\delta_{il}/2$ for smooth contours. P_{il}^* *and* u_{il}^* are respectively the fundamental solutions, which ones are shown in [12].

Linear elastic two dimensional domains can be analyzed by evaluating Eqn. (2) on the elements located at the body's boundary. However, for solids with cracks, using this equation for the discretization of all boundaries leads to singularities, because both crack surfaces are located on the same geometrical position. To deal with crack problems using BEM, many formulations have been proposed. The dual boundary element formulation stands out, as it applies to analysis of arbitrary crack growth. In this formulation, four algebraic relationships are required at each node along the crack path. To avoid redundant relationships, these four relationships are obtained from two different integral equations, which are written in terms of displacements, Eqn. (2), and tractions.

The hyper-singular integral representation at the boundary, in terms of tractions, can be obtained from Eqn. (2), which must be differentiated to obtain the integral representation in terms of strains. Then, Hooke's law is applied to obtain the integral representation in terms of stress. Finally, multiplication by the director cosines of the normal to crack surfaces at the collocation point leads to the traction representation, as follow:

$$\frac{1}{2}P_j(f) + \eta_k \,⨎_\Gamma S_{kj}(f,c)u_k(c)\,d\Gamma = \eta_k \fint_\Gamma D_{kj}(f,c)P_k(c)\,d\Gamma \tag{3}$$

where ⨎ is the integral of Hadamard finite part, the terms S_{kj} and D_{kj} contain the derivatives of P_{ij}^* and u_{ij}^* respectively, as indicated in [12]. In this paper, only linear boundary elements were used. High order elements could be used. However, this simple boundary element allow us calculate accurately the boundary values with low computational memory requirements. The singular integrals are evaluated in numerical forms, using sub-element procedure, whereas the hyper-singular integrals are calculated by analytical expressions. This procedure has shown to be accurate and efficient enough in arbitrary crack growth analyses.

3 Fracture mechanics model

The fatigue life prediction is a challenging problem in engineering design, inspection and maintenance. It is highly important to give accurate estimation of the life distribution of mechanical and structural components, in terms of the number of load cycles. For various kinds of materials, the crack growth rate can be modeled using the Paris' law [13]:

$$\frac{da}{dN} = C\Delta K^n \tag{4}$$

where a is the crack length, N is the number of loading cycles, C and n are material constants, and ΔK is the stress intensity factor range.

Stress intensity factors depend on the crack and structural geometry as well as on the stress field at the crack tip. For complex structures, these parameters can only be properly calculated by numerical methods. In this paper, stress intensity factors are evaluated through the displacement correlation technique using BEM model. For plane structures, stress intensity factors for modes I (opening) and II (sliding) are given throughout the following expressions:

$$K_I = \sqrt{\frac{2\cdot\pi}{r}}\frac{\xi}{(\kappa+1)}u_o \tag{5}$$

$$K_{II} = \sqrt{\frac{2\cdot\pi}{r}}\frac{\xi}{(\kappa+1)}u_s \tag{6}$$

where K_I and K_{II} are respectively stress intensity factors for modes I and II, u_o is the crack opening displacement, u_s is the crack sliding displacement, r is the distance between the crack tip and the computation point (i.e. mesh node) and ξ and κ are material properties. These variables are evaluated for six couples of mesh nodes near to the crack tip. Then, stress intensity factors at the crack tip are obtained by a local extrapolation process. This process has shown accurate according our results.

In mixed mode propagation, it is necessary to calculate the equivalent stress intensity factor K, and the crack propagation (or bifurcation) angle θ_p. For this purpose, the maximum circumferential stress criterion is adopted, which yields:

$$K = K_I \cos^3\left(\frac{\theta_p}{2}\right) - 3K_{II}\cos^2\left(\frac{\theta_p}{2}\right)\sin\left(\frac{\theta_p}{2}\right) \tag{7}$$

$$\tan\left(\frac{\theta_p}{2}\right) = \frac{1}{4}\left[\frac{K_I}{K_{II}} \pm \sqrt{\left(\frac{K_I}{K_{II}}\right)^2 + 8}\right] \tag{8}$$

For each load cycle, the equivalent stress intensity factor is evaluated at the minimum and maximum load levels, namely $K_{\min}$ and $K_{\max}$ respectively. If the equivalent stress intensity factor value is bigger than the material toughness, brittle failure is considered and the analysis is stopped; in this case, we talk about failure. When $K_{\max} < K_{Ic}$, the stress intensity factor range ΔK is computed and compared to the threshold ΔK_{th}. If $\Delta K \leq \Delta K_{th}$, no crack propagation is considered. Otherwise, the crack growth rate is computed using the Paris' law. The fatigue failure is defined by large crack propagation. As recommended by [14], when the crack growth rate da/dN is bigger than 0.1 mm/cycle, the analysis is stopped and structural failure takes place (the remaining life is negligible).

4 Reliability analysis

The reliability analysis aims at computing the failure probability Pf regarding a specific failure scenario, known as the limit state (note that reliability and failure probability are complementary).

The leading step in the reliability assessment is to identify the basic set of random variables $X = [x_1, x_2, ..., x_n]^T$ for which uncertainties have to be considered. For all these variables, probability distributions are attributed to model randomness. These probability distributions can be defined by physical observations, statistical studies, laboratory analysis and expert opinion. The number of random variables is an important parameter to determine the computing time consumed during the reliability analysis. In order to reduce the size of the random variable space, it is strongly recommended to consider as deterministic all variables whose uncertainties lead to minor effects on the value of the failure probability.

The second step consists in defining a number of potentially critical failure modes. For each of them, a limit state function G(X) separates the space into two regions: the safe domain, where G(X)>0, and the failure domain where G(X)<0. The boundary between these two domains is defined by G(X)=0, known as the limit state itself. It is worth to mention that an explicit expression of the limit state function is not possible, and only desired points can be computed by running the BEM analysis.

The failure probability is evaluated by the integral, [15]:

$$P_f = \int_{G\leq 0} f_X(x_1, x_2, \ldots, x_n)\ dx_1, dx_2, \cdots dx_n \tag{9}$$

where $f_X(x_1, x_2, \ldots, x_n)$ is the joint density function of the variables X. As the evaluation of the above integration is impossible in practice, alternative procedures have been developed on the basis of the concept of reliability index β, [16]. This variable is defined by the distance between the median point and the failure domain in the normalized space of random variables. The reliability index allows us to compute the failure probability, using the first order reliability method (FORM), as: $P_f = \Phi(-\beta)$, where $\Phi(\cdot)$ is the standard Gaussian cumulated distribution function.

When numerical mechanical methods are involved, the structural reliability analysis can be performed by one of two approaches: direct application of the reliability procedure using the mechanical analysis tool, or the use of the response surface method as an explicit representation of the structural behavior, in order to perform the reliability analysis. These two approaches will be discussed below.

5 Coupled BEM-reliability procedures

In order to consider random fatigue crack propagation, it is required to couple the reliability procedures with the BEM model. As mentioned above, this coupling can be performed by either the direct method or the response surface method.

5.1 Direct method (DM)

The basic procedure consists in directly coupling the reliability model with the mechanical model. As described in the previous section, the limit state function defines the safety and failure domains. For fatigue crack growth, this limit state function can be written in terms of number of load cycles:

$$G(X) = N_{\text{Resistant}}(X) - N_{\text{Applied}}(X) \tag{10}$$

where N*Resistant*(*X*) is the number of cycles corresponding to structural failure and N*Applied*(*X*) is the applied number of cycles during the service life. In order to give invariance measure of safety, the random variables, defined in the physical space, are transformed into independent standard Gaussian variables [16], by using appropriate probabilistic transformation. Figure 1 illustrates this transformation showing that the performance function G(*X*) in the physical space is transformed to H(*U*) in the standard normalized space, where $U = [u_1, u_2, \ldots, u_n]^T$ denote the standard Gaussian variables.

In this standard space, the reliability index β is given by the minimum distance between the failure domain and the origin of the standard space, therefore evaluated by solving the constrained optimization problem:

$$\text{find: } U^*$$
$$\text{which minimizes: } \beta = \sqrt{U^T \cdot U} \tag{11}$$
$$\text{subject to: } \mathrm{H}(U) = 0$$

The solution of this problem converges to the failure point nearest to the space origin, known as the design point or the most probable failure point X*. In the standard space, the distance between this point and the origin is the reliability index. The reliability index β can be found by applying the Rackwitz and Fiessler algorithm [17], directly to the mechanical model. As the number of cycles to failure is known point-by-point, the resistance N*Resistant*(*X*) is implicit, and therefore the derivatives of the limit state function can only be computed by the finite difference technique. In our case, the forward finite difference scheme was chosen because of its low computation cost. The numerical error due to finite difference may affect the convergence of the coupled procedure, as well as the precision of the solution, especially for nonlinear phenomena. However, for the problems studied in this paper, it was verified that this coupling procedure gives accurate results and stable convergence rate, with a reasonable number of mechanical analyses.

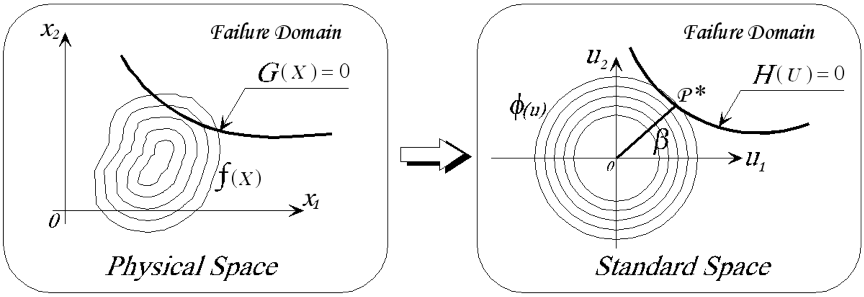


Figure 1: Probabilistic transformation from physical to standard space.

5.2 Response surface method (RSM)

The response surface method (RSM) is an efficient method for solving optimization problems, such one presented in Eqn. (11). The RSM allows us to replace complex models by approximate analytical functions based on the response values at various points in the design space. For reliability applications, the RSM is used to approximate the structural response at the vicinity of the most probable failure point, in terms of input variables related to geometrical data, material properties and boundary conditions. Naturally, the variables to be considered are those undergoing randomness within the reliability analysis. The set of realizations of these variables is named as experiment design (ED), which defines a set of structural responses from which a surface may be fitted using least square regression. Any shape surface, named response surface, can be adopted to represent the structural response. In this paper, the complete quadratic polynomial, defined in n-dimensional space, has been chosen to approximate the structural response.

The general procedure for evaluating the failure probability using response surface methods is divided into three steps:

1. In the first step, different sets of points are chosen according to the experiment design procedure. Each set constitutes the input values for which the mechanical response has to be computed. For the problem considered in this paper, the BEM model is performed to compute the mixed mode crack growth and the resisting number of load cycles corresponding to structural failure, for each set. During the iterative procedure, the points to be used in the ED are given in a hyper-cube centered at the current search point. The hyper-cube dimensions are given as a multiple of the variables standard-deviations. The mean values of the random variables are usually assumed as the first trial of the search point. After computing the mechanical responses for the selected points, the response surface can be approximated by polynomials identified by regression techniques.

2. The second step is defined by rewriting the limit state function in the standard normalized space using probabilistic transformations. Then, the minimum distance between the limit state function and the coordinate origin is calculated using an appropriate optimization procedure. This distance is the reliability index, β, as defined by Hasofer and Lind [16], and the design point as well as the direction cosines can be defined.

3. The third step is the estimation of the failure probability, which can be computed according to FORM approximation.

The procedure is iterative and it is continued by re-defining the ED at each new design point calculated. The center of the ED in the iteration *k* is the design point calculated in the iteration *k-1*. The convergence is given by the error measured between the reliability indexes of two successive iterations, in addition to the convergence of the design point coordinates.

In this procedure, the polynomial coefficients, a, are calculated by minimizing the quadratic error, ζ, between the exact responses, given by the BEM number of load cycles at failure $N_{\text{Resistant}}(w_k)$ at each ED point w_k, and the approximate response surface $RS(w_k)$ evaluated at the same points. The polynomial coefficients are obtained by minimizing:

$$\text{find: } a_c \text{ and } a_{ij}$$

$$\text{which minimizes: } \zeta = \sum_k \left[RS(w_k) - N_{\text{Resistant}}(w_k) \right]^2 \tag{12}$$

$$\text{where } RS(w_k) = a_c + \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_i x_j$$

It is to be noted that each numerical experiment to be used in the ED is obtained by complete fatigue crack growth analysis, in order to define the corresponding number of cycles. Therefore, each mechanical call represents a significant computing cost and experiments should be optimally designed, in order to reduce the numerical effort. In our case, the response surface represents a local approximation of the structural lifetime and is computed by:

$$G(X) = RS(X) - N_{\text{Applied}}(X) \tag{13}$$

Using the response surface, the design point search can be carried out by Rackwitz and Fiessler algorithm [17] which gives good results in this situation.

Beside its accuracy, the RSM is known to be robust for reliability analysis of complex structural systems, especially when high nonlinearities and bifurcations are involved. However, from the numerical point of view, this method becomes expensive when the number of random variables increases. For this reason, various experiment design schemes have been developed to reduce the number of the required points.

6 Application

A perforated panel is fixed at the bottom and subjected to uniform and cyclic tensile load at the top edge. An initial crack is located as shown in Fig. 2. The random variables considered in this analysis are the tensile load *P*, the hole diameter *D*, the location of the hole center with respect to the panel bottom, D_f, and the applied number of cycles N_{applied} (see Table 1).

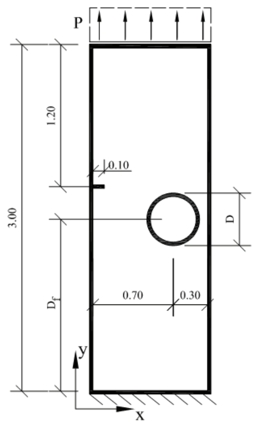
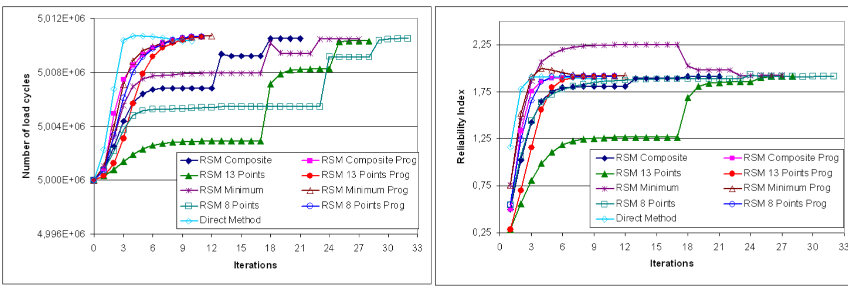


Figure 2: Perforated plate with initial crack. (dimensions in meter).

Table 1: Deterministic and random variables for this example.

Variable	Distribution	Mean	Standard Deviation
Young's modulus *E (MPa)*	deterministic	30 000	
Poisson's ratio *ν*	deterministic	0.20	
Paris'coefficient C $(m^{1\text{-}1.5n}/cycle/kN^n)$	deterministic	2.0×10^{-10}	
Paris' Law parameter *n*	deterministic	2.7	
Toughness K_c $(MN/m^{1.5})$	deterministic	104	
Hole diameter *D (m)*	normal	0.4	0.025
Hole location $D_f(m)$	normal	1.5	0.15
Applied load *P(kN/m)*	normal	5.00	0.80
Applied cycles N_{Cycles}^{Load} (cycles)	normal	5×10^6	10^5

The RSM, with various experiments design, and DM were applied to analyze this structure considering the four random variables described in Table 1. The convergence curves for two representative random variables, number of load cycles and reliability index, are shown in Fig. 3. Regarding these results, we observe that the convergence is achieved, for DM and RSM progressive resizing, with maximum 12 iterations, while the RSM stepped resizing needs not less than 21 iterations to the convergence. In this example, the DM has shown faster convergence than RSM approaches. The reliability index obtained is 1.912 that corresponds to a failure probability of 0.0279.



a) Convergence histories in Number of Load Cycles b) Convergence histories in Reliability Index

Figure 3: Convergence history for load cycles and reliability index.

In spite of DM and RSM with progressive resizing needed maximum 12 iterations to the convergence, the computing time associated is considerable different. Figure 4 shows the number of BEM runs for different reliability methods used. Concerning RSM analysis, the maximum and minimum BEM runs were observed when 13 Points and Minimum ED were adopted, respectively. Using the first one, it required 728 BEM runs, while with Minimum ED only 120 mechanical runs were demanded. Considering DM only 40 mechanical calls were required to achieve the convergence. Thus, the good performance of DM is confirmed, as it requires a low number of BEM runs when compared with RSM procedures.

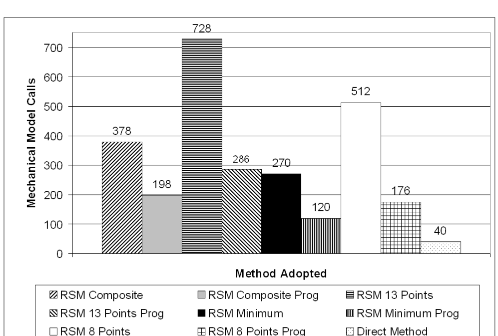


Figure 4: Number of BEM calls for the reliability analysis.

www.witpress.com, ISSN 1743-355X (on-line)

The crack growth path changes according the vertical position of hole, D_f . If the center hole position is aligned, or almost aligned, to the initial crack position, the crack grows up to the hole. However, below a certain position, the hole changes the stress field distribution in the structures as well as the crack growth path.

To better understand the role of the hole position, a parametric analysis has been performed by varying the mean hole position D_f from 1.00 m to 2.50 m (the other random and deterministic variables remains the same as described in Table 1. The reliability index and the most probable crack paths are shown in Fig. 5. We observe high sensitivity and dependence between reliability index and hole position. It can be seen that the minimum reliability index is achieved when the mean location is equal to 1.75 m. In this position, the reliability index calculated is 1.52.

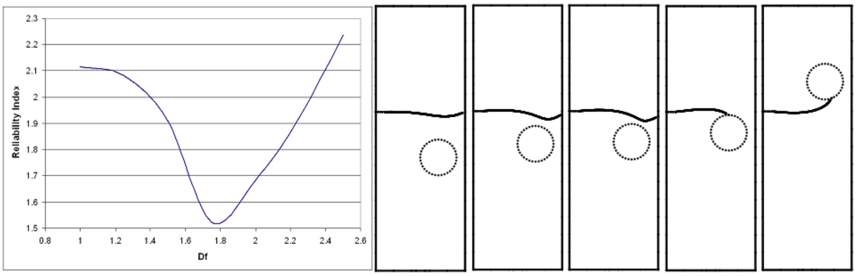


Figure 5: Influence of the hole position on the reliability index.

7 Conclusions

In this paper, a couple reliability and BEM model has been proposed for analysis of mixed mode crack propagation in structures subjected to fatigue. The BEM is an accurate approach to model random crack growth, especially in the framework of reliability analysis where many mechanical model runs are required. The DM and RSM have been applied to solve the reliability problem. The numerical application has shown good agreement between these two approaches. However, DM has demonstrated to be stable and more efficient than RSM.

The coupled model proposed is an interesting tool for probabilistic fatigue life assessment. Based on the reliability index results, inspections and maintenance plans can be developed and its costs, as well as the failure cost, can be considerably reduced.

Extension of the BEM code to consider multi materials and heterogeneities can be done in the future. In this case, structural systems can be simulated and a structural system reliability index can be achieved.

Acknowledgement

Sponsorship of this research project by the São Paulo State Foundation for Research – FAPESP – is greatly appreciated.

References

[1] Cisilino, A.P., Aliabadi, M.H. Three-dimensional boundary element analysis of fatigue crack growth in linear and non-linear fracture problems, Engineering Fracture Mechanics, 63, (1999), 713-733.
[2] Dell'erba, D.N., Aliabadi, M.H. Three-dimensional thermo-mechanical fatigue crack growth using BEM, International Journal of Fatigue, 22, (2000), 261-273.
[3] Leonel, E.D., Venturini, W.S. Dual boundary element formulation applied to analysis of multi-fractured domains, Engineering Analysis with Boundary Elements, 34, (2010), 1092-1099.
[4] Leonel, E.D., Venturini, W.S. Non-linear boundary element formulation with tangent operator to analyse crack propagation in quasi-brittle materials, Engineering Analysis with Boundary Elements, 34, (2010), 122-129.
[5] Bogdanoff, J.L., Kozin, F. Probabilistic models of cumulative damage. New York: Wiley. (1985).
[6] Wirsching, P.H., Torng, T.Y., Martin, W.S. Advanced fatigue reliability analysis. International Journal of Fatigue. 13, (1991), 389-483.
[7] Zhao, Y.X. A methodology for strain-based fatigue reliability analysis. Reliability Engineering and System Safety, 70, (2000), 205-213.
[8] Sobczyk, K., Spencer, B.F. Random Fatigue: From Data to Theory. Academic Press. (1992).
[9] Yu, L., Purnendu, K.D. Zheng, Y. A response surface approach to fatigue reliability of ship structures, Ships and Offshore Structures, 4, (2009), 253-259.
[10] Liu, W.K., Chen, Y., Belytschko, T., Lua, Y.J. Three Reliability methods for fatigue crack growth, Engineering Fracture Mechanics, 53, (1996), 733-752.
[11] Lua, Y.J., Liu, W.K., Belytschko, T. A stochastic damage model for the rupture prediction of a multi-phase solid. Part I1: statistical approach, Int. J. Fracture Mech. 55, (1992), 341-361.
[12] Hong, H.K., Chen, J.T. Derivations of Integral Equations of Elasticity, Journal of Engineering Mechanics, V.114, 6:1028-1044, 1988.
[13] Paris P.C., Erdogan F. A critical analysis of crack propagation laws. Journal of Basic Engineering, 85, (1963),528-534,.
[14] Pimenta, P.M., Ricardo, L.C.H. Camarao, A.F. Modern trends on fatigue, SAE Brasil International Conference on Fatigue, São Paulo, Brazil, (2001).
[15] Ditlevsen O., Madsen H.O. Structural reliability method. New York: John Wiley and Sons, (1996).
[16] Hasofer, A.M., Lind, N.C. Exact and invariant second moment code format, Journal of the Engineering Mechanics Division, ASCE. 100 (EM1), (1974), 111-121.
[17] Rackwitz, R., Fiessler, B. Structural reliability under combined random load sequences, Computers & Structures, 9, (1978), 489-494.