Efficient elasto-plastic analysis via an adaptive finite element-boundary element coupling method

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Abstract

This paper presents an adaptive FEM-BEM coupling method for elasto-plastic analysis. The proposed method is valid for both two- and three-dimensional applications. The method takes care of the evolution of the elastic and plastic regions and avoids some limitations of the standard FEM-BEM coupling approaches. It estimates the FEM and BEM sub-domains and automatically generates/adapts the FEM and BEM meshes/sub-domains, according to the state of computation. The method eliminates the cumbersome trial and error process in the identification of the FEM and BEM sub-domains in the standard FEM-BEM coupling approaches. An example application confirms the effectiveness of the proposed method.
Keywords: FEM, BEM, elasto-plasticity, adaptive coupling.

1 Introduction

There exist many application contexts where coupling of the finite element method (FEM) and the boundary element method (BEM) is, in principle, very attractive. Examples include, but not limited to, elasto-plastic applications with limited spread of plastic deformations. The FEM is utilized where the plastic material behaviour is expected to develop. The remaining bounded/unbounded linear elastic regions are best approximated by the BEM.

A crucial aspect of the existing (standard) FEM-BEM coupling approaches is that they require the user/analyst to predefine and manually localize the FEM and

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BEM sub-domains (prior to analysis). In an elasto-plastic analysis, it is difficult or impossible to predict regions where plasticity occurs. If the FEM sub-domain is not predefined in order to enclose the evolutional plastic regions, significant errors will be introduced to the conducted FEM-BEM coupling analysis. On the other hand, if the FEM sub-domain is notably over-estimated, the excessive number of degrees of freedom (in order to model the linear elastic part) will significantly increase the computational demand.

In pure BEM elasto-plastic analysis, the conventional numerical implementation requires the domain to be discretized into cells (predefined by the user/analyst). In this case the BEM loses its main advantage. Astrinidis et al. [1] presented adaptive discretization schemes that are based on a stress smoothing error criterion in the case of 2D elastic analysis, and on a total strain smoothing error criterion in the case of 2D elasto-plasticity. Maischak and Stephan [2] showed convergence for the boundary element approximation, obtained by the *hp*-version, for elastic contact problems, and derived a posteriori error estimates together with error indicators for adaptive *hp*-algorithms. Rebeiro et al. [3] developed a pure BEM approach to automatically generate the internal cells. Their approach considers cases when plasticity starts from the boundary. The stresses at the boundary nodes are computed and the internal cells are generated surrounding regions where plasticity is detected. The process is iterated until the detected plastic regions are fully discretized.

Brink et al. [4] investigated the coupling of mixed finite elements and Galerkin boundary elements in linear elasticity, taking into account adaptive mesh refinement based on a posteriori error estimators. Carstensen et al. [5] presented an *h*-adaptive FEM-BEM coupling approach (mesh refinement of the boundary elements and the finite elements) for the solution of visco-plastic and elasto-plastic interface problems. Mund and Stephan [6] derived a posteriori error estimates for nonlinear coupled FEM-BEM equations by using hierarchical basis techniques. They presented an approach for adaptive error control, which allows independent refinements of the finite and boundary elements.

Doherty and Deeks [7] developed an adaptive approach for analyzing 2D elasto-plastic unbounded media by coupling the FEM with the scaled boundary finite element method. The analysis begins with an "initial" finite element mesh that tightly encloses the load-medium interface, whereas the remainder of the problem is modelled using the semi-analytical scaled boundary finite element method. Load increments are applied, and if plasticity is detected in the outer band of finite elements, an additional band is added around the perimeter of the existing mesh. The scaled boundary finite element sub-domain is stepped out accordingly. However, this approach requires, in general, a preliminary knowledge of the parts of the domain that are likely to yield. Moreover, it requires additional iterations when plasticity is detected in the outer band of finite elements in order to accurately determine the computational sub-domains. This may end up in a very time-consuming process.

Elleithy and Langer [8] and Elleithy [9] presented an adaptive FEM-BEM coupling method for 2D elasto-plastic analysis. The method proposes the use of simple, fast post-calculations, based on energetic methods, and follows a simple hypothetical elastic boundary element computation in order to give fast and helpful estimation of the FEM and BEM sub-domains.

In this paper we present an alternative to the adaptive coupling method presented in [8,9]. The proposed method is valid for both two- and three-dimensional applications. The adaptive coupling method improves the estimation of regions where plastic material behaviour is going to develop. An outline of the paper is as follows. Section 2 briefly summarizes the conventional FEM-BEM coupling equations adopted in this investigation and the adaptive method of [8,9]. In the sequence, Section 3 presents the proposed adaptive FEM-BEM coupling method for elasto-plastic analysis. In Section 4, we present a numerical example that highlights the effectiveness of the adaptive coupling method.

2 Preliminaries

In this section, the conventional (direct) FEM-BEM coupling equations adopted in this investigation (without loss of generality) and the adaptive method of [8,9] are briefly described.

Elasto-plastic problems with limited spread of plastic strains lend themselves to a coupled approach. The FEM is utilized in regions where plastic material behaviour is expected to develop, whereas the complementary bounded/unbounded linear elastic region is approximated using the symmetric Galerkin BEM. The domain of the original problem Ω (with known boundary conditions specified on the entire boundary $\Gamma = \Gamma_N \cup \Gamma_D$) is decomposed into two sub-domains, namely ${}_F\Omega$ and ${}_B\Omega$, with the FEM-BEM coupling interface Γ_C. The coupled FEM-BEM equations, in incremental form, are solved at each iteration using a tangent FEM stiffness matrix K_T. The coupled FEM-BEM equations for a typical iteration may then be reformulated in the symmetric form

$$\begin{bmatrix} {}_F K_{T\,FF} & {}_F K_{T\,FC} & \\ {}_F K_{T\,CF} & {}_F K_{T\,CC} + {}_B K_{CC} & {}_B K_{CB} \\ & {}_B K_{BC} & {}_B K_{BB} \end{bmatrix} \begin{bmatrix} \Delta_F \underline{u}_F \\ \Delta \underline{u}_C \\ \Delta_B \underline{u}_B \end{bmatrix} = - \begin{bmatrix} \Delta_F \underline{\psi}_F \\ \Delta \underline{\psi}_C \\ \Delta_B \underline{\psi}_B \end{bmatrix}, \qquad (1)$$

where $\underline{u}$ and $\underline{\psi}$ are the displacements and the residual force vectors, respectively. The subscripts $(\,)_F$ and $(\,)_B$ indicate the displacement vectors (force vectors) not associated with the FEM and BEM sub-domains interface, respectively. The subscript $(\,)_C$ indicates those associated with the interface Γ_C.

Ref. [8,9] presented an adaptive FEM-BEM coupling method for solving problems in elasto-plasticity. The adaptive coupling method follows a linear hypothetical elastic computation at levels of loading specified by the user. The hypothetical elastic state of stresses is checked against yielding with a pseudo value of the material yield strength. An estimate of the regions sensible for FEM discretization is then derived. The FEM and BEM meshes are automatically generated. A coupled FEM-BEM stress analysis involving elasto-plastic deformations is then conducted. In order to determine the pseudo value of the material yield strength, an energy balance between the hypothetical elastic and elasto-plastic calculations was assumed in [8,9]

$$U_{\text{hyp elastic}} := \left(\int_\Omega \sigma_{ij}\varepsilon_{ij} dV\right)_{\text{hyp elastic}} \approx \left(\int_\Omega \sigma_{ij}\varepsilon_{ij} dV\right)_{\text{elasto-plastic}}, \tag{2}$$

where $U_{\text{hyp elastic}}$ elastic is the total hypothetical elastic strain energy. The pseudo value of the material yield strength $\sigma_{y \text{ pseudo}}$ is evaluated as follows

$$\frac{U_{\text{dist}}}{U_{\text{hyp elastic}}} \approx \frac{c(\sigma_y - \sigma_{y \text{ pseudo}})}{\sigma_y}, \tag{3}$$

where U_{dist} is the total strain energy that is vulnerable for redistribution due to plastic deformations and c is a constant that depends on the geometry of the stress-strain curve.

3 Adaptive FEM-BEM coupling method

In this section we present an adaptive FEM-BEM coupling method for elasto-plastic analysis. The method is valid for both 2D and 3D applications. It estimates the FEM and BEM sub-domains and automatically generates/adapts the FEM and BEM meshes/sub-domains, according to the state of computation. The adaptive coupling method improves the estimation of regions where plastic material behaviour is going to develop (regions where the FEM is employed). In the presence of plastic deformations in the FEM region, the solution there is obtained via an iterative scheme. Naturally, an improvement to the estimated FEM and BEM sub-domains will result in additional savings of required system resources and/or a higher potential advantage of eliminating the cumbersome trial and error process in the identification of the FEM and BEM sub-domains. Materials of von-Mises type are considered in this investigation.

The basic steps of implementation of the proposed adaptive FEM-BEM coupling method may be summarized as follows:

1. Levels of loading $(LL_1, LL_2, ..., LL_i, ...LL_m)$ are specified by the user/analyst in order to get an estimate of the FEM and BEM sub-domains (LL_m is the maximum level of loading for the problem at hand).

   If the user/analyst prefers to use a constant interface throughout the FEM-BEM coupling analysis, the maximum load level LL_m is specified (estimated FEM and BEM sub-domains will be utilized for all load increments).
2. For $k = 1,2,...,m$
   - 2.1. A hypothetical elastic stress state is determined with the load level LL_k via BEM elastic analysis with initial BEM discretization or FEM elastic analysis utilizing a FEM coarse mesh.
   - 2.2. Regions that violate the yield condition (utilizing the hypothetical elastic stresses of 2.1) are detected. A subsequent elastic analysis is conducted with a "modified" level of loading $LL_{k,\text{mod}}$, "effective"

material properties for the detected regions (effective Young's modulus $E_{k,eff}$ and Poisson ratio $\nu_{k,eff}$) and material properties E and ν for the remainder of the problem.

2.3. The hypothetical elastic state of stresses of step 2.2 is checked against the yield condition. FEM discretization is automatically generated for the regions that violate the yield condition. It may be useful to add a few bands of finite elements around the perimeter of the discretized FEM sub-domain. Consequently, the BEM discretization is generated so as to represent best the remaining bounded/unbounded linear elastic regions (fig. 1).

2.4. Coupled FEM-BEM stress analysis involving elasto-plastic deformations is conducted for the current load increment.

2.5. A repetition of step 2.4 is required for the next load increment if the current state of computation in addition to the load increment is less than or equal to LL_k, else go to step 2.1.

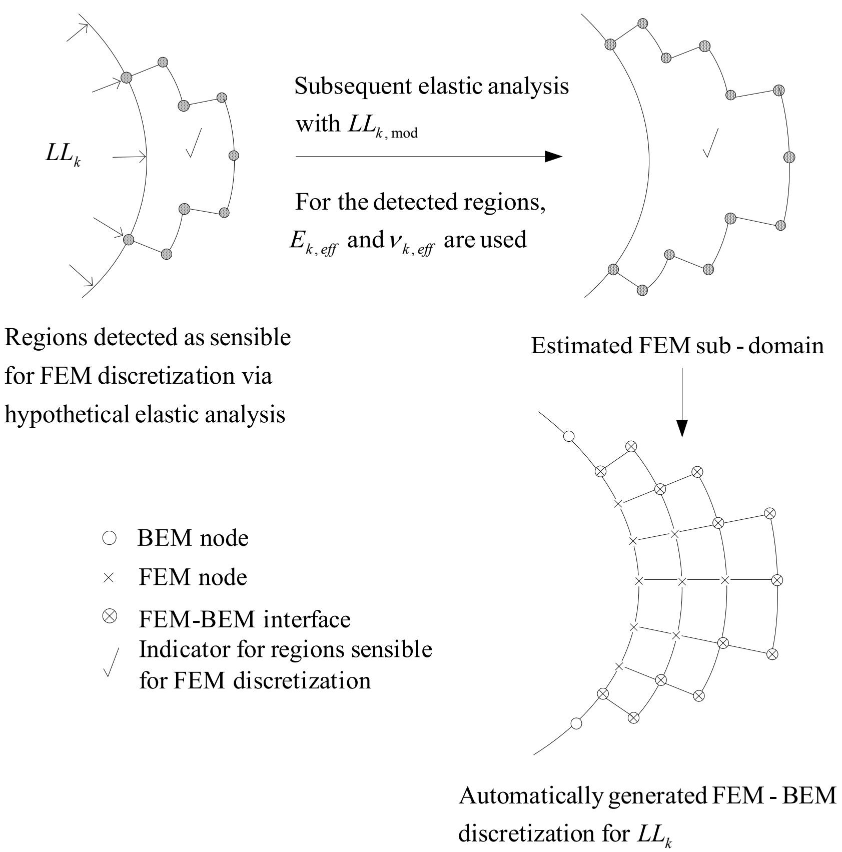

Figure 1: Estimated FEM and BEM sub-domains.

It should be emphasized that steps 1-2.3 are carried out for the sole purpose of estimating and adapting the FEM and BEM sub-domains according to the state of computation. Modified load levels $LL_{k,\,\text{mod}}$ and effective material properties ($E_{k,\,eff}$ and $\nu_{k,\,eff}$) are not involved in carrying out step 2.4.

In the remainder of this section we will elaborate more on the determination of the modified level of loading $LL_{k,\,\text{mod}}$ and effective material properties ($E_{k,\,eff}$ and $\nu_{ki,\,eff}$) at a typical level of loading LL_k.

The simplicity of linear elastic analysis has motivated some researchers to attempt solving elasto-plastic problems by adapting a modified form of available elastic solutions (see, e.g. references [10,11]). Linear elastic analysis in an iterative manner with a complete spatial distribution of updated material properties is conducted at each iteration in order to approximately simulate elasto-plastic behaviour. The schemes for updating the material properties include projection, arc length, and energy methods (fig. 2). Material points with the same stress level are represented by single points (e.g. a, b, e and f) on the uniaxial stress-strain curve (fig. 3).

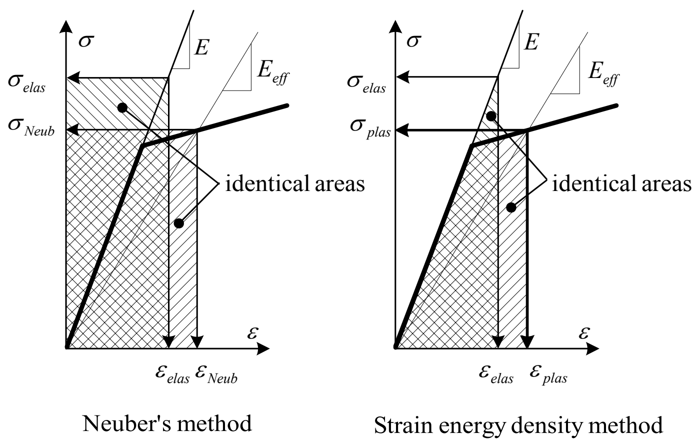

Figure 2: Energetic methods (Neuber and strain energy density methods).

Let us consider materials of von-Mises type obeying a bilinear strain hardening rule. Neuber’s method (fig. 2) assumes an energy balance between the strain energy density corresponding to the elasto-plastic stress-strain state and the hypothetical elastic strain energy density (same geometry submitted to the same loading [10,11]). For uni-dimensional states of stress, it is assumed that the product of stress and strain in elasticity is locally identical to the same product calculated by means of an elasto-plastic analysis. For tri-dimensional states of stress, the fundamental hypothesis may be written as [10,11]

$$(\sigma_{ij}\varepsilon_{ij})_{\text{elasto-plastic}} \cong (\sigma_{ij}\varepsilon_{ij})_{\text{hyp elastic}}. \qquad (4)$$

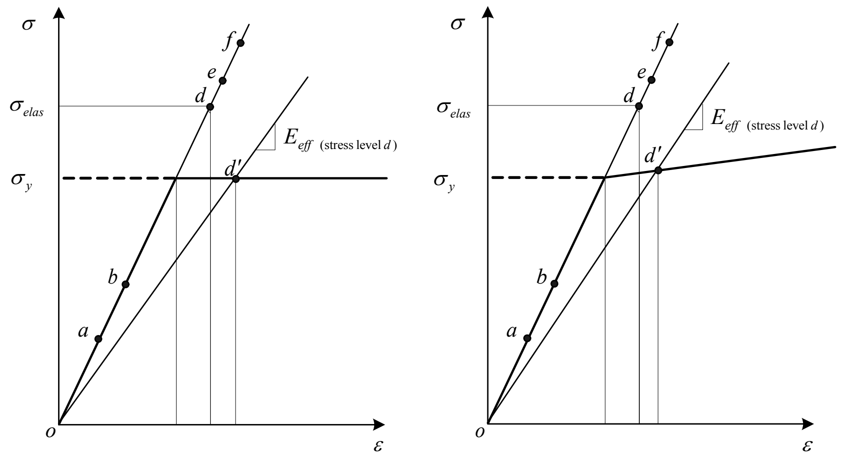

Figure 3: Effective Young's modulus E_{eff}.

From a virtual work principle we utilize the global formulation (eqn (2)). We further assume that there exists a stress level with which effective material properties are determined (stress level d, fig. 3). Next, at a typical level of loading LL_k we define the total strain energy that is vulnerable for redistribution due to plastic deformations $U_{k,\ \text{dist}}$ as the total hypothetical strain energy of the regions that violate the yield condition (step 2.2, basic steps of implementation of the adaptive coupling method)

$$U_{k,\ \text{dist}} = \int_{\Omega} ((\sigma_{ij}\varepsilon_{ij})_{\text{hyp elastic}})_x \kappa_x dV, \tag{5}$$

where $\kappa_x = 1$ if $((\sigma_{ij}\varepsilon_{ij})_{\text{hyp elastic}} - \sigma_y \varepsilon_y)_x > 0$, otherwise $\kappa_x = 0$ and σ_y is the uniaxial material yield strength. $\varepsilon_y = \sigma_y / E$. The effective Young's modulus $E_{k,\ eff}$ is then evaluated (fig. 2,3) as follows

$$\frac{U_{k,\text{dist}}}{U_{k,\ \text{hyp elastic}}} \approx \frac{U_{k,\text{dist (stress level d)}}}{U_{k,\text{hyp elastic (stress level d)}}} = \frac{cE_{k,eff}}{E}, \tag{6}$$

where c is a constant. For perfect plasticity and isotropic hardening plasticity models, it is concluded from fig 3 that a reasonable and conservative value is $c = 1$. The effective Poisson ratio $\nu_{k,eff}$ is obtained from equations adopted in iterative elastic analyses in order to simulate elasto-plastic behaviour [10]

$$1/E_{k,eff} = 1/E + 2\phi_k/3 \tag{7}$$

and

$$\nu_{k,eff} = E_{eff}(\nu/E + \phi_k/3). \tag{8}$$

For the determination of the modified level of loading $LL_{k,\,\mathrm{mod}}$, we further investigate the uniaxial stress-strain curve. Relating the hypothetical elastic stress state curve to that of the effective material parameters, it may be easily concluded that

$$LL_{k,\,\mathrm{mod}} = LL_i \sqrt{E} \Big/ \sqrt{E_{i,\,eff}} \,. \tag{9}$$

Compared to the adaptive coupling method of [8,9], the presented method involves additional FEM or BEM elastic iterations. These elastic iterations involved in estimation of the FEM and BEM sub-domains are more than rewarded by an improved estimate of the FEM and BEM sub-domains provided by the proposed adaptive method.

4 Example application

In this section we present an example application that highlights the effectiveness of the adaptive FEM-BEM coupling method presented in Section 3.

The square plate with a centred elliptical defect (fig. 4) is subjected to uniformly distributed tensile loads over the two pairs of the opposing ends (equal biaxaial tension). The applied tractions $P = 100\,\mathrm{x}10^6$ N/m^2 are scaled with the load factor λ. The elastic material properties of the plate are described by Young's modulus ($E = 206.9\mathrm{x}10^9$ N/m^2) and Poisson's ratio ($\nu = 0.29$). Material of von-Mises type is considered ($\sigma_y = 450\mathrm{x}10^6$ N/m^2), with no hardening effect ($H = 0$), as a yield function and plane strain loading conditions. Due to symmetry, only one quarter of the problem is modelled.

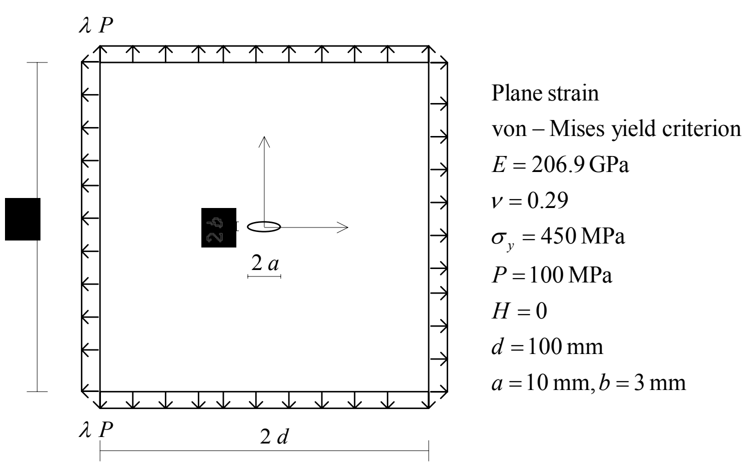

Figure 4: Plate with a centred elliptical defect.

The problem is solved by means of the adaptive coupling method presented in Section 3. The loads are applied incrementally. The elastic prediction and estimates of the regions sensible for discretization by the FEM obtained via the

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adaptive coupling method are shown in (fig. 5). The coupled FEM-BEM solutions are obtained with the automatically generated FEM and BEM discretization for the particular values of λ . The FEM discretization is generated over the regions that are estimated as sensible for FEM discretization, while the BEM mesh is generated to represent the remaining linear elastic region. Fig. 5 further shows the yielded regions obtained using the adaptive coupled FEM-BEM method for the selected values of λ . The results clearly show that the adaptive FEM-BEM coupled method employs smaller FEM sub-domains. Moreover, the method is practically advantageous as it does not necessitate the predefinition and manual localization of the FEM and BEM sub-domains.

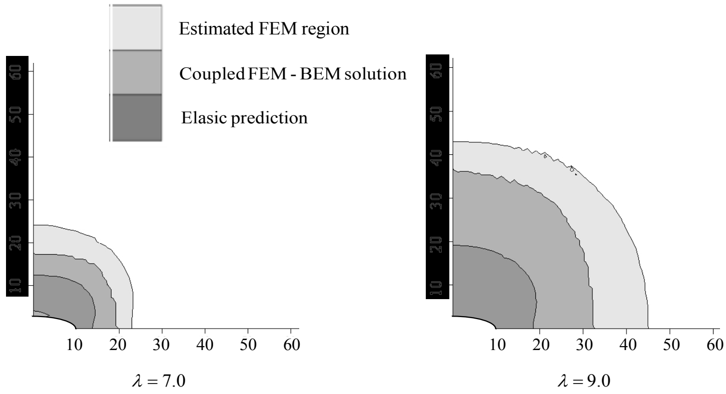

Figure 5: Estimated FEM regions and yielded regions (adaptive FEM-BEM coupling).

5 Conclusion and outlook

The present adaptive coupling method is practically advantageous as it does not necessitate predefinition and manual localization of the FEM and BEM sub-domains. Moreover, the method is computationally efficient as it substantially decreases the size of FEM meshes, which plainly leads to reduction of required system resources and gain in efficiency. The numerical results in 2D elasto-plastic analysis confirm the effectiveness of the proposed method. The extension of the adaptive FEM-BEM coupling method to 3D elasto-plastic applications is currently under investigation. As in the 2D analysis, the FEM sub-domain discretization is progressively adapted and automatically generated to include regions where plasticity occurs, according to the state of computation. Preliminary results indicate the practicality and the efficiency of the FEM-BEM coupling method for 3D elasto-plastic applications.

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