# A stochastic BEM formulation for vibro-acoustic analysis of structures in the mid-to-high frequency range

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## Abstract

Predicting the acoustic behaviour of a complex structure in the mid-frequency range is a challenging task due to the complexity of the structure and the involved physical phenomena. Since, in the high- and even mid-frequency range, the variability of the product may have a significant effect on its NVH behaviour, it is important to include this sensitiveness in numerical models. In other respects a detailed description of the structures is necessary for structure borne contributions.

The approach proposed in this paper provides a robust method for the vibro-acoustic analysis of 3D acoustic domains coupled with structural components in the mid-frequency range.

The method is based on a probabilistic approach which introduces uncertainties in the Boundary Element formulation: random geometrical parameters are introduced into the integral representations in order to model the increasing sensitivity of the harmonic vibrational responses to any parameter perturbation when the frequency increases.

The stochastic formulation is solved in terms of the expectations of the square boundary kinematic unknowns and is valid in the entire frequency domain.

The effectiveness of the formulation is demonstrated in this paper with a numerical application to a 3D model of an acoustic cavity coupled with a vibrating structure and with an internal source.
*Keywords: boundary elements, acoustic, mid-frequency, stochastic, parametric modelling.*

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## 1 Introduction

The mid-frequency (MF) domain is a transition region; the behaviour of the structure cannot be regarded as completely deterministic nor statistical. This frequency field is usually defined as the domain for which a complex vibrating structure may be subdivided into two different parts: one is stiff and exhibits a deterministic low-frequency (LF) behaviour, the other one of them is flexible and presents a high frequency (HF) behaviour. This combination of high and low frequency behaviours implies that the modal contribution may not be neglected, as in a HF model [1], but on the other hand, the classical LF predictive tools [2–4] are ineffective for modelling the HF contribution. For many complex engineering structures, such as vehicles, trains, truck cabins, etc., it is relevant to develop design tools addressing specifically the MF range. For treating this specific problematic, different approaches have been investigated. These can be organized into three categories:

- Effective alternative deterministic methods which aim at extending conventional low-frequency element-based methods to higher frequencies. Current research on enhancing the BEM focuses on Fast Multi-pole methods [5, 6]. Alternative wave based (WB) schemes have been developed which, thanks to their implementation into the so-called hybrid FE-WB approaches, increase the efficiency of the conventional FEM [7].
- Novel energy methods which aim at improving and extending the high-frequency methods towards lower frequencies. One example is the Virtual SEA [8]. The method aims at treating the structure-borne contribution from a few hundred Hertz upwards, by means of the SEA. The novelty of the method consists in the development of a frequency dependent procedure for defining the SEA subsystems. A promising approach is the wave and finite element (WFE) method to modelling the dynamics of structures which are piecewise homogeneous or periodic in one or two dimensions or which are axisymmetric [9].
- Hybrid approaches that combine deterministic and energy descriptions. Langley and Shorter developed a hybrid method which couples FE and SEA formulations [10]. The stiff components (low modal density) are modelled with FE and the flexible components (high modal density) are analysed with the SEA. We should also mention the enhanced deterministic prediction methods applied to create more accurate Statistical Energy Analysis models or hybrid FE-SEA models [11, 12].

This paper aims at presenting the development and the application of a new formulation for the vibro-acoustic analysis of acoustic domain and structures in the MF field. The formulation extends the applicability of classical Boundary Element Method introducing geometric uncertainties in the boundary description. In the section two of this paper the basis of the SBEM is reviewed. In section 3, two numerical applications dealing with acoustical 3D applications are proposed to illustrate the effectiveness of the formulation.

## 2 SBEM formulation

The Stochastic Boundary Element Formulation (SBEM) is able to treat the vibrational behaviour of a structure in the complete frequency range. This feature is due to the fact that the underlying assumptions of this formulation are very general, and are linked to the material description of the structure rather than to the frequency field of investigation. Nevertheless, this formulation shall not be utilized in the low frequency domain because the assumptions of the method have no influence in this range and moreover the number of unknowns is greater than the one for the classical Boundary Element Method. In the high frequency field (the domain where all the subsystems have a high frequency behaviour), the SBEM is suitable and can be used to describe the behaviour of the complete structure in a much more convenient way than it could be carried out with a classical FEM or BEM. In the mid-frequency field (the domain where some of the subsystems have a high frequency behaviour and the others show a low frequency behaviour), the SBEM can be utilized to describe the whole structure, or can be coupled with classical deterministic methods to obtain an hybrid formulation.

The SBEM is based on a boundary integral formulation and a statistical approach to account for uncertainties in the structural parameters. The underlying idea is that a structure always encounters physical uncertainties which play an increasing role when the frequency increases. According to Fahy and Mohammed [13], the differences among systems which share the same design characteristics, and the effects of these differences on the vibrational behaviour are individually unpredictable in the HF (high frequency) range, therefore a probabilistic model is appropriate. Thus, introducing randomness to the geometrical or/and material properties of the structure leads to a precise description of the deterministic LF (low frequency) response and a smooth response in the high-frequency field corresponding to the average of the strongly oscillating vibratory response. In between, a transition zone is observed in which the response gradually shifts from the deterministic to the average response.

The starting point of Stochastic Boundary Element Method (SBEM) is a formulation developed in the late Nineties [14, 15]. Since 2004 A. Pratellesi et al. have been working in order to test and eventually extend the capabilities and applicability of the stochastic formulation to deal with vibro-acoustic problems [16, 17]. The SBEM formulation, initially developed for academic structures, has been extended and completed to deal with hybrid problems, to be coupled with FEM. The previously referenced papers deal with 2D applications; in this paper the authors present the first results of the SBEM application to 3D acoustic and vibroacoustic problems.

### 2.1 High frequency modeling by the SBEM

In the high-frequency field, the vibrational response of a structure is extremely sensitive to small perturbations of its geometrical and material properties. This phenomenon has been illustrated by Manohar and Keane [18], who calculated

 
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the successive probability density functions of the eigenfrequencies of a beam, for which a random parameter is introduced in the definition of its mass density. Thus, solving the usual constitutive equations describing the vibrational behaviour of the structure, by means of a usual numerical solver, is generally meaningless. To overcome this problem, randomness is introduced to the description of the geometry of the structure and a formulation showing explicitly the expectations of the usual kinematic unknowns, with respect to the randomness, is derived. This randomness should not affect the response in the low-frequency field, on the other hand, the aim is to obtain a smooth response in the high-frequency field highlighting the overall trend of the fast varying deterministic behaviour. In other respects, writing a first order moment (FOM, itis the expectation of the boundary variables) formulation is useless since these variables diminish to zero when the frequency rises. Therefore, the formulation must be written on the second order unknowns (SOM, the expectation of the square modulus or cross product of the boundary variables).

## 2.2 The random formulation

The initial stage for deriving the SIF equations is a direct boundary integral formulation. The random parameters are introduced on the geometrical description of the structures, the boundary parameters are supposed to be randomly known and are written: $\widetilde{x}_i = x_i + \epsilon_i$ where $x_i$ is the deterministic value of the parameter, while $\epsilon_i$ is the zero mean random variable.To solve the formulation a statistic probability distribution should be introduced, section 2.3. A randomness is then applied to the locations of the loadings. The collocational method is employed and allows to solve the integral problem using a discrete set of equations evaluated at nodal points. $N_u$ and $N_T$ are respectively the number of boundary elements defined for $\partial\tilde{\Omega}_u$ and $\partial\tilde{\Omega}_T$. As an illustration, the equation evaluated at point $\tilde{\boldsymbol{\xi}} \in \partial\tilde{\Omega}_u$ is reported, eq. (1): $u_i$ is the kinematic unknown (e.g. pressure, displacement) for element $i$, $T_i$ is the first order derivative of the boundary unknown, $G$ denotes the Green kernel, $dG$ is the first order derivative of the Green kernel, $\hat{u}$ and $\hat{T}$ are the imposed boundary conditions; $\tilde{u}_j$ is the boundary random unknown at element $j$.

$$\frac{1}{2}u_\xi = \int_{\tilde{\Omega}_f} f(\mathbf{y}) \cdot G(\mathbf{y}, \tilde{\boldsymbol{\xi}})\mathrm{d}\Omega + \sum_{j=1}^{N_T} \int_{\partial\tilde{\Omega}_j} [u_j \cdot dG(\mathbf{x}, \tilde{\boldsymbol{\xi}}) - \hat{T}_j \cdot G(\mathbf{x}, \tilde{\boldsymbol{\xi}})]\mathrm{d}\partial\Omega$$
$$+ \sum_{k=1}^{N_u} \int_{\partial\tilde{\Omega}_k} [\hat{u}_k \cdot dG(\mathbf{x}, \tilde{\boldsymbol{\xi}}) - T_k \cdot G(\mathbf{x}, \tilde{\boldsymbol{\xi}})]\mathrm{d}\partial\Omega. \qquad (1)$$

The aim is to derive an integral representation whose unknowns are the expectations of the cross-products of the force and displacement unknowns. Therefore, for any boundary location $\tilde{\boldsymbol{\xi}} \in \partial\tilde{\Omega}$, the right- and left-hand sides of eq. (1) are multiplied by the conjugate of the random boundary unknown at the same spatial position. The expectations of the equations are finally considered. They are represented by $\langle - \rangle$.

To solve the $N_u + N_T$ equations some statistical assumptions for limiting the number of unknowns were defined. These assumptions define the correlation of the different variables appearing in the equations above. They are based on a physical interpretation of the integral equations.

### 2.3 Analytical representation of the random boundaries in 2D and 3D models

In order to solve the SBEM set of equations, the numerical evaluation of the expectations of boundary and domain integrals must be carried out. The explicit expression for 3D applications of the stochastic Green function, for $i \neq j$, is:

$$\left\langle \int_{\partial\tilde{\Omega}} G(\widetilde{x}_i, \widetilde{x}_j) d\partial\Omega \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\partial\tilde{\Omega}} G(\widetilde{x}_i, \widetilde{x}_j) p(\epsilon_i) p(\epsilon_j) d\epsilon_i d\epsilon_j d\partial\Omega. \quad (2)$$

The integration paths in 2D and 3D applications, are random, therefore it is not possible to commutate the expectation and the integral operators (as it is possible for rods and beams, 1D elements). To simplify this evaluation, it can be shown that the random variable can be judiciously chosen in order to get rid of the randomness in the integration path by means of a change of a variable [16].

Using a parametric representation of the random boundaries as a function of the uncertain parameter $\varepsilon$, the integration path on the right side of eq. (2) does not depend on the random variable any more. Therefore, it is possible to switch the integration and the expectation operators.

Since the strength of the formulation relies on the way uncertainties are modelled, and on the assumption described in section 2.2, the most important modelling choices, requirements and consequences on the results are listed:

- uncertainties are introduced only in the local geometric description of the boundary locations. Each element is randomized and its randomness is independent from the uncertainties introduced on the surrounding elements;
- the triple integral on the right side of eq. (2) is integrated by means of Gauss quadrature rule: we have chosen 4 points for the surface integration, and 3 to 7 points on the uncertain parameter range for the double integration of the probability distribution, depending on the shape of the function and on the required accuracy;
- the Probability Distributions can assume many different shapes (Gaussian, Triangular, Rectangular, Hyperbolic....), provided that it can be expressed as a mathematical function to carry out the expectation integral. For the applications example reported in section 3 we have chosen a triangular distribution, zero centred, crisp value equal $1/a$, lower limit $-a$ and upper limit $a$, unitary area;
- node locations are kept fixed for numerical reasons: this enables the easy coupling with FEM velocity for structure models, and allows the parametric description of the boundary. This means that the model preserves its shape, and only local variability is introduced. The variability in the distance

between the field and source points interacts with the wavelength in the high frequency range and affects the response of the structure;

- the introduction of uncertainties in the material, which influences strongly the response of the structure, cannot be taken into account with the actual formulation and assumptions; this is mainly due to the fact that the material properties affect all the responses at a nodal location and correlation/de-correlation rules cannot be introduced for them as explained in section 2.2.

## 3 Applications

### 3.1 3D acoustic cavity with internal pressure source

The first application is a rectangular acoustic cavity, meshed with quadrilateral linear elements, with rigid wall boundary conditions, $v_i = 0$ for $i = 1 : N$, where $N$ is the number of nodes of the model. The cavity is excited by a unit pressure acoustic source $P_0$ which is located inside the domain, figure 1(a). The value of $P_0$ is deterministic and constant with frequency, its location is uncertain.

The geometrical, and material properties of the acoustic domain are summarised in table 1.

This simple example is intended to model the influence of the uncertainties on the boundary and on the excitation locations. From a physical point of view it can be useful to model the performances of rooms and acoustic cavities when excited by an acoustic source; introducing absorption as boundary impedance conditions it can be also useful to model the efficiency of the absorption treatment in the mid and high-frequency range.

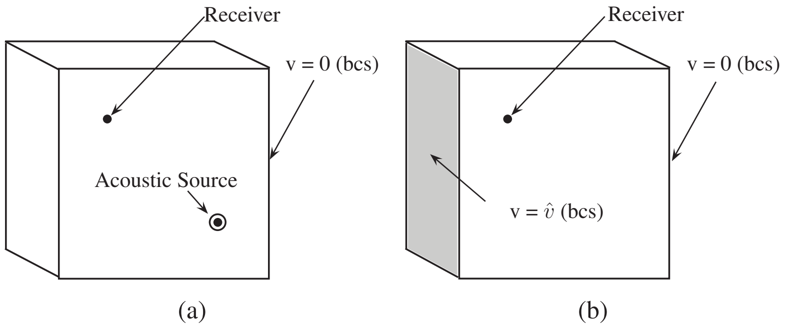


Figure 1: Layout of the numerical applications: (a) Acoustic cavity with Velocity Boundary conditions and Internal Acoustic Source, (b) Acoustic cavity with Velocity Boundary conditions.

In figure 2 the effect of the uncertainties in the evaluation of the G integral as a function of the frequency is reported: the G function is evaluated between


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Table 1: SBEM numerical application 1: material and analysis data.

| | |
|---|---|
| Dimension of the box | 1x1x0.5 m |
| Position of the acoustic Source | (0.2, 0.2, 0.1) m |
| Position of the receiver | (0.7, 0.8, 0.3) m |
| Speed of Sound | 330 m/s |
| Fluid Density | 1.3 kg/m$^3$ |
| Characteristic value of the uncertainties, $a$ | 0.2 |

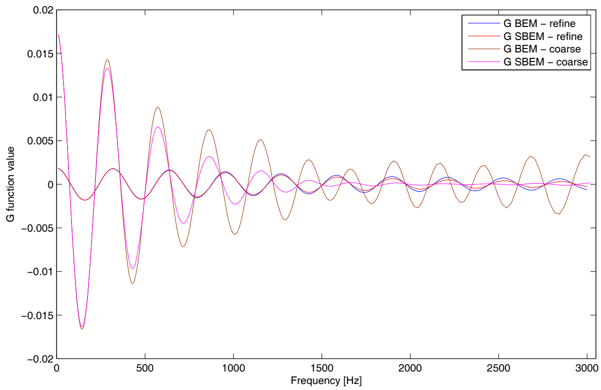


Figure 2: Influence of the uncertainties on the Green functions terms, with respect to two different mesh sizes.

two distinct boundary locations of the model, and as we can clearly see, as the frequency increases, the effect of the uncertainties starts to smooth the response. The comparison is made between two different meshes, one very coarse and one refined.

In figure 3 are reported the response curve at the receiver point obtained respectively with a BEM code, a full SBEM code with both First Order and Second Order Moments included into the formulation. At first, one can globally state that the influence of the randomness increases with frequency. The SBEM curves give a precise representation of the modal behaviour in the low-frequency range. On the other hand, the high-frequency behaviour of the random formulation is smooth and only delivers information for the general trend of the frequency variation of the boundary unknowns.

In figure 4 we can clearly see the effect of the uncertainties on the first order moment response, which is compared with a BEM solution and a SBEM solution with only second order moment terms. In the mid- and high-frequency range the simple expectation of the direct BEM equation it is not representative of the response of a uncertain structure, since the response vanishes to zero compared

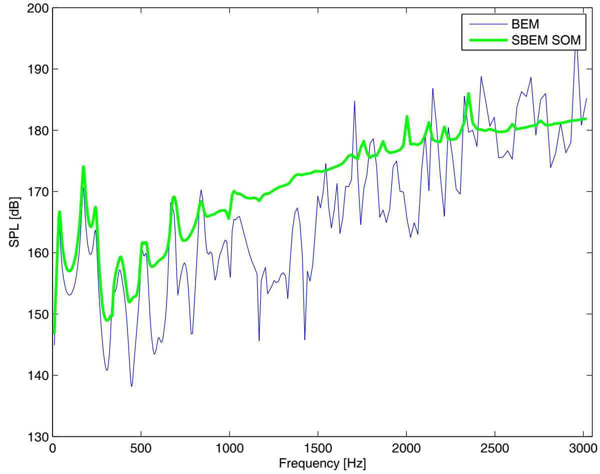


Figure 3: SPL at the receiver point for Acoustic source problem: comparison between BEM and SBEM SOM solutions.

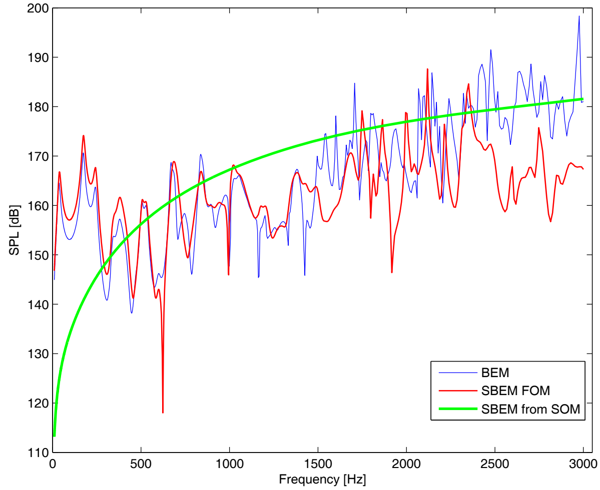


Figure 4: SPL at the receiver point for Acoustic source problem: comparison between BEM, SBEM FOM and SBEM SOM solutions with only SOM.

to the deterministic response. As expected, the SBEM second order response is smooth and effectively gives the correct trend of the response in the high-frequency field. On the other hand, not taking into account the first order moments leads

Table 2: SBEM numerical application 1: comparison of CPU time.

| Code | DOFs of the model | scaled CPU time / frequency step |
|---|---|---|
| DBEM | 184 | 0.0125 |
| DBEM (reference solution) | 1300 | 1 |
| SBEM | 184 | 0.22 |

to removing the peaks which were predicted before by the SBEM in the low-frequency field, and the results become inaccurate in this domain.

From a computational point of view the SBEM formulation is more complex than classical deterministic BEM, but we should also notice that the high frequency trend can be obtained with a coarse mesh, sized to reach the frequency value where the uncertainties start to play their role on the response; the mesh does not need to be scaled according to the higher frequency. Moreover the response is robust against uncertainties and is representative of the ensemble of structures identified by the uncertainties.

A comparison in terms of time for each frequency step is reported in table 2. The confrontation is intended to evaluate the complexity of the SBEM code, developed and implemented in Matlab, with a Direct BEM code which is also implemented by the authors in Matlab, and is the starting point for the development of the Stochastic code. The BEM model, which has been validated with commercial BEM solvers, has been chosen as reference time for the evaluation of SBEM performances. The results show that the SBEM code is penalized in the low-frequency and small models application, but has some advantages over the classical methods when dealing with high frequency applications since it does not need a very refined mesh. Moreover if compared to classical sampling methods for uncertainties modelling, it does not require to calculate and average over a population of structures, since the averaging effect is included in the expected terms. Only one run of the program for each frequency step is necessary to obtain a statistical response.

If compared to SEA, the SBEM suffers from complexity, but has also the advantage to be valid on the whole frequency range; moreover homogeneous quantities at the interface like force and displacement (FEM-SBEM variables) can be coupled easily.

### 3.2 3D acoustic cavity with velocity boundary conditions

The second application is a rectangular acoustic cavity, meshed with quadrilateral linear elements, with rigid wall boundary conditions, except one side which is coupled with a vibrating structure and has $v = \hat{v}$ boundary conditions, figure 1(b). For sake of simplicity the Boundary velocity has been kept constant over the frequency range and equal to 1 m/s: this condition can be easily changed to the real structure coupling by multiplying the input unit velocity vector by the corresponding velocity vector evaluated with FEM. The geometrical, and material properties of the acoustic domain can be found in table 1. The input velocity conditions are kept deterministic, while the boundary locations are randomized.

This application has been carried out to evaluate the accuracy of the response, the sensitiveness to uncertainties in the geometric description and also the potentialities of the formulation, since the SOM formulation is different from the previous application.

In this case the SBEM formulation is more complex than in the first application and requires more equations to be solved: this is due to the assumptions which rule the correlation among the unknowns, and define the vibrating structure as a primary source for the acoustic cavity.

The coupled formulation of SBEM is based on the classical FEM-BEM formulation for fluid structure interaction: in the FOM solution it could be fully coupled, while in the SOM solution it has one way coupling because the structure needs to be solved first.

As expected, in figure 5 we can clearly see the effect of the uncertainties on the first order moment response, which is compared with a BEM solution and a SBEM solution with only second order moment terms. Moreover the SBEM second order response is smooth and effectively gives the correct trend of the response in the high-frequency field. The FOM response becomes not representative of the response as the frequency increases.

The proposed applications can turn out to be simple and academic, but they have been chosen to test the formulation, which is still in its development phase for the

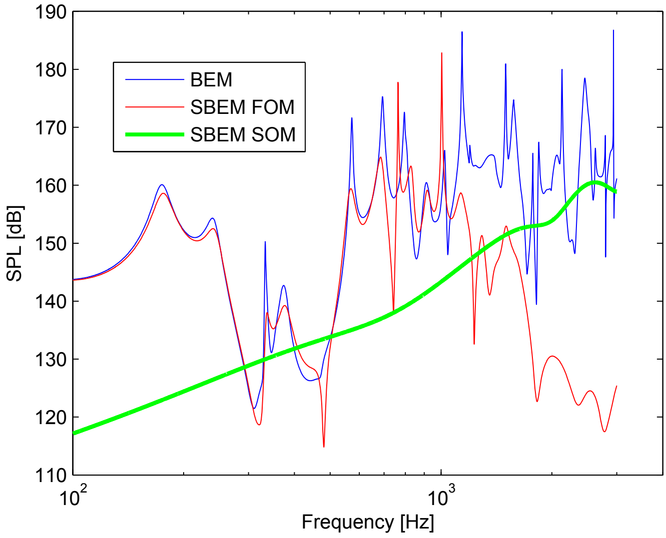


Figure 5: SPL at the receiver point for Velocity problem: comparison between BEM, SBEM FOM and SBEM SOM solution.

 
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3D models: some numerical issues regarding convergence, stability and influence of the uncertainties are being solved and implemented.

## 4 Conclusions and further research

In conclusion, two numerical applications illustrated the relevancy of this formulation, an acoustic domain with internal acoustic source, and an acoustic domain coupled with a vibrating structure. In both applications, the peaks of the response corresponding to the low frequency modes of the acoustic domain are precisely described, while the densely spaced peaks belonging to the high frequency behaviour are smoothed.

The computational details reported in the article helps to understand the efficiency and the actual limits of the formulation.

It has also been proven that the velocity approach, which is the most interesting one from an industrial point of view, can be correctly handled by means of the SBEM formulation; this application have also highlighted some open points which need to be solved in order to proceed to a possible industrial utilization of the stochastic approach. First of all the assumptions, which controls the correlation, need to be revised to handle correctly a random domain which is completely coupled (all sides) with a deterministic vibrating structure. Otherwise this would lead to a prohibitively large number of equations for the full stochastic formulation. Secondly also the parametrization of the boundaries as function of the uncertainties, has to be improved to account also for small shape modifications.

The SBEM method has proved to be consistent in dealing with vibro-acoustic problems in the mid-frequency range. As a promising approach it can also lead to many possible research topics intended to integrate and extend the stochastic modelling towards the well-established classical and the novel numerical methods.

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