Regularization of the hypersingular integrals in 3-D problems of fracture mechanics

V. V. Zozulya
Centro de Investigación Científica de Yucatán A.C., Calle 43, No 130, Colonia Chuburná de Hidalgo, C.P. 97200, Mérida, Yucatán, México

Abstract

This article considers the hypersingular integrals, which arise when the boundary integral equation (BIE) methods are used to solve fracture mechanics problems . The methodology of hypersingular integral regularization developed in our previous publications is based on theory of distribution and Green's theorems. In the case of piecewise constant and piecewise linear approximations the hypersingular integrals are transformed into the regular contour integrals that can be easily calculated analytically or numerically.
Keywords: weakly singular, singular, hypersingular integrals, boundary integral equations, fracture mechanics.

1 Introduction

A huge amount of publications is devoted to the boundary integral equation methods (BEM) and its application science and engineering. One of the main problems arising in numerical solution of the BIE by the BEM is a calculation of the divergent integrals. Different methods have been developed for regularization of the divergent integrals [7]. The hypersingular integrals had been considered by Hadamard in the sense of finite part (*FP*) in [4]. The theory of distributions allows us to study the divergent integrals and integral operators with kernels containing different kind of singularities in the same way as the regular integrals.

Analysis of the most known methods used for treatment of the different divergent integrals has been done in our previous publications. It was shown that theory of distributions provides a unified approach for the study of the divergent integrals and integral operators with kernels containing different kind of singularities. We applied the theory of distribution approach for the first time in [8], then it was further developed in [9–11] and was applied for static and

WIT Transactions on Modelling and Simulation, Vol 47, © 2008 WIT Press
www.witpress.com, ISSN 1743-355X (on-line)
doi:10.2495/BE080221

dynamic problems of fracture mechanics in [17] and [18] respectively. See also monograph [3] and review articles [4, 5, 12] for details and further references.

In the present paper regularization methods developed in [9] are applied for the regularization of the hypersingular integrals that arise when the BIE methods are used for solution of the 3-D fracture mechanics problems.

2 Boundary integral equations

Let us consider a plane crack with a surface ∂V in a three-dimensional linearly elastic homogeneous isotropic space $\Re^3$. We introduce Cartesian coordinates system, with x_1 and x_2 axes in the plane of the crack, and the x_3 axis perpendicular to this plane. In [3–5] it was shown that the BIE that relate load $p_i(\mathbf{y})$ on the crack faces and crack opening $\Delta u_j(\mathbf{x})$ may be written in the following form

$$p_i(\mathbf{y}) = -\int_{\partial V} F_{ij}(\mathbf{x},\mathbf{y})\Delta u_j(\mathbf{x})\, d\mathrm{S}\,. \tag{1}$$

The kernels $F_{ij}(\mathbf{x},\mathbf{y})$ in the BIE (1) may be presented in the form

$$F_{11} = \frac{\mu}{4\pi(1-\upsilon)}\left[\frac{(1-2\upsilon)}{r^3} + 3\upsilon\frac{(x_1-y_1)^2}{r^5}\right],\ F_{12} = \frac{\mu\upsilon}{4\pi(1-\upsilon)}\frac{(x_1-y_1)(x_2-y_2)}{r^5},$$

$$F_{22} = \frac{\mu}{4\pi(1-\upsilon)}\left[\frac{(1-2\upsilon)}{r^3} + 3\upsilon\frac{(x_2-y_2)^2}{r^5}\right],\ F_{33} = \frac{\mu}{4\pi(1-\upsilon)}\frac{1}{r^3}\ . \tag{2}$$

where μ and υ are elastic module and Poison ration, r is a distance between points $\mathbf{x}$ and $\mathbf{y}$

$$r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$$

3 The BEM equations

In order to transform the BIE into finite dimensional BEM equations we have to split the boundary ∂V into finite elements, which are called boundary elements (BE).

$$\partial V = \bigcup_{n=1}^{N} \partial V_n, \quad \partial V_n \cap \partial V_k = \varnothing, \quad \text{if } n \neq k.$$

On each BE we shall choose Q nodes of interpolation and shape functions $\varphi_{nq}(\mathbf{x})$. Then the vectors of displacements discontinuity and traction on the BE ∂V_n will be represented approximately in the form

$$\begin{aligned} \Delta u_i(\mathbf{x}) &\approx \sum_{q=1}^{Q} \Delta u_i^n(\mathbf{x}_q)\varphi_{nq}(\mathbf{x}),\ \mathbf{x}\in\partial V_n \\ p_i(\mathbf{x}) &\approx \sum_{q=1}^{Q} p_i^n(\mathbf{x}_q)\varphi_{nq}(\mathbf{x})\ \ \mathbf{x}\in\partial V_n \end{aligned} \tag{3}$$

and on the hold boundary ∂V in the form

$$\Delta u_i(\mathbf{x}) \approx \sum_{n=1}^{N}\sum_{q=1}^{Q} \Delta u_i^n(\mathbf{x}_q)\varphi_{nq}(\mathbf{x}), \ \mathbf{x} \in \bigcup_{n=1}^{N} \partial V_n,$$
$$p_i(\mathbf{x}) \approx \sum_{n=1}^{N}\sum_{q=1}^{Q} p_i^n(\mathbf{x}_q)\varphi_{nq}(\mathbf{x}), \quad \mathbf{x} \in \bigcup_{n=1}^{N} \partial V_n \tag{4}$$

Substituting expressions (4) in (1) gives us the BE equations which relate the vectors of displacements discontinuity and traction on the crack surface in the form

$$p_i^m(\mathbf{y}_r) = -\sum_{n=1}^{N}\sum_{q=1}^{Q} F_{ji}^n(\mathbf{y}_r, \mathbf{x}_q)\Delta u_j^n(\mathbf{x}_q) \tag{5}$$

where

$$F_{ji}^n(\mathbf{y}_r, \mathbf{x}_q) = \int_{\partial V_n} F_{ji}(\mathbf{y}_r, \mathbf{x})\varphi_{nq}(\mathbf{x})dS. \tag{6}$$

More detailed information on transition from the BIE to the BEM equations can be found in [1,2].

4 Piecewise constant approximation

In the application of in the BEM, it is necessary to calculate the divergent integrals over any triangular, rectangular or polygonal elements. The piecewise constant approximation is the simplest one. Interpolation functions in this case do not depend on the BE form and dimension of the domain. They have the form

$$\varphi_{nq}(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in \partial V_n, \\ 0 & \forall \mathbf{x} \notin \partial V_n. \end{cases} \tag{7}$$

In order to simplify situation we transform origin of the global system of coordinates at the nodal point, where $\mathbf{y}^0 = 0$. Regular representations for the hypersingular integrals in (6) have the form

$$J_3^{0,0} = F.P.\int_{S_n} \frac{dS}{r^3} = -\int_{\partial S_n} \frac{r_n}{r^3} dl, \ J_5^{2,0} = F.P.\int_{S_n} \frac{x_1^2}{r^5} dS = \int_{\partial S_n} \left(\frac{r_n}{3r^3} - \frac{x_2^2 r_n}{r^5} - \frac{2x_1 n_1}{3r^3} \right) dl,$$
$$J_5^{0,2} = F.P.\int_{S_n} \frac{x_2^2}{r^5} dS = \int_{\partial S_n} \left(\frac{r_n}{3r^3} - \frac{x_1^2 r_n}{r^5} - \frac{2x_2 n_2}{3r^3} \right) dl, \tag{8}$$
$$J_5^{1,1} = F.P.\int_{S_n} \frac{x_1 x_2}{r^5} dS = \int_{\partial S_n} \left(\frac{x_1 x_2 r_n}{r^5} - \frac{r_+}{3r^3} \right) dl$$

where $r_n = x_\alpha n_\alpha$, $r_+ = x_1 n_2 + x_2 n_1$, $r_- = x_2 n_1 - x_1 n_2$. See [10, 11, 13] for details.

Let us consider the contour ∂V_n as a polygon with K vertexes us it is shown on fig. 1.

In this case

$$F_{ji}^n(\mathbf{y}_r, \mathbf{x}_q) = \sum_{k=1}^{K} \int_{l_k} F_{ji}(\mathbf{y}_r, \mathbf{x})dl. \tag{9}$$

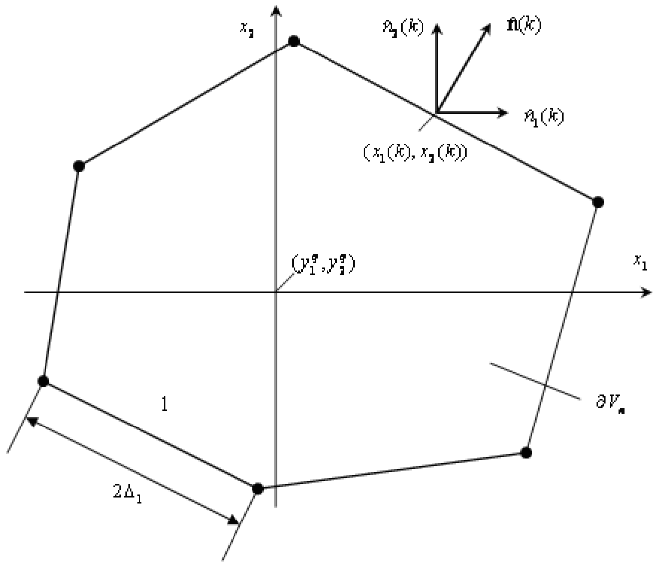

Figure 1.

Coordinates of an arbitrary point on the contour ∂V_n may be represented in the form

$$x_1(\xi) = x_1(k) - \xi\Delta_k \hat{n}_2 \text{ and } x_2(\xi) = x_2(k) + \xi\Delta_k \hat{n}_1 \tag{10}$$

where $x_1(k)$ and $x_2(k)$ are the coordinates of the middle of k-th side of the contour, $\hat{\mathbf{n}}(\hat{n}_1, \hat{n}_2)$ is a unit vector normal to the contour and $\xi \in [-1,1]$ is a parameter of integration along the k-th side, $2\Delta_k$ is the length of a k-th side.

These are some useful notations which will be used bellow

$$r(\xi) = \sqrt{\Delta_k^2 \xi^2 + 2\xi\Delta_k r_-(k) + r^2(k)}\ , r(k) = \sqrt{x_1^2(k) + x_2^2(k)}\ , r_n(k) = x_\alpha(k)\hat{n}_\alpha(k),$$
$$r_\tau(k) = x_1(k)\hat{n}_1(k) - x_2(k)\hat{n}_2(k),\ r_+(k) = x_1(k)\hat{n}_2(k) + x_2(k)\hat{n}_1(k), \tag{11}$$

Using these notations the integrals in (8) may be represented in a convenient for the calculation form

$$J_{0,3}^{0,0}(k) = -r_n(k) I_{3,0},$$

$$J_{0,5}^{2,0}(k) = \frac{1}{3} r_n(k) I_{3,0} - r_n(k)\left(x_2^2(k) I_{5,0} - 2\hat{n}_1(k) x_2(k) I_{5,1} - \hat{n}_1^2(k) I_{5,2}\right) - $$
$$-\frac{2}{3}\left(\hat{n}_1(k)\left(x_2(k) I_{3,0} - \hat{n}_2(k) I_{3,1}\right)\right)$$

$$J_{0,5}^{0,2}(k) = \frac{1}{3} r_n(k) I_{3,0} - r_n(k)\left(x_1^2(k) I_{5,0} + 2\hat{n}_2(k) x_1(k) I_{5,1} - \hat{n}_2^2(k) I_{5,2}\right) - \tag{12}$$
$$-\frac{2}{3}\left(\hat{n}_2(k)\left(x_1(k) I_{3,0} + \hat{n}_2(k) I_{3,1}\right)\right)$$

$$J_{0,5}^{1,1}(k) = r_n(k)\left(x_1(k)x_2(k)I_{5,0} + r_\tau(k)I_{5,1} - \hat{n}_1(k)\hat{n}_2(k)I_{5,1}\right) - \\ -\frac{1}{3}(r_+(k)I_{5,0} + (\hat{n}_1^2(k) - \hat{n}_2^2(k))I_{5,1}$$

Integrals in (12) may be calculated analytically

$$I_{1,0} = \Delta_k \int_{-1}^{1} \frac{1}{r(\xi)} d\xi = \ln\left|r_\tau(k) + \Delta_k \xi + r(\xi)\right|\Big|_{-1}^{1},$$

$$I_{1,1} = (\Delta_k)^2 \int_{-1}^{1} \frac{\xi}{r(t)} d\xi = r(\xi)\Big|_{-1}^{1} - r_\tau(k)I_{1,0},$$

$$I_{3,0} = \Delta_k \int_{-1}^{1} \frac{1}{r(\xi)^3} d\xi = \frac{1}{r^2(k) - r_\tau^2(k)} \left.\frac{\Delta_k \xi + r_\tau(k)}{r(\xi)}\right|_{-1}^{1},$$

$$I_{3,1} = (\Delta_k)^2 \int_{-1}^{1} \frac{\xi}{r(\xi)^3} d\xi = -\frac{1}{r^2(k) - r_\tau^2(k)} \left.\frac{r_\tau(k)\Delta_k \xi + r^2(k)}{r(\xi)}\right|_{-1}^{1},$$

$$I_{5,0} = \Delta_k \int_{-1}^{1} \frac{1}{r(\xi)^5} d\xi = \frac{1}{r^2(k) - r_\tau^2(k)} I_{3,0} + \frac{1}{2\left(r^2(k) - r_\tau^2(k)\right)} \left.\frac{\Delta_k \xi + r_\tau(k)}{r(\xi)^3}\right|_{-1}^{1}, \quad (13)$$

$$I_{5,1} = (\Delta_k)^2 \int_{-1}^{1} \frac{\xi}{r(\xi)^5} d\xi = r_\tau(k)I_{5,0} - \left.\frac{1}{3r(\xi)^3}\right|_{-1}^{1},$$

$$I_{5,2} = (\Delta_k)^3 \int_{-1}^{1} \frac{\xi^2}{r(\xi)^5} d\xi = \left.\frac{(\Delta_k \xi + r_\tau(k))^3}{3(r^2 - r_\tau^2)r(\xi)^3}\right|_{-1}^{1} - 2r_\tau(k)I_{5,1} - r^2(k)I_{5,0}.$$

Then fundamental solutions (9) may be represented in the form

$$F_{11}^n\left(\mathbf{y}_r, \mathbf{x}_q\right) = \frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{K} J_{0,3}^{0,0}(k) + 3\nu \sum_{k=1}^{K} J_{0,5}^{0,2}(k)\right],$$

$$F_{22}^n\left(\mathbf{y}_r, \mathbf{x}_q\right) = \frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{K} J_{0,3}^{0,0}(k) + 3\nu \sum_{k=1}^{K} J_{0,5}^{2,0}(k)\right], \quad (14)$$

$$F_{33}^n\left(\mathbf{y}_r, \mathbf{x}_q\right) = -\frac{\mu}{4\pi(1-\upsilon)} \sum_{k=1}^{K} J_{0,3}^{0,0}(k), \; F_{12}^n\left(\mathbf{y}_r, \mathbf{x}_q\right) = \frac{\mu\upsilon}{4\pi(1-\upsilon)} \sum_{k=1}^{K} J_{0,5}^{1,1}(k)$$

It is important to mention that here all calculations can be done analytically, no numerical integration is needed.

5 Piecewise linear approximation

Let us consider rectangle BE that is shown on fig. 2.
The quadrilateral BE is defined by it's angular nodes and its shape functions are

$$\varphi_1 = 1/4\left(1-\xi_1\right)\left(1-\xi_2\right) \qquad \varphi_2 = 1/4\left(1+\xi_1\right)\left(1-\xi_2\right)$$
$$\varphi_3 = 1/4\left(1+\xi_1\right)\left(1+\xi_2\right) \qquad \varphi_4 = 1/4\left(1-\xi_1\right)\left(1+\xi_2\right) \quad (15)$$

According to (5) and (6) in this case we have to calculate

$$F_{ji}^{n}\left(\mathbf{y}_{r},\mathbf{x}_{q}\right)=\sum_{q=1}^{4}\int_{\partial V_n}F_{ji}\left(\mathbf{y}_{r},\mathbf{x}\right)\varphi_{q}\left(\mathbf{x}\right)dS=\sum_{q=1}^{4}\sum_{k=1}^{4}\int_{l_k}F_{ji}\left(\mathbf{y}_{r},\mathbf{x}\right)\varphi_{q}\left(\mathbf{x}\right)dl. \qquad (16)$$

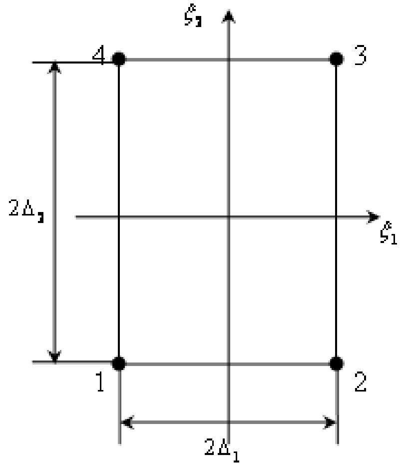

Figure 2.

Regular representations for the hypersingular integrals in (6) in this case have the form

$$J_{q,3}^{0,0}=F.P.\int_{S_n}\frac{\varphi_q(\xi)}{r^3}dS=-\int_{\partial S_n}\left(\varphi_q(\xi)\frac{r_n}{r^3}+\frac{1}{r}\partial_n\varphi_q(\xi)\right)dl\,, \qquad (17)$$

$$J_{q,5}^{2,0}=F.P.\int_{S_n}\varphi_q(\xi)\frac{x_1^2}{r^5}dS=\int_{\partial S_n}\left(\varphi_q(\xi)\left(\frac{r_n}{3r^3}-\frac{x_2^2 r_n}{r^5}-\frac{2x_1 n_1}{3r^3}\right)-\left(\frac{1}{r}+\frac{x_1^2}{r^3}\right)\partial_n\varphi_q(\xi)\right)dl\,,$$

$$J_{q,5}^{0,2}=F.P.\int_{S_n}\varphi_q(\xi)\frac{x_2^2}{r^5}dS=\int_{\partial S_n}\left(\varphi_q(\xi)\left(\frac{r_n}{3r^3}-\frac{x_1^2 r_n}{r^5}-\frac{2x_2 n_2}{3r^3}\right)-\left(\frac{1}{r}+\frac{x_2^2}{r^3}\right)\partial_n\varphi_q(\xi)\right)dl\,,$$

$$J_{q,5}^{1,1}=F.P.\int_{S_n}\varphi_q(\xi)\frac{x_1x_2}{r^5}dS=\int_{\partial S_n}\left(\varphi_q(\xi)\left(\frac{x_1x_2 r_n}{r^5}-\frac{r_+}{3r^3}\right)-\frac{x_1x_2}{r^3}\partial_n\varphi_q(\xi)\right)dl$$

In order to simplify situation we transform origin of the global system of coordinates at the middle point of re. The coordinate axes x_1 and x_2 are located in the plane of the BE and coincide with the local ones ξ_1 and ξ_2, while the axis x_3 is perpendicular to that plane. Coordinates of the nodal points are

$$(x_1^1=-\Delta_1, x_2^1=-\Delta_2),\ (x_1^2=\Delta_1, x_2^2=-\Delta_2),$$
$$(x_1^3=\Delta_1, x_2^3=\Delta_2),\ (x_1^4=\Delta_1, x_2^4=-\Delta_2)$$

These are some useful notations which will be used bellow

$$x_1 = \Delta_1(1+\xi_1)\,,\ x_2 = \Delta_2(1+\xi_2)\,,\ dS = \Delta_1\Delta_2 d\xi_1 d\xi_2\,,\ dl = \sqrt{dx_1^2 + dx_2^2}$$

$$r(\boldsymbol{\xi}, \mathbf{y}^q) = \sqrt{(x_1 - y_1^q)^2 + (x_2 - y_2^q)^2} = \sqrt{(\Delta_1(1+\xi_1) - y_1^q)^2 + (\Delta_2(1+\xi_2) - y_2^q)^2}\ ,$$

$$r_n = (x_\alpha - y_\alpha^q)n_\alpha\,,\ r_+ = (x_1 - y_1^q)n_2 + (x_2 - y_2^q)n_1\,,\ r_- = (x_2 - y_2^q)n_1 - (x_1 - y_1^q)n_2$$

$$\partial_n\varphi_1(\boldsymbol{\xi}) = \frac{1}{4}(r_+(\boldsymbol{\xi}) - \hat{n}_1 - \hat{n}_2)\,,\ \partial_n\varphi_2(\boldsymbol{\xi}) = \frac{1}{4}(-r_+(\boldsymbol{\xi}) + \hat{n}_1 - \hat{n}_2)\,, \tag{18}$$

$$\partial_n\varphi_3(\boldsymbol{\xi}) = \frac{1}{4}(r_+(\boldsymbol{\xi}) + \hat{n}_1 + \hat{n}_2)\,,\ \partial_n\varphi_4(\boldsymbol{\xi}) = \frac{1}{4}(-r_+(\boldsymbol{\xi}) - \hat{n}_1 + \hat{n}_2)\,.$$

Now we will calculate integrals (17) that are included in representation (16) for fundamental solutions. Let point $\mathbf{y}^0$ is located at the left low corner of the rectangle (point 1). Its coordinates are $\mathbf{y}^0 = (-\Delta_1, -\Delta_2)$. Other point can be considered in the same way. In this case a situation is not symmetrical we have to do calculations side by side. Order of summation in (16) can be changed, therefore we first will calculate integrals in (17) for every shape function and then their sums

$$J_{I,n}^{l,m}(k) = \sum_{q=1}^{4} J_{q,n}^{l,m} \tag{19}$$

Side 1-2. In this case $n_1 = 0$, $n_2 = -1$, $\xi_2 = -1$. The main parameters defined by (18) are

$$x_1 = \Delta_1(1+\xi_1)\,,\ x_2 = -\Delta_2\,,\ dl = \Delta_1 d\xi_1\,,\ r(\xi_1) = \Delta_1(1+\xi_1)\,,\ r_+ = -\Delta_1(1+\xi_1)\,,$$

$$r_- = \Delta_1(1+\xi_1)\,,\ r_n = 0\,,\ \varphi_1 = 1/2\left(1-\xi_1\right),\ \varphi_2 = 1/2\left(1+\xi_1\right),\ \varphi_3 = 0\,,\ \varphi_4 = 0\,,$$

$$\partial_n\varphi_1(\boldsymbol{\xi}) = 1/4(1-\xi_1)\,,\partial_n\varphi_2(\boldsymbol{\xi}) = 1/4(1+\xi_1)\,,\partial_n\varphi_3(\boldsymbol{\xi}) = -1/4(1+\xi_1)\,,$$

$$\partial_n\varphi_4(\boldsymbol{\xi}) = -1/4(1-\xi_1)$$

The sums (19) in this case are

$$J_{I,3}^{0,0}(1) = \sum_{q=1}^{4} J_{q,3}^{0,0} = 0\,,\ J_{I,5}^{2,0}(1) = \sum_{q=1}^{4} J_{q,5}^{2,0} = 0\,,\ J_{I,5}^{0,2}(1) = \sum_{q=1}^{4} J_{q,5}^{0,2} = 0\,,$$

$$J_{I,5}^{1,1}(1) = \sum_{q=1}^{4} J_{q,5}^{1,1} = -\frac{1}{6\Delta_1}$$

Side 2-3. In this case $n_1 = 1$, $n_2 = 0$, $\xi_1 = 1$, The main parameters defined by (18) are

$$x_1 = \Delta_1\,, x_2 = \Delta_2(1+\xi_2)\,,\ dl = \Delta_2 d\xi_2\,,\ r(\xi_2) = \sqrt{4\Delta_1^2 + \Delta_2^2(1+\xi_2)^2}\ ,\ r_n = 2\Delta_1\,,$$

$$r_+ = \Delta_2(1+\xi_2)\,,\ r_- = \Delta_2(1+\xi_2)\,,\ \varphi_1 = 0\,,\ \varphi_2 = 1/2\left(1-\xi_2\right),\ \varphi_3 = 1/2\left(1+\xi_2\right),$$

$$\varphi_4 = 0\,,\ \partial_n\varphi_1(\boldsymbol{\xi}) = -1/4(1-\xi_2)\,,\ \partial_n\varphi_2(\boldsymbol{\xi}) = 1/4(1-\xi_2)\,,$$

$$\partial_n\varphi_3(\boldsymbol{\xi}) = 1/4(1+\xi_2)\,,\partial_n\varphi_4(\boldsymbol{\xi}) = -1/4(1+\xi_2)$$

The sums (19) in this case are

$$J_3^{0,0}(2) = \sum_{q=1}^{4} \Big(\varepsilon_q^2 \Big(J_{0,1}^{0,0}(2) + \zeta_q^2 J_{0,3}^{0,0}\Big|_1(2) \Big) + \\ + \frac{1}{2}(r_n^q(2) I_{1,0}(2) + \xi_1^q \xi_2^q (\hat{n}_1(2) + \hat{n}_2(2)) I_{1,0}\Big|_1(2)) \Big)$$

$$J_5^{2,0}(2) = \frac{1}{2} \sum_{q=1}^{4} \Big(\varepsilon_q^2 \Big(J_{0,5}^{2,0}(2) + \zeta_q^2 J_{0,5}^{2,0}\Big|_1(2) \Big) - \\ - \frac{1}{2}\Big(r_n^q(2) J_3^{2,0}(2) + \xi_1^q \xi_2^q (\hat{n}_1(2) + \hat{n}_2(2)) x_2(2) J_3^{2,0}\Big|_1(2) \Big) \Big)$$

$$J_{1,5}^{0,2}(2) = \frac{1}{6} \sum_{q=1}^{4} \Big(\varepsilon_q^2 \Big(J_{0,5}^{0,2}(2) + \zeta_q^2 J_{0,5}^{0,2}\Big|_1(2) \Big) - \\ - \frac{1}{2}\Big(r_n^q(2) J_3^{0,2}(2) + \xi_1^q \xi_2^q (\hat{n}_1(2) + \hat{n}_2(2)) x_2(2) J_3^{0,2}\Big|_1(2) \Big) \Big)$$

$$J_{1,5}^{1,1}(2) = \frac{1}{6} \sum_{q=1}^{4} \Big(\varepsilon_q^2 \Big(J_{0,5}^{1,1}(2) + \zeta_q^2 J_{0,5}^{1,1}\Big|_1(2) \Big) - \\ - \frac{1}{2}\Big(r_n^q(2) J_3^{1,1}(2) + \xi_1^q \xi_2^q (\hat{n}_1(2) + \hat{n}_2(2)) x_2(2) J_3^{1,1}\Big|_1(2) \Big) \Big)$$

Here $\varepsilon_q^k = \begin{cases} 1 & \text{for} \quad k = q \ \& \ k = q+1 \\ 0 & \text{otherwise} \end{cases}$, $I_{m,l-1}\Big|_1 = (\Delta_k)^l \int_{-1}^{1} \frac{\xi^l}{r^m(\xi)} d\xi$,

$$J_3^{2,0}(k) = x_1^2(k) I_{3,0} + 2\hat{n}_2(k) x_1(k) I_{3,1} - \hat{n}_2^2(k) I_{3,2} + I_{1,0}(k),$$

$$J_3^{0,2}(k) = x_2^2(k) I_{3,0} - 2\hat{n}_1(k) x_2(k) I_{3,1} - \hat{n}_1^2(k) I_{3,2} + I_{1,0}(k).$$

Side 3-4. In this case $n_1 = 0$, $n_2 = 1$, $\xi_2 = 1$. The main parameters defined by (18) are

$$x_1 = \Delta_1(1+\xi_1),\ x_2 = \Delta_2,\ dl = \Delta_1 d\xi_1,\ r(\xi_1) = \sqrt{4\Delta_2^2 + \Delta_1^2(1+\xi_1)^2},\ r_n = 2\Delta_2,$$

$$r_+ = \Delta_1(1+\xi_1),\ r_- = -\Delta_1(1+\xi_1),\ \varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 1/2(1+\xi_1),$$

$$\varphi_4 = 1/2(1-\xi_1),\ \partial_n\varphi_1(\xi) = -1/4(1-\xi_1),\ \partial_n\varphi_2(\xi) = -1/4(1+\xi_1),$$

$$\partial_n\varphi_3(\xi) = 1/4(1+\xi_1),\ \partial_n\varphi_4(\xi) = 1/4(1-\xi_1)$$

The sums (19) in this case are

$$J_3^{0,0}(3) = \sum_{q=1}^{4} \Big(\varepsilon_q^3 \Big(J_{0,1}^{0,0}(2) + \zeta_q^3 J_{0,3}^{0,0}\Big|_1(3) \Big) + \\ + \frac{1}{2}(r_n^q(3) I_{1,0}(3) + \xi_1^q \xi_2^q (\hat{n}_1(3) + \hat{n}_2(3)) I_{1,0}\Big|_1(3)) \Big)$$

$$J_5^{2,0}(3) = \frac{1}{2} \sum_{q=1}^{4} \Big(\varepsilon_q^3 \Big(J_{0,5}^{2,0}(3) + \zeta_q^3 J_{0,5}^{2,0}\Big|_1(3) \Big) -$$

$$-\frac{1}{2}\left(r_n^q(3)J_3^{2,0}(3)+\xi_1^q\xi_2^q(\hat{n}_1(3)+\hat{n}_2(3))x_2(3)J_3^{2,0}\Big|_1(3)\right)\Bigg)$$

$$J_{1,5}^{0,2}(3)=\frac{1}{6}\sum_{q=1}^{4}\Bigg(\varepsilon_q^3\left(J_{0,5}^{0,2}(3)+\zeta_q^3 J_{0,5}^{0,2}\Big|_1(3)\right)-$$

$$-\frac{1}{2}\left(r_n^q(3)J_3^{0,2}(3)+\xi_1^q\xi_2^q(\hat{n}_1(3)+\hat{n}_2(3))x_2(3)J_3^{0,2}\Big|_1(3)\right)\Bigg)$$

$$J_{1,5}^{1,1}(3)=\frac{1}{6}\sum_{q=1}^{4}\Bigg(\varepsilon_q^3\left(J_{0,5}^{1,1}(3)+\zeta_q^2 J_{0,5}^{1,1}\Big|_1(3)\right)-$$

$$-\frac{1}{2}\left(r_n^q(3)J_3^{1,1}(3)+\xi_1^q\xi_2^q(\hat{n}_1(3)+\hat{n}_2(3))x_2(3)J_3^{1,1}\Big|_1(3)\right)\Bigg)$$

Side 4-1. In this case $n_1=-1$, $n_2=0$, $\xi_1=-1$. The main parameters defined by (18) are

$$x_1=-\Delta_1,\ x_2=\Delta_2(1+\xi_2),\ dl=\Delta_2 d\xi_2\ r(\xi_2)=\Delta_2(1+\xi_2),\ r_n=0,$$

$$r_+=-\Delta_2(1+\xi_2),\ r_-=0,\ \varphi_1=1/2(1-\xi_2),\ \varphi_2=0,\ \varphi_3=0,\ \varphi_4=1/2(1+\xi_2),$$

$$\partial_n\varphi_1(\boldsymbol{\xi})=1/4(1-\xi_2),\ \partial_n\varphi_2(\boldsymbol{\xi})=-1/4(1-\xi_2),\ \partial_n\varphi_3(\boldsymbol{\xi})=-1/4(1+\xi_2),$$

$$\partial_n\varphi_4(\boldsymbol{\xi})=1/4(1+\xi_2)$$

The sums (19) in this case are

$$J_{I,3}^{0,0}(4)=\sum_{q=1}^{4}J_{q,3}^{0,0}=0,\ J_{I,5}^{2,0}(4)=\sum_{q=1}^{4}J_{q,5}^{2,0}=0,\ J_{I,5}^{0,2}(4)=\sum_{q=1}^{4}J_{q,5}^{0,2}=0,$$

$$J_{I,5}^{1,1}(4)=\sum_{q=1}^{4}J_{q,5}^{1,1}=\frac{1}{6\Delta_2}$$

Substituting all obtained for each side equations in (16) finally we have

$$F_{11}^n\left(\mathbf{y}_r,\mathbf{x}_q\right)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{K}J_{I,3}^{0,0}(k)+3\nu\sum_{k=1}^{K}J_{I,5}^{0,2}(k)\right],$$

$$F_{22}^n\left(\mathbf{y}_r,\mathbf{x}_q\right)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{K}J_{I,3}^{0,0}(k)+3\nu\sum_{k=1}^{K}J_{I,5}^{2,0}(k)\right],\qquad(20)$$

$$F_{33}^n\left(\mathbf{y}_r,\mathbf{x}_q\right)=-\frac{\mu}{4\pi(1-\upsilon)}\sum_{k=1}^{K}J_{I,3}^{0,0}(k),\ F_{12}^n\left(\mathbf{y}_r,\mathbf{x}_q\right)=\frac{\mu\upsilon}{4\pi(1-\upsilon)}\sum_{k=1}^{K}J_{I,5}^{1,1}(k)$$

It is important to mention that here all calculations can be done analytically, no numerical integration is needed.

References

[1] Brebbia C.A., Dominguez J. *Boundary elements. An introductory course*, WIT Press, Southampton, 1998

[2] Brebbia C.A., Telles J.C.F. and Wrobel L.C. *Boundary element techniques. Theory and applications in engineering.* Springer-Verlag, Berlin. 1984

[3] Guz A.N. and Zozulya V.V., *Brittle fracture of constructive materials under dynamic loading*, Kiev, Naukova Dumka, 1993, (in Russian).
[4] Guz A.N. and Zozulya V.V. Fracture dynamics with allowance for a crack edges contact interaction, *International Journal of Nonlinear Sciences and Numerical Simulation,* **2**(3), pp. 173–233, 2001.
[5] Guz A.N. and Zozulya V.V. Elastodynamic unilateral contact problems with friction for bodies with cracks, *International Applied Mechanics,* **38**(8), pp. 3–45, 2002.
[6] Hadamard J., *Le probleme de cauchy et les eguations aux devivees partielles lineaires hyperboliques,* Herman, Paris, 1932, [English translation, Dover, New York, 1952], [Russian translation: Nauka, Moscow, 1978]
[7] Tanaka M., Sladek V. and Sladek J., Regularization techniques applied to boundary element methods. *Applied Mechanics Review*, **47**, pp. 457–499, 1994.
[8] Zozulya V.V., Integrals of Hadamard type in dynamic problem of the crack theory. *Doklady Academii Nauk. UkrSSR, Ser. A. Physical Mathematical & Technical Sciences*, **2**, pp.19–22, 1991, (in Russian).
[9] Zozulya V.V. Regularization of the divergent integrals. I. General consideration. *Electronic Journal of Boundary Elements,* **4**(2), pp. 49–57, 2006.
[10] Zozulya V.V. Regularization of the divergent integrals. II. Application in Fracture Mechanics. *Electronic Journal of Boundary Elements,* **4**(2), pp. 58–56, 2006.
[11] Zozulya V.V. and Gonzalez-Chi P.I. Weakly singular, singular and hypersingular integrals in elasticity and fracture mechanics, *Journal of the Chinese Institute of Engineers,* **22**(6), pp. 763–775, 1999.
[12] Zozulya V.V. and Gonzalez-Chi P.I. Dynamic fracture mechanics with crack edges contact interaction, *Engineering Analysis with Boundary Elements,* **24**(9), pp. 643–659, 2000.
[13] Zozulya V.V. and Lukin A.N. Solution of three-dimensional problems of fracture mechanics by the method of integral boundary equations. *International Applied Mechanics,* **34**(6), pp. 544–551, 1998.
[14] Zozulya V.V. and Men'shikov V.A. Solution of tree dimensional problems of the dynamic theory of elasticity for bodies with cracks using hypersingular integrals, *International Applied Mechanics*, **36**(1), pp. 74–81, 2000.