Efficient Boundary Element Method for a focused domain

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Abstract

An efficient Boundary Element Method for focused domain is developed. In this method, whole boundaries are divided into near boundaries which are near to the focused domain and far boundaries which are sufficiently far from it. We set up boundary integral equations and express these integrals which contain unknown quantities on the far boundaries as low order multipole moments, approximately. Thus the number of unknowns and boundary integral equations set up are decreased drastically by this method, and enables us to compute them efficiently. When unknown quantities are required only in a specific domain, especially on large-scale boundary value problems, this method enables us to compute them efficiently. The capability of this method is verified with some numerical experiments.
Keywords: Boundary Element Method, 2D potential problem, focused domain, multipole expansion, generalized inverse matrix.

1 Introduction

The Boundary Element Method (BEM) is one of the major numerical solutions for boundary value problems and the Fast Multipole Boundary Element Method (FMBEM), is in widespread use as an efficient solution for large-scale boundary value problems.

The BEM is generally used to obtain all unknown quantities with uniform accuracy in a whole analytical domain. However, in reality, there are not a few cases where the unknown quantities are required only in a specific domain, that is, the focused domain for practical purpose. For example, we can point to corrosion or anticorrosion analysis for the evaluation of anti-corrosion effect on specific parts of structures such as boats and ships, and elastic analysis for the evaluation of

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strength on stress concentration zones. In such cases, it is advantageous to compute more efficiently unknown quantities only in focused domain with the same accuracy as the conventional BEM[1].

Based on the above-mentioned background, we developed a new efficient boundary element method for a focused domain.

In this method, a whole boundary is divided into near boundaries which are near to a focused domain and far boundaries which are sufficiently far from it (Fig. 1). Then, we set up boundary integral equations (BIEs) whose collocation points are on all the elements of the near boundaries and express these integrals which contain unknown quantities on far boundaries as low order multipole moments by using a multipole expansion of the fundamental solution. Moreover, we also set up boundary integral equations whose weighting functions are harmonic functions by the number of those multipole moments. The boundary integrals which contain unknown quantities on the far boundaries are expressed approximately as the multipole moments by using a generalized inverse matrix. Thus the number of unknowns and boundary integral equations set up are decreased drastically by this method, and enables us to compute them efficiently (Fig. 2).

This paper presents the efficient boundary element method for the focused domain, that is the "Focused Domain Efficient Boundary Element Method (FD–EBEM)". The capability of this method extended to 2D potential problems is verified with some numerical experiment.

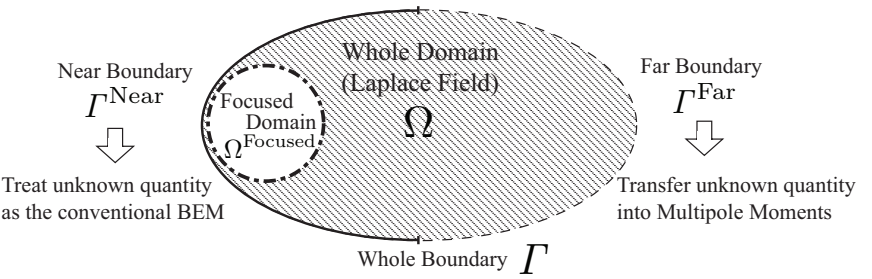

Figure 1: Division of boundary into near and far boundaries.

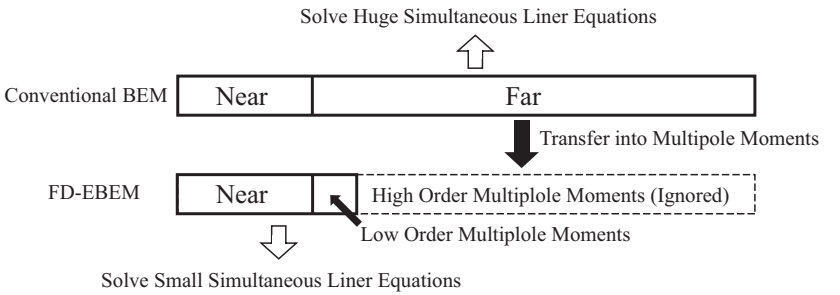

Figure 2: Reduction of the number of unknown quantities.

2 BIE formulation and the conventional BEM approach

The BIE formulation and its discretization using the conventional BEM for 2D potential problems are summarized in this section. They are fundamental to FD–EBEM.

2.1 Setup of boundary value problem

For simplicity, consider the following Laplace equation governing a potential problem in a 2D domain Ω (Fig. 3):

$$\nabla^2 u(z) = 0 \qquad (\forall z \in \Omega) \tag{1}$$

under the boundary condition $u(z_0) = u' \ (z_0 \in \Gamma)$. $u(z)$ is potential field in domain Ω, Γ is the boundary of the Ω, and *the symbol prime*($'$) quantities indicate given values on the boundary.

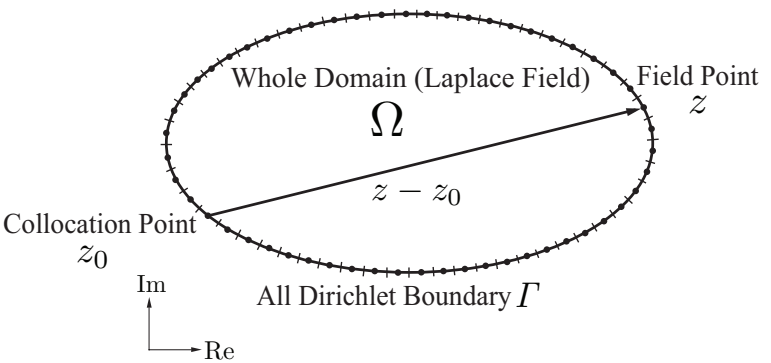

Figure 3: Domain Ω and boundary Γ (discretized).

2.2 BIE of potential problems

The BIE for this boundary value problem can be expressed as the following formula:

$$c(z)u(z) = \mathrm{Re}\left[\int_\Gamma u^*(z_0, z)q(z)|\mathrm{d}z| - \int_\Gamma q^*(z_0, z)u(z)|\mathrm{d}z|\right], \tag{2}$$

where q the flux $q(z) = \partial u(z)/\partial n(z)$, $n(z)$ outward normal, z_0 the collocation point, z field point and $c(z_0)$ coefficients that are equal to $1/2$ if $z_0 \in \Gamma$ and Γ is smooth around z_0, or 1 if $z_0 \in \Omega$; $u^*(z_0, z)$ and $q^*(z_0, z)$ are the fundamental solution given by:

$$u^*(z_0, z) = -\frac{1}{2\pi}\ln(z - z_0), \qquad q^*(z, z_0) = -\frac{n(z)}{2\pi(z - z_0)}. \tag{3}$$

2.3 Discretization

In this subsection, discretization with constant boundary elements is explained for simplicity. The boundary Γ is divided into N line elements Γ_k $(k = 1, 2, \cdots, N)$ and one node z_0^j is placed on each element. We obtain the following discretized equation of BIE (2) in matrix form:

$$\boldsymbol{G}_\Gamma \boldsymbol{q}_\Gamma = \boldsymbol{H}_\Gamma \boldsymbol{u}'_\Gamma, \tag{4}$$

where

$$\boldsymbol{H}_\Gamma = \begin{bmatrix} h_{11} & \cdots & h_{1N} \\ \vdots & \ddots & \vdots \\ h_{N1} & \cdots & h_{NN} \end{bmatrix}, \qquad \boldsymbol{G}_\Gamma = \begin{bmatrix} g_{11} & \cdots & g_{1N} \\ \vdots & \ddots & \vdots \\ g_{N1} & \cdots & g_{NN} \end{bmatrix},$$

$$h_{jk} = \frac{\delta_{jk}}{2} + \int_{\Gamma_k} q^*(z_0^j, z)|\mathrm{d}z|, \qquad g_{jk} = \int_{\Gamma_k} u^*(z_0^j, z)|\mathrm{d}z|, \tag{5}$$

$$\boldsymbol{u}'_\Gamma = \begin{Bmatrix} u'_1 \\ \vdots \\ u'_N \end{Bmatrix}, \qquad \boldsymbol{q}_\Gamma = \begin{Bmatrix} q_1 \\ \vdots \\ q_N \end{Bmatrix}, \tag{6}$$

u_k and q_k $(k = 1, 2, \ldots, N)$ are nodal values of u and q on the element Γ_k, respectively. Obviously, the construction of matrix $\boldsymbol{H}_\Gamma$ and $\boldsymbol{G}_\Gamma$ requires $O(N^2)$ operations using the two expressions in Eq. (5) and the size of the required memory for storing them is also $O(N^2)$ since they are in general non-symmetric and dense matrices. The solution of system in Eq. (4) using direct solvers such as Gauss elimination is even worse, requiring $O(N^3)$ operations because of this general matrix. That is why the conventional BEM is not so efficient for large-scale problems, despite its robustness in the meshing stage as compared with other domain based methods [2].

3 FD–EBEM formulation for 2D potential problems

As stated in the previous section, the conventional BEM is not so efficient for large-scale problems because of requiring vast operations and memory. FMBEM was developed for getting over this weak point of the conventional BEM. But, it is hard to say that FMBEM is always efficient, especially, in the case where the unknown quantities are required only in a specific domain, because, FMBEM is the method for obtaining all unknown quantities with uniform accuracy in a whole analytical domain. In such cases, FD–EBEM enables us to compute efficiently the unknown quantities only in a specific domain. In this section, we explain the formulation of FD–EBEM for 2D potential problems.

3.1 Setup of boundary value problem

For simplicity, consider the same Laplace equation (1) governing potential problem in a 2D domain Ω (Fig. 4) under the boundary condition $u(z_0) = u' \ (z_0 \in \Gamma)$ as given in the previous section. We also assume that a specific domain is focused domain $\Omega^{\mathrm{Focused}}$ in the whole domain Ω and potential $u(z_0)$, $z_0 \in \Omega^{\mathrm{Focused}}$ is a requisite unknown quantity. Far boundary Γ^{Far} is sufficiently far from the focused domain $\Omega^{\mathrm{Focused}}$. Far boundary Γ^{far} and the center of multipolar moments z_c are determined so that $\forall z \in \Gamma^{\mathrm{Far}} (\subseteq \Gamma)$ satisfy the following formula for $\forall z_0 \in \Omega^{\mathrm{Focused}}$:

$$|z - z_c| \ll |z_0 - z_c|, \tag{7}$$

Additionally, the boundary near the focused domain $\Omega^{\mathrm{Focused}}$ is denoted near boundary $\Gamma^{\mathrm{Near}} \left(= \Gamma \cap \overline{\Gamma^{\mathrm{Far}}}\right)$, and $\exists z_0 \in \Gamma^{\mathrm{Near}}$ satisfies Eq. (7) for $\forall z \in \Gamma^{\mathrm{Far}} (\subseteq \Gamma)$.

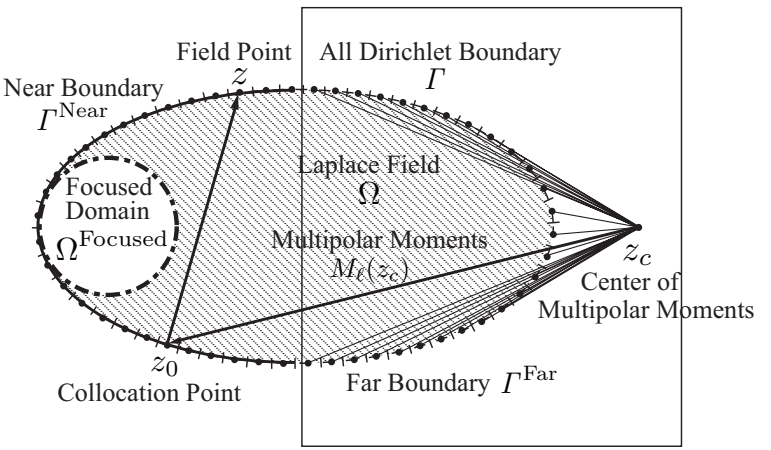

Figure 4: Focused domain and classification of boundary.

3.2 Division of boundary integral into near and far boundary

Now, we deal with the boundary value problem where the potential u on the whole boundary Γ is given. Thus, the boundary integral contains unknown quantity in the first right-hand term of Eq. (2), where $z_0 \in \Gamma$. In this method, the first right-hand

term of Eq. (2) is devided into the near boundary Γ^{Near} and the far boundary Γ^{Far} part as follows:

$$c(z_0)u(z_0) = \mathrm{Re}\left[\int_{\Gamma^{\mathrm{Near}}} u^*(z_0,z)\cdot q(z)|\mathrm{d}z| + \int_{\Gamma^{\mathrm{Far}}} u^*(z_0,z)\cdot q(z)|\mathrm{d}z| - \int_{\Gamma} q^*(z_0,z)\cdot u(z)|\mathrm{d}z|\right]. \quad (8)$$

3.3 Multipole expansion of boundary integral on far boundary

The displacement component of the fundamental solution in complex notation $u^*(z_0,z)$ can be transformed as the following formula:

$$u^*(z_0,z) = -\frac{1}{2\pi}\left\{\log|z_0-z_c| + \log\left(1-\frac{z-z_c}{z_0-z_c}\right)\right\} \quad (9)$$

In the case where z_0, z and z_c satisfy Eq. (7), we can apply the following Taylor series expansion:

$$\log(1-\xi) \approx \sum_{\ell=1}^{L}\frac{\xi^\ell}{\ell} \qquad |\xi|<1, \quad (10)$$

and we obtain

$$u^*(z_0,z) \approx -\frac{1}{2\pi}\left\{\log|z_0-z_c| - \sum_{\ell=1}^{L}\frac{1}{(z_0-z_c)^\ell}\cdot\frac{(z-z_c)^\ell}{\ell}\right\} \quad (11)$$

Therefore, the second right-hand term of Eq. (8) is described by Eq. (11) as follows.

$$\int_{\Gamma^{\mathrm{Far}}} u^*(z_0,z)\cdot q(z)|\mathrm{d}z| \approx \frac{1}{2\pi}\sum_{\ell=0}^{L} O_\ell(z_0-z_c)M_\ell(z_c) \quad (12)$$

where

$$M_\ell(z_c) = \int_{\Gamma^{\mathrm{Far}}}\frac{(z-z_c)^\ell}{\ell}\cdot q(z)|\mathrm{d}z|, \quad O_\ell(z) = \begin{cases}\log|z| & (\ell=0)\\ z^{-\ell} & (\ell\geq 1)\end{cases}. \quad (13)$$

In the case where z_0 and z satisfy the following formula : Eq. (14), the finite series of Eq. (12) is known to be sufficiently convergent by approximately 15 terms.

$$|z_0-z_c| \geq 3|z-z_c| \quad (14)$$

It means that the finite series of Eq. (12) is sufficiently convergent by a few terms in the case where the collocation point z_0 is placed in the focused domain $\Omega^{\mathrm{Focused}}$ and on the near boundary Γ^{Near} but it is not convergent in the case the collocation point is placed on the far boundary Γ^{Far}.

3.4 BIE whose Collocation Point is placed on near boundary

When the collocation point z_0 is placed on the near boundary Γ^{Near} and the field point z is placed on the far boundary Γ^{Far}, z and z_0 can satisfy Eq. (7). Therefore Eq. (8) can be transformed into the following formula:

$$c(z_0)u(z_0) \approx \mathrm{Re}\left[\int_{\Gamma^{\mathrm{Near}}} u^*(z_0,z)\cdot q(z)|\mathrm{d}z| - \int_{\Gamma} q^*(z_0,z)\cdot u(z)|\mathrm{d}z| + \frac{1}{2\pi}\sum_{\ell=0}^{L} O_\ell(z_0-z_c)M_\ell(z_c)\right] \quad (15)$$

Consider the case where we discretize boundary Γ by constant boundary elements and set BIEs whose collocation points are placed on elements of the near boundary Γ^{Near} only. In this case, we can't solve the simultaneous linear equations due to insufficiency of equations by the increment of multipolar moments, or L. Additional equations for this insufficiency will be demonstrated in the next subsection.

3.5 BIE whose weighting functions are harmonic functions

We set up boundary integral equations whose weighting functions are harmonic functions, that is $O_\ell(z-z_c)$ by the number of those multipole moments, or L. In this case, BIEs is the following formula:

$$\int_{\Gamma} O_\ell(z-z_c)\cdot q(z)|\mathrm{d}z| - \int_{\Gamma^{\mathrm{Near}}} P_\ell(z-z_c)\cdot u(z)|\mathrm{d}z| - \int_{\Gamma^{\mathrm{Far}}} P_\ell(z-z_c)\cdot u(z)|\mathrm{d}z| = 0 \qquad (\ell = 0,1,\cdots,L) \quad (16)$$

where

$$P_\ell(z) = \frac{\partial O_\ell(z)}{\partial n(z)} = -\frac{\ell\cdot n(z)}{z^{\ell+1}} \quad (17)$$

The process for expressing the integral in Eq. (16) which contains unknown quantities on far boundaries as these multipolar moments approximately is demonstrated below. We discretize the boundary by constant boundary elements and obtain the discretized BIE in matrix form:

$$\begin{bmatrix}\boldsymbol{K}_{\Gamma^{\mathrm{Near}}} & \boldsymbol{K}_{\Gamma^{\mathrm{Far}}}\end{bmatrix}\begin{Bmatrix}\boldsymbol{q}_{\Gamma^{\mathrm{Near}}} \\ \boldsymbol{q}_{\Gamma^{\mathrm{Far}}}\end{Bmatrix} = \boldsymbol{J}_\Gamma \boldsymbol{u}'_\Gamma, \quad (18)$$

where

$$\boldsymbol{K}_\Gamma = \begin{bmatrix}\boldsymbol{k}_{00} & \cdots & \boldsymbol{k}_{0N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{k}_{L0} & \cdots & \boldsymbol{k}_{LN}\end{bmatrix} \qquad \boldsymbol{J}_\Gamma = \begin{bmatrix}\boldsymbol{j}_{01} & \cdots & \boldsymbol{j}_{0N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{j}_{L0} & \cdots & \boldsymbol{j}_{LN}\end{bmatrix}, \quad (19)$$

$$\boldsymbol{k}_{\ell k} = \begin{Bmatrix} \mathrm{Re}\left[\int_{\Gamma_k} P_\ell(z-z_c)|\mathrm{d}z|\right] \\ \mathrm{Im}\left[\int_{\Gamma_k} P_\ell(z-z_c)|\mathrm{d}z|\right] \end{Bmatrix}, \quad \boldsymbol{j}_{\ell k} = \begin{Bmatrix} \mathrm{Re}\left[\int_{\Gamma_k} O_\ell(z-z_c)|\mathrm{d}z|\right] \\ \mathrm{Im}\left[\int_{\Gamma_k} O_\ell(z-z_c)|\mathrm{d}z|\right] \end{Bmatrix}. \tag{20}$$

$\mathrm{Re}[z]$ and $\mathrm{Im}[z]$ represents a real and an imaginary part of z. We discretize Eq. (13) by constant boundary elements in the same way.

$$\boldsymbol{M} = \boldsymbol{I}_{\Gamma^{\mathrm{Far}}} \boldsymbol{q}_{\Gamma^{\mathrm{Far}}}, \tag{21}$$

where

$$\boldsymbol{I}_\Gamma = \begin{bmatrix} \boldsymbol{i}_{01} & \cdots & \boldsymbol{i}_{0N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{i}_{L1} & \cdots & \boldsymbol{i}_{LN} \end{bmatrix}, \qquad \boldsymbol{i}_{\ell k} = \begin{Bmatrix} \mathrm{Re}\left[\int_{\Gamma_k} \frac{(z-z_c)^\ell}{\ell}|\mathrm{d}z|\right] \\ \mathrm{Im}\left[\int_{\Gamma_k} \frac{(z-z_c)^\ell}{\ell}|\mathrm{d}z|\right] \end{Bmatrix},$$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_0 \\ \vdots \\ \boldsymbol{M}_L \end{bmatrix}, \qquad \boldsymbol{M}_\ell = \begin{bmatrix} \mathrm{Re}[M_\ell(z_c)] \\ \mathrm{Im}[M_\ell(z_c)] \end{bmatrix}.$$

Let us assume q_j can be described as the following formula by using $\boldsymbol{I}^+_{\Gamma^{\mathrm{Far}}}$, Moore-Penrose type generalized inverse matrix of $\boldsymbol{I}_{\Gamma^{\mathrm{Far}}}$ in Eq. (21) :

$$\boldsymbol{q}_{\Gamma^{\mathrm{Far}}} \approx \boldsymbol{I}^+_{\Gamma^{\mathrm{Far}}} \boldsymbol{M} = \boldsymbol{I}^T_{\Gamma^{\mathrm{Far}}} \left(\boldsymbol{I}_{\Gamma^{\mathrm{Far}}} \cdot \boldsymbol{I}^T_{\Gamma^{\mathrm{Far}}}\right)^{-1} \boldsymbol{M}, \tag{22}$$

where $\boldsymbol{A}^T$ is the transposed matrix of A. Therefore the third left-handed term of Eq. (16) is:

$$\boldsymbol{K}_{\Gamma^{\mathrm{Far}}} \boldsymbol{q}_{\Gamma^{\mathrm{Far}}} \approx \boldsymbol{K}_{\Gamma^{\mathrm{Far}}} \boldsymbol{I}^T_{\Gamma^{\mathrm{Far}}} \left(\boldsymbol{I}_{\Gamma^{\mathrm{Far}}} \cdot \boldsymbol{I}^T_{\Gamma^{\mathrm{Far}}}\right)^{-1} \boldsymbol{M}. \tag{23}$$

3.6 Simultaneous linear equations

As mentioned above, this method drastically reduces the number of unknown quantities and the BIEs set up. We can set up the simultaneous linear equations by discretizing Eq. (15) and Eq. (16), and substituting Eq. (23) as the following:

$$\begin{bmatrix} \boldsymbol{G}_{\Gamma^{\mathrm{Near}}} & \boldsymbol{O} \\ \boldsymbol{K}_{\Gamma^{\mathrm{Near}}} & \boldsymbol{K}_{\Gamma^{\mathrm{Far}}} \boldsymbol{I}^T_{\Gamma^{\mathrm{Far}}} \left(\boldsymbol{I}_{\Gamma^{\mathrm{Far}}} \cdot \boldsymbol{I}^T_{\Gamma^{\mathrm{Far}}}\right)^{-1} \end{bmatrix} \begin{Bmatrix} \boldsymbol{q}_{\Gamma^{\mathrm{Near}}} \\ \boldsymbol{M} \end{Bmatrix} \approx \begin{bmatrix} \boldsymbol{H}_\Gamma \\ \boldsymbol{J}_\Gamma \end{bmatrix} \boldsymbol{u}'_\Gamma, \tag{24}$$

where

$$\boldsymbol{O} = \begin{bmatrix} \boldsymbol{O}_{10} & \cdots & \boldsymbol{O}_{1L} \\ \vdots & \ddots & \vdots \\ \boldsymbol{O}_{10} & \cdots & \boldsymbol{O}_{1L} \end{bmatrix}, \qquad \boldsymbol{O}_{j\ell} = \begin{Bmatrix} \mathrm{Re}[O_\ell(z_0^j - z_c)] \\ -\mathrm{Im}[O_\ell(z_0^j - z_c)] \end{Bmatrix}. \tag{25}$$

We can compute unknown quantities q on the near boundary Γ^{Near} and multipolar moments $M_l(z_c)$ by solving Eq. (24). In the case where a collocation point z_0 is placed in the focused domain $\Omega^{\mathrm{Focused}}$, $c(z_0) = 1$ in Eq. (15), hence unknown quantity u in $\Omega^{\mathrm{Focused}}$ can be computed with the same level of accuracy as the conventional BEM.

4 Numerical results

In order to examine the accuracy and efficiency of FD–EBEM, we carry out a numerical experiment with this method and the conventional BEM.

4.1 Setup boundary value problem

We consider a tube-shaped Laplace field with a square hole as shown in Fig. 5. Boundary conditions on the inner square and outer circle are Dirichlet conditions as shown in Fig. 6. The requisite unknown quantity is assumed to be potential u in focused domain $\Omega^{\mathrm{Focused}}$ shown in Fig. 5. The inner square and outer circle are far boundary Γ^{Far} and near boundary Γ^{Near}, respectively. The center of multipole moments z_c is placed at the center of the outer circle so that $\forall z_0 \in \Omega^{\mathrm{Focused}}$ satisfy Eq. (14) for $\forall z \in \Gamma^{\mathrm{Far}}$. The number of multipole moments is set to be 15.

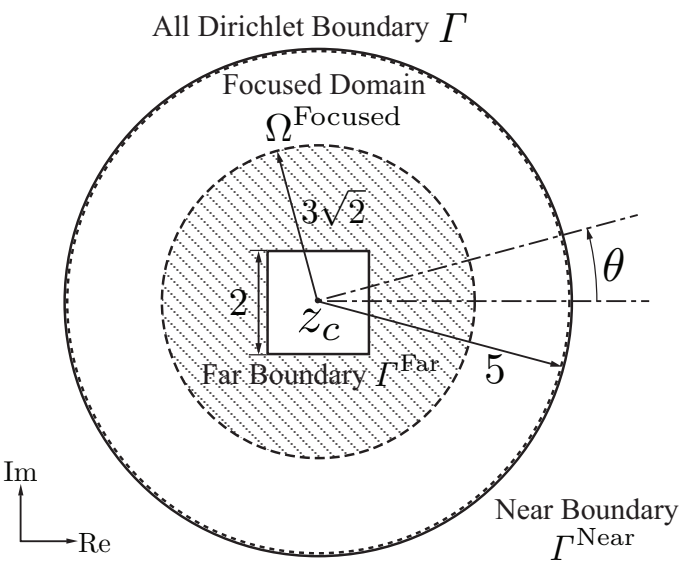

Figure 5: Model for the numerical experiment.

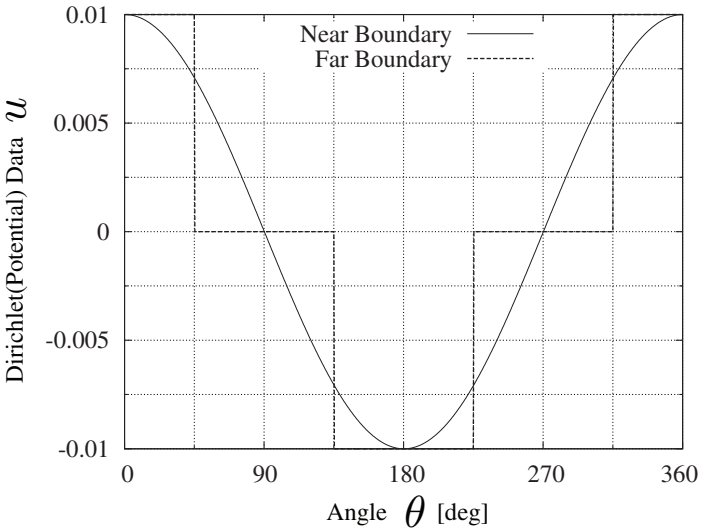

Figure 6: Boundary condition.

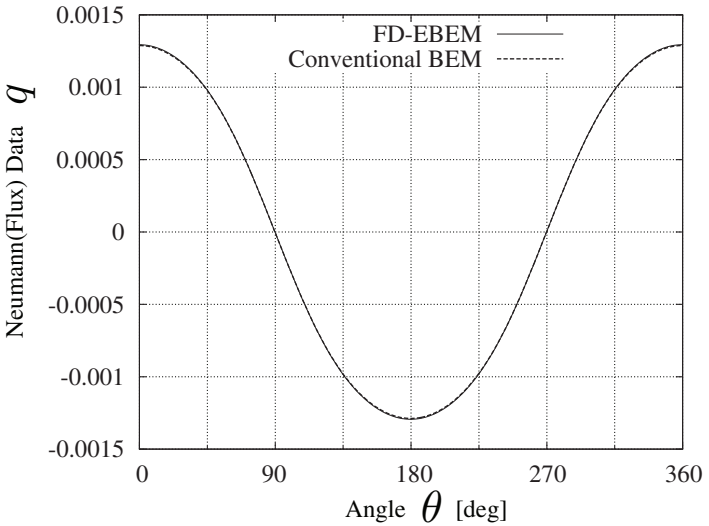

Figure 7: Neumann data (flux) q on near boundary Γ^{Near}.

4.2 Verification of accuracy

For the problem indicated in the previous subsection, we divided Γ^{Near} and Γ^{Far} into 1000 and 5000 constant line elements, respectively, and computed flux q on Γ^{Near} and potential u by this method and the conventional BEM.

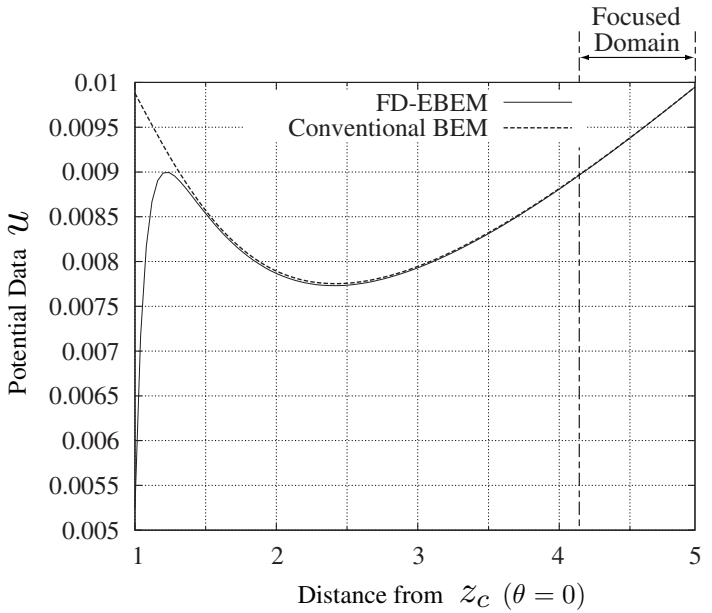

Figure 8: Potential data u in domain $\Omega(\theta = 0)$.

Fig. 7 compares this method with the conventional BEM in the computational results of flux q on the near boundary Γ^{Near}. Good agreement in computational results is observed.

Fig. 8 compares this method with the conventional BEM in the computational results of potential u on the line of parameter $\theta = 0$ within the whole domain Ω. Though this method differs from the conventional BEM in the computational results external to the focused domain $\Omega^{\mathrm{Focused}}$, they are in good agreement in the focused domain $\Omega^{\mathrm{Focused}}$.

It seems that this method is capable of computing flux q on the near boundary Γ^{Near} and potential u in the focused domain $\Omega^{\mathrm{Foacused}}$ with the same sufficient accuracy as the conventional BEM.

4.3 Verification of efficiency

For the problem indicated in the previous subsection, we divided Γ^{Near} and Γ^{Far} into constant line elements and computed unknown quantities by this method and the conventional BEM.

The number of elements on Γ^{Near} fixed at 1000 and the number of elements on Γ^{Far} is varied from 1000 to 10000. All the computations in this subsection were done on Intel(R) Core(TM) i7–965 Extreme Edition(3.2 GHz).

Fig. 9 shows the relationship between computational time and the total number of elements N. It demonstrates that the computational cost of this method is about

$O(N)$, while the cost of the conventional BEM is about $O\left(N^3\right)$. This method is more efficient as the number of elements on far boundaries Γ^{Far} is larger.

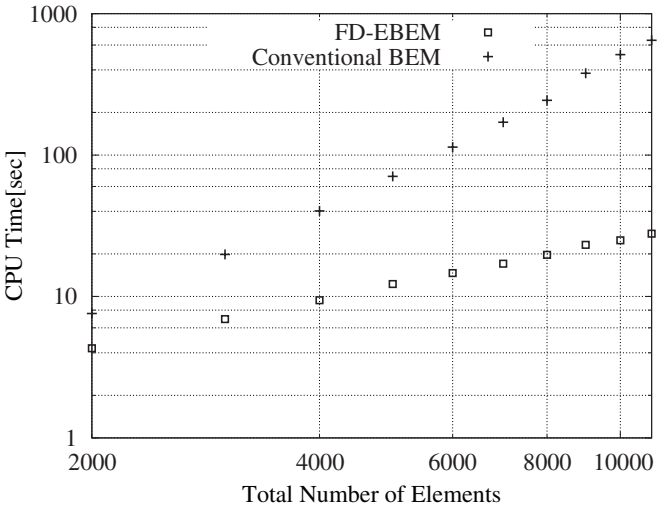

Figure 9: Comparison of CPU time between FD–EBEM and the conventional BEM.

5 Conclusions

In this paper, we presents the efficient boundary element method for the focused domain, that is the "Focused Domain Efficient Boundary Element Method (FD–EBEM)". We validated FD–EBEM in accuracy and efficiency by carrying out a numerical experiment.

There are many practical applications in which unknown quantities only in a specific domain are required. This method can extended not only to potential problems but also any other boundary value problems such as elastostatic analysis.

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