Free vibration analysis of a circular plate with multiple circular holes by using the addition theorem and direct BIEM

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Abstract

The purpose of this paper is to propose an analytical method to solve natural frequencies and natural modes of a circular plate with multiple circular holes by using the null field integral formulation, the addition theorem and complex Fourier series. Owing to the addition theorem, all kernel functions are represented in the degenerate form and further transformed into the same polar coordinate from each local coordinate at center of all circles. Not only avoiding the computation of the principal value but also the calculation of higher-order derivatives in the plate problem can be easily determined. According to the specified boundary conditions, a coupled infinite system of simultaneous linear algebraic equations is derived and its solution can be obtained in an analytical way. The direct searching approach is adopted to determine the natural frequency through singular value decomposition (SVD). After determining the unknown Fourier coefficients, the corresponding mode shapes are obtained by using the direct boundary integral equations for domain points. Some numerical results are presented. Moreover, the inherent problem of spurious eigenvalue using integral formulation is investigated and the SVD updating technique is adopted to suppress the occurrence of spurious eigenvalues. Excellent accuracy, fast rate of convergence and high computational efficiency are the main features of the present method.

Keywords: direct boundary integral equation, null-field integral equation, addition theorem, complex Fourier series, vibration, spurious eigenvalue, SVD updating technique.

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1 Introduction

Circular plates with multiple circular holes are widely used in engineering structures [1], e.g. missiles, aircraft, etc. These holes usually cause the change of natural frequency as well as the decrease of load carrying capacity. It is important to comprehend the associated effects in the work of mechanical design or flight control of the structure. As quoted by Leissa and Narita [2]: "the free vibrations of circular plates have been of practical and academic interest for at least a century and a half", we revisit this problem by proposing an analytical method.

Over the past few decades, most of the researches have focused on the analytical solutions for natural frequencies of the circular or annular plates [3–6]. Recently some researchers intended to extend an annular plate [7, 8] to the plate with an eccentric hole. Lee *et al.* [9, 10] proposed a semi-analytical approach to solve the free vibration analysis of a circular plate with multiple holes and pointed out some insufficient accurate results in [8] after careful comparisons.

It is clear that the number of variables can be dramatically decreased by using boundary element method (BEM) or boundary integral equation method (BIEM). For BEM applications to plate problems, readers may consult with the review article [11]. By using the BIEM to analytically solve the problem of plate with multiple holes, two questions need to be solved. One is the improper integral in the boundary integral equation; the other is that both field point and source point are located on different circular boundaries. These problems have been treated by using the degenerate kernel and tensor transformation [9, 10], respectively. But tensor transformation accompanied with the higher order derivative increases the complexity of computation and then affect the accuracy of its solution [9]. In addition, the collocation method in [9, 10] belongs to point-matching approach instead of analytical derivation. It also increases the effort of computation since boundary nodes for collocation are required.

This paper presents an analytical approach to solve the multiple-connected domain plate problem by using the null field integral formulation, addition theorem and complex Fourier series. Some numerical computations are presented. Moreover, the SVD updating technique [10] is employed to suppress the appearance of spurious eigenvalues.

2 Problem statement and direct boundary integral formulations

2.1 Problem statement of plate eigenproblem

As shown in fig.1, a uniform thin circular plate with H circular holes centred at O_k (k=0,1,…,H) has a domain Ω which is enclosed with boundary

$$B=\bigcup_{k=0}^{H} B_k \tag{1}$$

where R_k denotes the radius of the kth circle and O_0 is the position vector of the outer circular boundary of the plate. The governing equation of the free flexural vibration for this plate is expressed as:

$$\nabla^4 u(x) = \lambda^4 u(x), \qquad x \in \Omega \tag{2}$$

where ∇^4 is the biharmonic operator, u is the lateral displacement, $\lambda^4 = \omega^2 \rho_0 h / D$, λ is the dimensionless frequency parameter, ω is the circular frequency, ρ_0 is the volume density, h is the plate thickness, $D = Eh^3/12(1-\mu^2)$ is the flexural rigidity of the plate, E denotes the Young's modulus and μ is the Poisson's ratio

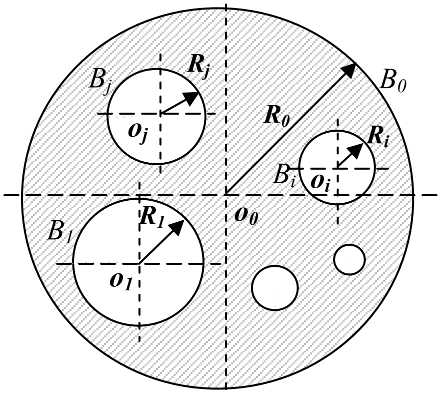

Figure 1: Problem statement for an eigenproblem of a circular plate with multiple circular holes.

2.2 Direct boundary integral formulation

The integral representation for the plate problem can be derived from the Rayleigh-Green identity [12] as follows:

$$\begin{aligned} u(x) = &\int_B U(s,x)v(s)dB(s) - \int_B \Theta(s,x)m(s)dB(s) \\ &+ \int_B M(s,x)\theta(s)dB(s) - \int_B V(s,x)u(s)dB(s), \qquad x \in \Omega \end{aligned} \tag{3}$$

where B is the boundary of the domain Ω, $u(x)$ is the displacement, s and x mean the source and field points, respectively. $U(s, x)$, $\Theta(s, x)$, $M(s, x)$ and $V(s, x)$ in eqn. (3) are kernel functions. The kernel function $U(s, x)$ in eqn. (3),

$$U(s,x) = \frac{1}{8\lambda^2 D}\left[Y_0(\lambda r) - iJ_0(\lambda r) + \frac{2}{\pi}K_0(\lambda r) \right], \tag{4}$$

is the fundamental solution which satisfies

$$\nabla^4 U(s,x) - k^4 U(s,x) = \delta(s-x) \tag{5}$$

where $\delta(s-x)$ is the Dirac-delta function, $J_0(\lambda r)$, $Y_0(\lambda r)$ and $K_0(\lambda r)$ are Bessel functions, $r \equiv |s-x|$ and $i^2=-1$. The other three kernel functions, $\Theta(s, x)$, $M(s, x)$ and $V(s, x)$, in eqn. (3) can be obtained by applying the following slope, moment and effective shear operators defined by

$$K_\Theta = \frac{\partial(\cdot)}{\partial n} \tag{6}$$

$$K_{\mathrm{M}} = -D\left[\mu\nabla^2(\cdot) + (1-\mu)\frac{\partial^2(\cdot)}{\partial n^2}\right] \tag{7}$$

$$K_{\mathrm{V}} = -D\left[\frac{\partial}{\partial n}\nabla^2(\cdot) + (1-\mu)\frac{\partial}{\partial t}\left(\frac{\partial}{\partial n}\left(\frac{\partial}{\partial t}(\cdot)\right)\right)\right] \tag{8}$$

to the kernel $U(s, x)$ with respect to the source point, where $\partial/\partial n$ and $\partial/\partial t$ are the normal and tangential derivatives, respectively, ∇^2 means the Laplacian operator.

2.3 Null-field integral equations

The null-field integral equation for displacement can be derived from eqn. (3) and by moving the field point outside the domain. It is expressed as follows:

$$\begin{aligned} 0 = &\int_B U(s,x)v(s)dB(s) - \int_B \Theta(s,x)m(s)dB(s) \\ &+ \int_B M(s,x)\theta(s)dB(s) - \int_B V(s,x)u(s)dB(s), \quad x \in \Omega^C \cup B, \end{aligned} \tag{9}$$

where Ω^C is the complementary domain of Ω. It is noted that once kernel functions are expressed in proper degenerate forms, the field points can be exactly located on the real boundary, that is $x \in \Omega^C \cup B$.

2.4 Degenerate kernels and Fourier series for boundary densities

In the polar coordinate, the field point and source point can be expressed as (ρ, ϕ) and (R, θ), respectively. By employing the addition theorem [13], the kernel function $U(s, x)$ is expanded in the series form as follows:

$$U: \begin{cases} U^I(s,x) = \dfrac{1}{8\lambda^2 D}\displaystyle\sum_{m=-\infty}^{\infty} \{J_m(\lambda\rho)[Y_m(\lambda R) - iJ_m(\lambda R)] + \frac{2}{\pi} I_m(\lambda\rho)K_m(\lambda R)\}e^{im(\varphi-\theta)}, \ \rho < R \\ U^E(s,x) = \dfrac{1}{8\lambda^2 D}\displaystyle\sum_{m=-\infty}^{\infty} \{J_m(\lambda R)[Y_m(\lambda\rho) - iJ_m(\lambda\rho)] + \frac{2}{\pi} I_m(\lambda R)K_m(\lambda\rho)\}e^{im(\varphi-\theta)}, \ \rho \geq R \end{cases} \tag{10}$$

where the superscripts "I" and "E" denote the interior and exterior cases for $U(s, x)$ degenerate kernel to distinguish $\rho < R$ and $\rho > R$, respectively, as shown in fig. 2.

The displacement $u(s)$, slope $\theta(s)$, moment $m(s)$ and shear force $v(s)$ along the circular boundaries in the null-field integral equations can be expanded in terms of complex Fourier series, respectively, as shown below:

$$u^k(s_k)=\sum_{n=-\infty}^{\infty} a_n^k e^{in\theta_k} \quad , \qquad s_k \in B_k, \qquad k=0,\ldots,H \tag{11}$$

$$\theta^k(s_k)=\sum_{n=-\infty}^{\infty} b_n^k e^{in\theta_k} \quad , \qquad s_k \in B_k, \qquad k=0,\ldots,H \tag{12}$$

$$m^k(s)=\sum_{n=-\infty}^{\infty} c_n^k e^{in\theta_k} \quad , \qquad s_k \in B_k, \qquad k=0,\ldots,H \tag{13}$$

$$v^k(s)=\sum_{n=-\infty}^{\infty} d_n^k e^{in\theta_k} \quad , \qquad s_k \in B_k, \qquad k=0,\ldots,H \tag{14}$$

where a_n^k, b_n^k, c_n^k and d_n^k are the complex Fourier coefficients of the *k*th circular boundary and θ_k is its polar angle.

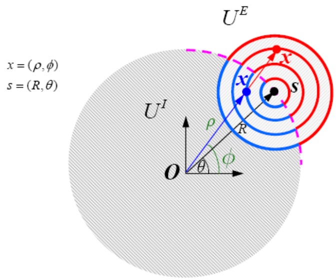

Figure 2: Degenerate kernel for $U(s, x)$.

3 Analytical eigensolution for a circular plate with multiple circular holes

To present formulation clearly and without loss of geniality, a clamped circular plate ($u^0 = \theta^0 = 0$) with H circular holes subject to free edge ($m^k = v^k = 0$; $k=1,\ldots,H$) is considered. Considering the null field near the circular boundary B_0, eqn. (9) can be explicitly expressed as:

$$\begin{aligned}0=&\int_{B_0} U^E(s_0,x_0)v^0(s_0)dB_0(s_0)-\int_{B_0}\Theta^E(s_0,x_0)m^0(s_0)dB_0(s_0)\\&-\left[\sum_{k=1}^{H}\int_{B_k} M^E(s_k,x_k)\theta^k(s_k)dB_k(s_k)-\int_{B_k}V^E(s_k,x_k)u^k(s_k)dB_k(s_k)\right]\end{aligned} \tag{15}$$

By substituting the degenerate kernels, such as eqn. (10), and eqns. (11)-(14) into the eqn. (15), employing the analytical integration along each circular boundary and applying the orthogonal property, eqn. (15) yields

$$\begin{aligned}0 = {} & \frac{\pi R_0}{4\lambda^2 D}\sum_{m=-\infty}^{\infty}\{J_m(\lambda R_0)[Y_m(\lambda\rho_0)-iJ_m(\lambda\rho_0)]+\frac{2}{\pi}I_m(\lambda R_0)K_m(\lambda\rho_0)\}d_m^0 \mathrm{e}^{im\phi_0} \\ & -\frac{\pi R_0}{4\lambda D}\sum_{m=-\infty}^{\infty}\{J'_m(\lambda R_0)[Y_m(\lambda\rho_0)-iJ_m(\lambda\rho_0)]+\frac{2}{\pi}I'_m(\lambda R_0)K_m(\lambda\rho_0)\}c_m^0 \mathrm{e}^{im\phi_0} \\ & -\sum_{k=1}^{H}\Bigg[-\frac{\pi R_k}{4\lambda^2}\sum_{m=-\infty}^{\infty}\{J_m(\lambda\rho_k)[\alpha_m^Y(\lambda R_k)-i\alpha_m^J(\lambda R_k)]+\frac{2}{\pi}I_m(\lambda\rho_k)\alpha_m^K(\lambda R_k)\}b_m^k \mathrm{e}^{im\phi_k} \\ & +\frac{\pi R_k}{4\lambda^2}\sum_{m=-\infty}^{\infty}\{J_m(\lambda\rho_k)[\beta_m^Y(\lambda R_k)-i\beta_m^J(\lambda R_k)]+\frac{2}{\pi}I_m(\lambda\rho_k)\beta_m^K(\lambda R_k)\}a_m^k \mathrm{e}^{im\phi_k}\Bigg]\end{aligned} \tag{16}$$

where the (ρ_0,ϕ_0), (ρ_1,ϕ), …, (ρ_H,ϕ_H) are the coordinates for the field point x with respect to each center of circles. From eqns. (7) and (8), the moment and the effective shear operators, $\alpha_m^X(\lambda\rho)$ and $\beta_m^X(\lambda\rho)$ are defined as, respectively,

$$\alpha_m^X(\lambda\rho) = D\left\{(1-\mu)\frac{X'_m(\lambda\rho)}{\rho}-\left[(1-\mu)\frac{m^2}{\rho^2}\mp\lambda^2\right]X_m(\lambda\rho)\right\} \tag{17}$$

$$\beta_m^X(\lambda\rho) = D\left\{\left[m^2(1-\mu)\pm(\lambda\rho)^2\right]\frac{X_m{}'(\lambda\rho)}{\rho^2}-m^2(1-\mu)\frac{X_m(\lambda\rho)}{\rho^3}\right\} \tag{18}$$

where the upper (lower) signs refer to $X = J$, Y, (I, K), respectively. The Bessel's differential equations have been used to simplify $\alpha_m^X(\lambda\rho)$ and $\beta_m^X(\lambda\rho)$.

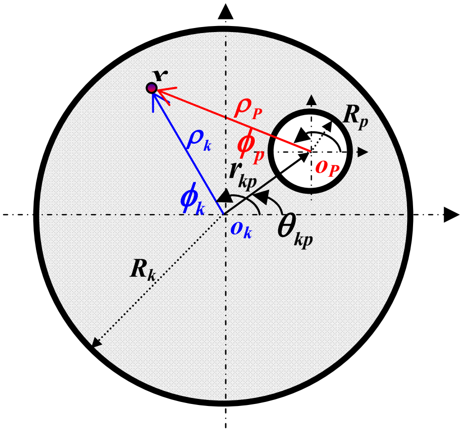

Figure 3: Notation of Graf's addition theorem for Bessel functions.

Based on the Graf's addition theorem for Bessel functions given in [13, 14], we can express the theorem in the following form

$$J_m(\lambda\rho_p)\mathrm{e}^{im\phi_p} = \sum_{n=-\infty}^{\infty}J_{m-n}(\lambda r_{pk})\mathrm{e}^{i(m-n)\theta_{pk}}J_n(\lambda\rho_k)e^{in\phi_k} \tag{19}$$

$$I_m(\lambda\rho_p)\mathrm{e}^{im\phi_p} = \sum_{n=-\infty}^{\infty}I_{m-n}(\lambda r_{pk})\mathrm{e}^{i(m-n)\theta_{pk}}I_n(\lambda\rho_k)e^{in\phi_k} \tag{20}$$

$$Y_m(\lambda\rho_p)\mathrm{e}^{im\phi_p}=\begin{cases}\sum\limits_{n=-\infty}^{\infty}Y_{m-n}(\lambda r_{pk})\mathrm{e}^{i(m-n)\theta_{pk}}J_n(\lambda\rho_k)e^{in\phi_k}, & \rho_k<r_{pk}\\ \sum\limits_{n=-\infty}^{\infty}J_{m-n}(\lambda r_{pk})\mathrm{e}^{i(m-n)\theta_{pk}}Y_n(\lambda\rho_k)e^{in\phi_k}, & \rho_k>r_{pk}\end{cases} \tag{21}$$

$$K_m(\lambda\rho_p)\mathrm{e}^{im\phi_p}=\begin{cases}\sum\limits_{n=-\infty}^{\infty}(-1)^nK_{m-n}(\lambda r_{pk})\mathrm{e}^{i(m-n)\theta_{pk}}I_n(\lambda\rho_k)e^{in\phi_k}, & \rho_k<r_{pk}\\ \sum\limits_{n=-\infty}^{\infty}(-1)^{m-n}I_{m-n}(\lambda r_{pk})\mathrm{e}^{i(m-n)\theta_{pk}}K_n(\lambda\rho_k)e^{in\phi_k}, & \rho_k>r_{pk}\end{cases} \tag{22}$$

where (ρ_p,ϕ_p)and (ρ_k,ϕ_k) shown in fig. 3 are the polar coordinates of a general field point $\boldsymbol{x}$ with respect to O_p and O_k, which are origins of two polar coordinate system and (r_{pk},θ_{pk}) are the polar coordinates of O_k with respect to O_p.

By using the addition theorem for Bessel functions $J_m(\lambda\rho_k)$, $Y_m(\lambda\rho_k)$ and $K_m(\lambda\rho_k)$, eqn. (16) near the circular boundary B_0 under the condition of $\rho_0>r_{k0}$ can be expanded as follows:

$$\begin{aligned}0=&\frac{\pi R_0}{4\lambda^2D}\sum_{m=-\infty}^{\infty}\{J_m(\lambda R_0)[Y_m(\lambda\rho_0)-iJ_m(\lambda\rho_0)]+\frac{2}{\pi}I_m(\lambda R_0)K_m(\lambda\rho_0)\}d_m^0\mathrm{e}^{im\phi_0}\\&-\frac{\pi R_0}{4\lambda D}\sum_{m=-\infty}^{\infty}\{J_m'(\lambda R_0)[Y_m(\lambda\rho_0)-iJ_m(\lambda\rho_0)]+\frac{2}{\pi}I_m'(\lambda R_0)K_m(\lambda\rho_0)\}c_m^0\mathrm{e}^{im\phi_0}\\&+\sum_{k=1}^{H}\Bigg[\frac{\pi R_k}{4\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_m^Y(\lambda R_k)-i\alpha_m^J(\lambda R_k)]\sum_{n=-\infty}^{\infty}J_{m-n}(\lambda r_{k0})\mathrm{e}^{i(m-n)\theta_{k0}}J_n(\lambda\rho_0)\\&\qquad+\frac{2}{\pi}\alpha_m^K(\lambda R_k)\sum_{n=-\infty}^{\infty}I_{m-n}(\lambda r_{k0})\mathrm{e}^{i(m-n)\theta_{k0}}I_n(\lambda\rho_0)\}\mathrm{e}^{in\phi_0}b_m^k\\&-\frac{\pi R_k}{4\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta_m^Y(\lambda R_k)-i\beta_m^J(\lambda R_k)]\sum_{n=-\infty}^{\infty}J_{m-n}(\lambda r_{k0})\mathrm{e}^{i(m-n)\theta_{k0}}J_n(\lambda\rho_0)\\&\qquad+\frac{2}{\pi}\beta_m^K(\lambda R_k)\sum_{n=-\infty}^{\infty}I_{m-n}(\lambda r_{k0})\mathrm{e}^{i(m-n)\theta_{k0}}I_n(\lambda\rho_0)\}\mathrm{e}^{in\phi_0}a_m^k\Bigg]\end{aligned} \tag{23}$$

Furthermore, eqn. (23) can be rewritten as

$$0=\sum_{m=-\infty}^{\infty}\mathrm{e}^{im\phi_0}\left\langle A_m^0(\lambda\rho_0)d_m^0+B_m^0(\lambda\rho_0)c_m^0+\sum_{k=1}^{H}\left[\sum_{n=-\infty}^{\infty}A_{mn}^k(\lambda\rho_0)b_n^k+\sum_{n=-\infty}^{\infty}B_{mn}^k(\lambda\rho_0)a_n^k\right]\right\rangle \tag{24}$$

where

$$A_m^0(\lambda\rho_0)=\frac{\pi R_0}{4\lambda^2D}\{J_m(\lambda R_0)[Y_m(\lambda\rho_0)-iJ_m(\lambda\rho_0)]+\frac{2}{\pi}I_m(\lambda R_0)K_m(\lambda\rho_0)\} \tag{25}$$

$$B_m^0(\lambda\rho_0)=-\frac{\pi R_0}{4\lambda D}\{J_m'(\lambda R_0)[Y_m(\lambda\rho_0)-iJ_m(\lambda\rho_0)]+\frac{2}{\pi}I_m'(\lambda R_0)K_m(\lambda\rho_0)\} \tag{26}$$

$$\begin{aligned}A_{mn}^k(\lambda\rho_0)=&\frac{\pi R_k}{4\lambda^2}\mathrm{e}^{i(n-m)\theta_{k0}}\{J_{n-m}(\lambda r_{k0})\alpha_n^J(\lambda R_k)[Y_m(\lambda\rho_0)-iJ_m(\lambda\rho_0)]\\&+\frac{2}{\pi}(-1)^{n-m}I_{n-m}(\lambda r_{k0})\alpha_n^I(\lambda R_k)K_m(\lambda\rho_0)\}\end{aligned} \tag{27}$$

$$B_{mn}^{k}(\lambda\rho_0) = -\frac{\pi R_k}{4\lambda^2} e^{i(n-m)\theta_{k0}} \{J_{n-m}(\lambda r_{k0})\beta_n^J(\lambda R_k)[Y_m(\lambda\rho_0) - iJ_m(\lambda\rho_0)] + \frac{2}{\pi}(-1)^{n-m} I_{n-m}(\lambda r_{k0})\beta_n^I(\lambda R_k)K_m(\lambda\rho_0)\} \tag{28}$$

By differentiating eqn. (24) with respect to ρ_0, the equation for the slope θ near the circular boundary B_0 is given as

$$0 = \sum_{m=-\infty}^{\infty} e^{im\phi_0} \left\langle C_m^0(\lambda\rho_0)d_m^0 + D_m^0(\lambda\rho_0)c_m^0 + \sum_{k=1}^{H}\left[\sum_{n=-\infty}^{\infty} C_{mn}^k(\lambda\rho_0)b_n^k + \sum_{n=-\infty}^{\infty} D_{mn}^k(\lambda\rho_0)a_n^k\right]\right\rangle \tag{29}$$

where $C_m^0(\lambda\rho_0)$, $D_m^0(\lambda\rho_0)$, $C_{mn}^k(\lambda\rho_0)$ and $D_{mn}^k(\lambda\rho_0)$ can be obtained by differentiating $A_m^0(\lambda\rho_0)$, $B_m^0(\lambda\rho_0)$, $A_{mn}^k(\lambda\rho_0)$ and $B_{mn}^k(\lambda\rho_0)$ in eqns. (25-28) with respective to ρ_0.

Similarly, considering the null field points near the circular boundary B_p ($p=1,\ldots,H$), eqn. (9) can be explicitly expressed as:

$$0 = \int_{B_0} U^I(s_0,x_0)v^0(s_0)dB_0(s_0) - \int_{B_0} \Theta^I(s_0,x_0)m^0(s_0)dB_0(s_0) - \left[\sum_{k=1}^{H}\int_{B_k} M^\gamma(s_k,x_k)\theta^k(s_k)dB_k(s_k) - \int_{B_k} V^\gamma(s_k,x_k)u^k(s_k)dB_k(s_k)\right] \tag{30}$$

where $\gamma = I$, $k=p$; $\gamma = E$, $k \neq p$.

By substituting degenerate kernel functions and complex Fourier series into eqn. (30) and then applying the addition theorem, eqn. (30) yields

$$0 = \sum_{m=-\infty}^{\infty} e^{im\phi_p} \left\langle E_m^p(\lambda\rho_p)d_m^p + F_m^p(\lambda\rho_p)c_m^p + \sum_{\substack{k=0\\k\neq p}}^{H}\left[\sum_{n=-M}^{M} E_{mn}^k(\lambda\rho_p)b_n^k + \sum_{n=-\infty}^{\infty} F_{mn}^k(\lambda\rho_p)a_n^k\right]\right\rangle \tag{31}$$

where

$$E_m^p(\lambda\rho_p) = \frac{\pi R_p}{4\lambda^2}\{J_m(\lambda\rho_p)[\alpha_m^Y(\lambda R_p) - i\alpha_m^J(\lambda R_p)] + \frac{2}{\pi}I_m(\lambda\rho_p)\alpha_m^K(\lambda R_p)\} \tag{32}$$

$$F_m^p(\lambda\rho_p) = -\frac{\pi R_p}{4\lambda^2}\{J_m(\lambda\rho_p)[\beta_m^Y(\lambda R_p) - i\beta_m^J(\lambda R_p)] + \frac{2}{\pi}I_m(\lambda\rho_p)\beta_m^K(\lambda R_p)\} \tag{33}$$

$$E_{mn}^k(\lambda\rho_p) = \begin{cases} \frac{\pi R_k}{4\lambda^2 D} e^{i(n-m)\theta_{kp}} \{J_{n-m}(\lambda r_{kp})J_m(\lambda\rho_p)[Y_n(\lambda R_k) - iJ_n(\lambda R_k)] \\ \quad + \frac{2}{\pi}I_{n-m}(\lambda r_{kp})I_m(\lambda\rho_p)K_n(\lambda R_k)\}, & k=0 \\ \frac{\pi R_k}{4\lambda^2} e^{i(n-m)\theta_{kp}} \{J_m(\lambda\rho_p)\alpha_n^J(\lambda R_k)[Y_{n-m}(\lambda r_{kp}) - iJ_{n-m}(\lambda r_{kp})] \\ \quad + \frac{2}{\pi}(-1)^m I_m(\lambda\rho_p)\alpha_n^I(\lambda R_k)K_{n-m}(\lambda r_{kp})\}, & k \neq 0,p \end{cases} \tag{34}$$

$$F_{mn}^{k}(\lambda\rho_p)=\begin{cases}-\dfrac{\pi R_k}{4\lambda D}\mathrm{e}^{i(n-m)\theta_{kp}}\{J_{n-m}(\lambda r_{kp})J_m(\lambda\rho_p)[Y_n'(\lambda R_k)-iJ_n'(\lambda R_k)] \\ \qquad\qquad +\dfrac{2}{\pi}I_{n-m}(\lambda r_{kp})I_m(\lambda\rho_p)K_n'(\lambda R_k)\}, \qquad k=0 \\ -\dfrac{\pi R_k}{4\lambda^2}\mathrm{e}^{i(n-m)\theta_{kp}}\{J_m(\lambda\rho_p)\beta_n^J(\lambda R_k)[Y_{n-m}(\lambda r_{kp})-iJ_{n-m}(\lambda r_{kp})] \\ \qquad\qquad +\dfrac{2}{\pi}(-1)^m I_m(\lambda\rho_p)\beta_n^I(\lambda R_k)K_{n-m}(\lambda r_{kp})\}, \qquad k\neq 0,p\end{cases} \quad (35)$$

By differentiating eqn. (31) with respect to ρ_p, the equation of the slope θ near the circular boundary B_p is given as

$$0=\sum_{m=-\infty}^{\infty}\mathrm{e}^{im\phi_p}\left\langle G_m^p(\lambda\rho_p)d_m^p+H_m^p(\lambda\rho_p)c_m^p+\sum_{\substack{k=0\\k\neq p}}^{H}\left[\sum_{n=-M}^{M}G_{mn}^k(\lambda\rho_p)b_n^k+\sum_{n=-\infty}^{\infty}H_{mn}^k(\lambda\rho_p)a_n^k\right]\right\rangle \quad (36)$$

where $G_m^p(\lambda\rho_p), H_m^p(\lambda\rho_p), G_{mn}^k(\lambda\rho_p)$ and $H_{mn}^k(\lambda\rho_p)$ can be obtained by differentiating $E_m^p(\lambda\rho_p), F_m^p(\lambda\rho_p)$, $E_{mn}^k(\lambda\rho_p)$ and $F_{mn}^k(\lambda\rho_p)$ in eqns. (32-35) with respective to ρ.

By setting ρ_p to R_p and applying the orthogonal property of $\{\mathrm{e}^{im\phi_p}\}$ ($p=0, 1,\ldots,H$), eqns.(24), (29), (31) and (36) yield , for m=0, ±1, ±2, …; n=0, ±1, ±2, …; $p=1,\ldots,H$,

$$\begin{cases}A_m^0(\lambda R_0)d_m^0+B_m^0(\lambda R_0)c_m^0+\sum_{k=1}^{H}\left[\sum_{n=-\infty}^{\infty}A_{mn}^k(\lambda R_0)b_n^k+\sum_{n=-\infty}^{\infty}B_{mn}^k(\lambda R_0)a_n^k\right]=0 \\ C_m^0(\lambda R_0)d_m^0+D_m^0(\lambda R_0)c_m^0+\sum_{k=1}^{H}\left[\sum_{n=-\infty}^{\infty}C_{mn}^k(\lambda R_0)b_n^k+\sum_{n=-\infty}^{\infty}D_{mn}^k(\lambda R_0)a_n^k\right]=0 \\ E_m^p(\lambda R_p)d_m^p+F_m^p(\lambda R_p)c_m^p+\sum_{\substack{k=0\\k\neq p}}^{H}\left[\sum_{n=-M}^{M}E_{mn}^k(\lambda R_p)b_n^k+\sum_{n=-\infty}^{\infty}F_{mn}^k(\lambda R_p)a_n^k\right]=0 \\ G_m^p(\lambda R_p)d_m^p+H_m^p(\lambda R_p)c_m^p+\sum_{\substack{k=0\\k\neq p}}^{H}\left[\sum_{n=-M}^{M}G_{mn}^k(\lambda R_p)b_n^k+\sum_{n=-\infty}^{\infty}H_{mn}^k(\lambda R_p)a_n^k\right]=0\end{cases} \quad (37)$$

eqn. (37) results in a couple infinite system of simultaneous linear algebraic equations for the coefficients a_m^k, b_m^k, c_m^k and d_m^k, k=0,…, H. In the following computation, only the finite M terms are used in eqn. (37). According to the direct-searching scheme [12], the eigenvalue are obtained by applying the SVD technique [15] to the matrix from eqn. (37). Once the eigenvalues are found, the associated mode shapes can be obtained by substituting the corresponding boundary eigenvectors (i.e. the complex Fourier series representing the fictitious boundary density) into the boundary integral equations.

4 Spurious eigenvalue for multiply-connected plate eigenproblem using BEM

In this section, SVD updating technique is adopted to suppress the appearance of spurious eigenvalue which cause the present method fail. The concept of this technique is to provide sufficient constrains to overcome the rank deficiency of the system. Since four null field integral equations for the plate formulation are provided [10], there are 6 (C (4,2)) options for choosing any two equations to solve the eigenproblem. The $U\Theta$ formulation in section 3 uses the first and the second equations. To provide sufficient constrains, the *UM* formulation is alternative, which uses the first and the third equations. Applying the moment operator of eqn. (17), to eqns. (24) and (31), respectively, yield

$$0=\sum_{m=-\infty}^{\infty} \mathrm{e}^{im\phi_0} \left\langle P_m^0(\lambda\rho_0)d_m^0 + Q_m^0(\lambda\rho_0)c_m^0 + \sum_{k=1}^{H}\left[\sum_{n=-\infty}^{\infty} P_{mn}^k(\lambda\rho_0)b_n^k + \sum_{n=-\infty}^{\infty} Q_{mn}^k(\lambda\rho_0)a_n^k\right]\right\rangle \quad (38)$$

$$0=\sum_{m=-\infty}^{\infty} \mathrm{e}^{im\phi_p} \left\langle S_m^p(\lambda\rho_p)d_m^p + T_m^p(\lambda\rho_p)c_m^p + \sum_{\substack{k=0\\k\neq p}}^{H}\left[\sum_{n=-M}^{M} S_{mn}^k(\lambda\rho_p)b_n^k + \sum_{n=-\infty}^{\infty} T_{mn}^k(\lambda\rho_p)a_n^k\right]\right\rangle \quad (39)$$

where $P_m^0(\lambda\rho_0)$, $Q_m^0(\lambda\rho_0)$, $P_{mn}^k(\lambda\rho_0)$, $Q_{mn}^k(\lambda\rho_0)$, $S_m^p(\lambda\rho_p)$, $T_m^p(\lambda\rho_p)$, $S_{mn}^k(\lambda\rho_p)$ and $T_{mn}^k(\lambda\rho_p)$ can be obtained by applying the moment operator to $A_m^0(\lambda\rho_0)$, $B_m^0(\lambda\rho_0)$, $A_{mn}^k(\lambda\rho_0)$, $B_{mn}^k(\lambda\rho_0)$, $E_m^p(\lambda\rho_p)$, $F_m^p(\lambda\rho_p)$, $E_{mn}^k(\lambda\rho_p)$ and $F_{mn}^k(\lambda\rho_p)$ with respect to ρ.

By setting ρ_P to R_P and applying the orthogonal property of $\{\mathrm{e}^{im\phi_P}\}$ ($p=0, 1,\ldots,H$), eqns.(24), (38), (31) and (39) yield , for m=0, ±1, ±2, …; n=0, ±1, ±2, …; $p=1,\ldots,H$,

$$\begin{cases} A_m^0(\lambda R_0)d_m^0 + B_m^0(\lambda R_0)c_m^0 + \sum_{k=1}^{H}\left[\sum_{n=-\infty}^{\infty} A_{mn}^k(\lambda R_0)b_n^k + \sum_{n=-\infty}^{\infty} B_{mn}^k(\lambda R_0)a_n^k\right]=0 \\ P_m^0(\lambda R_0)d_m^0 + Q_m^0(\lambda R_0)c_m^0 + \sum_{k=1}^{H}\left[\sum_{n=-\infty}^{\infty} P_{mn}^k(\lambda R_0)b_n^k + \sum_{n=-\infty}^{\infty} Q_{mn}^k(\lambda R_0)a_n^k\right]=0 \\ E_m^p(\lambda R_p)d_m^p + F_m^p(\lambda R_p)c_m^p + \sum_{\substack{k=0\\k\neq p}}^{H}\left[\sum_{n=-M}^{M} E_{mn}^k(\lambda R_p)b_n^k + \sum_{n=-\infty}^{\infty} F_{mn}^k(\lambda R_p)a_n^k\right]=0 \\ S_m^p(\lambda R_p)d_m^p + T_m^p(\lambda R_p)c_m^p + \sum_{\substack{k=0\\k\neq p}}^{H}\left[\sum_{n=-M}^{M} S_{mn}^k(\lambda R_p)b_n^k + \sum_{n=-\infty}^{\infty} T_{mn}^k(\lambda R_p)a_n^k\right]=0 \end{cases} \quad (40)$$

which is called the *UM* formulation. By using the concept of SVD technique of updating terms, combing eqns. (37) and (40) can filter out spurious eigenvalues.

5 Numerical results and discussions

To demonstrate the proposed method, the FORTRAN code was implemented to determine natural frequencies and modes of a circular plate with multiple circular holes. The same problem was independently solved by using FEM (the ABAQUS software [16]) for comparison. The thickness of plate is 0.002m and the Poisson ratio μ=1/3. The general-purpose linear triangular elements of type S3 were employed to model the plate problem by using ABAQUS. Although the thickness of the plate is 0.002 m, these elements do not suffer from transverse shear locking based on the theoretical manual of ABAQUS [16].

A circular plate with three holes [10] is considered. The radii of holes are 0.4m, 0.2m and 0.2m and the coordinates of the center are (0.5,0), (−0.3,0.4) and (−0.3, −0.4), respectively. Fig. 4 shows the former six natural frequency parameters versus the number of terms of Fourier series *N*. It indicates that the proposed solution promptly converges with few terms of Fourier series. By taking thirteen terms of Fourier series (*N*=13), fig. 5 shows the minimum singular value versus the frequency parameter by using the $U\Theta$ formulation, *UM* formulation and SVD updating technique for a circular clamped plate with three circular holes. Since the direct-searching scheme is used, the drop location indicates the eigenvalue. The spurious eigenvalues of 5.5811 and 7.9906 occur when using the *UM* and $U\Theta$ formulation, respectively. They are found to be the true eigenvalue of simply supported and clamped circular plate with a radius of 0.4 m, respectively. It demonstrates that the spurious eigenvalue can be filtered out by using the SVD updating technique. The same problem is also solved by using ABAQUS and its model has 308960 elements in order to obtain comparable results for comparison. The former five natural frequency parameters and modes by using the present method, the semi-analytical method [10] and FEM are shown in fig. 6. Good agreement between the results of the present method and those of ABAQUS is observed.

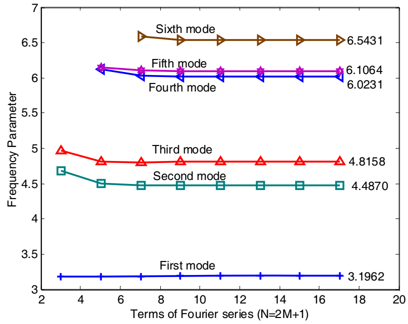

Figure 4: Natural frequency parameter versus the number of terms of Fourier series.

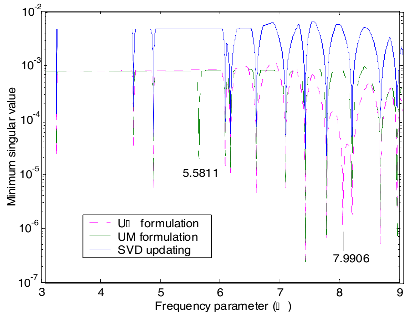

Figure 5: The minimum singular value versus the frequency parameter by using different methods.

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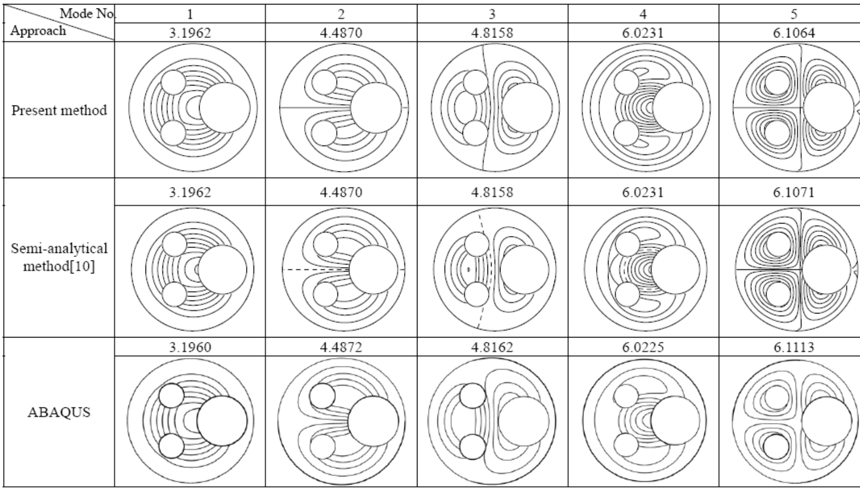

Figure 6: The former five natural frequency parameters and mode shapes by using the present method, semi-analytical method and FEM.

6 Concluding remarks

Natural frequencies and natural modes of a circular plate with multiple circular holes have been obtained by using the null field integral formulation and the addition theorem. Based on the addition theorem, two critical problems of improper integrals in the boundary integration and the higher derivative in the multiply-connected domain problems were successively treated in a novel way. By matching the specified boundary conditions, an analytical solution can be derived from a coupled infinite systems of simultaneous linear algebraic equations. By using the direct-searching method, numerical eigensolutions were given from the corresponding truncated finite system. The proposed results match well with those provided by the FEM using many elements to obtain acceptable data for comparison. In addition, the SVD technique of updating terms was employed to filter out spurious eigenvalues. Numerical results show good accuracy and fast rate of convergence thanks to the analytical method.

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