Two-parameter concept for anisotropic cracked structures

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Abstract

For two-dimensional anisotropic elasticity the M-integral was evolved, in terms of the boundary element method serving for numerical determination of the T stress which represents a notable parameter for fracture evaluation of cracked solid bodies on top of the stress intensity factor. T-stress issues for the crack face pressure instances are employed as the reference solutions to infer weight functions, e.g. to obtain T-stress results for thick-walled cylinders weakened by an internal edge crack subject to any complicated loading. Some examples are demonstrated.
Keywords: complex stress distribution, crack face, material anisotropy, remote load, self-regularization.

1 Introduction

Besides the stress intensity factor, the T-stress is the other parameter reflected in fracture evaluation. The path-independent *M-integral* approach to interpret the T-stress, in conformity with Shah et al. [1], is augmented to use plane, in general anisotropic cracked constituents. It is realized in the boundary element method. An example is demonstrated to indicate the accuracy of the quantification formulated and its suitability. The numerical projects derived are illustrative of the fact that material anisotropy has indeed a substantial influence on the T stress.

To assess the *M integral*, conceptions for the stresses and displacements at interior points are required. When being the interior points very close to the boundary, some inconveniences in the standard BEM representation arise. A technique to get over this near-singularity problem was issued as early as

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several years ago, namely for isotropic analysis. In so doing, a chart was worked out to get the continuous form of the Somigliana's identity by using a simple solution commensurable to a rigid body movement. The self-regularization scheme can be performed to obtain interior point field quantities in an anisotropic body. It makes the application of comparatively coarse mesh layouts possible, for the boundary even if the interior point is located extremely near it.

The implementation is based on the quadratic isoparametric element. Standard and traction singular quarter point elements were applied contiguous to the crack tip for fracture mechanics study. To check up path independence of the T-stress solutions obtained applying the *M integral*, no less than two circular round the crack tip were selected for the interpretation of the integral.

For numerical determination of the T stress, the *M integral* represents an effective methodology. This paper launches the *M integral* for two-dimensional anisotropic elasticity being realized in context with the BEM.

Collapse of thick-walled pressurized cylinders is frequently owing to the presence of internal or external cracks in practice. A long internal single radial crack may be treated as an edge crack in two dimensions. It has been shown that an internally pressurized cracked cylinder can be considered as a "low-constrained" geometry. The fracture toughness measured from "high-constrained" test specimens may be conservative when applied to this constituent. On that account, precision T stress solutions for thick-walled cylinders with an internal radial crack are desirable to reliably predict the failure loading.

This paper presents T-stress solutions for a cylinder afore-said, namely using the BEM and the contour integral approach, with the loading being an example of crack face pressures which are realized by polynomial stress distributions.

Notwithstanding that the superposition method can be applied to estimate the T-stress using available T-stress solutions for simple loading conditions, it is not possible to cover the complete range of loading conditions in engineering. The weight function method was initiated to be one of the most powerful concepts to obtain the stress intensity factors for more intricate problems in Li et al. [2]. The T-stress weight functions are derived from T-stress solutions for two reference loading conditions, which correspond to the cases when the cracked cylinder is subject to a constant and to a linear variation of the applied stress on the crack faces.

2 BEM interpretation

It is known that the solution for the displacements $u_i(p)$ and stress $\sigma_{ij}(p)$ at an interior point, *p*, of a domain can be derived from the Somigliana´s identities, namely in the form

$$u_j(p)=\int_S t_i(Q)U_{ji}(p,Q)dS(Q)-\int_S u_i(Q)T_{ji}(p,Q)dS(Q) \tag{1}$$

and

$$\sigma_{ij}(p)=\int_S u_k(Q)S_{kij}(p,Q)dS(Q)-\int_S t_k(Q)D_{kij}(p,Q)dS(Q), \tag{2}$$

where U_{ji}, T_{ji}, D_{kij} and S_{kij} are the fundamental solutions, $t_i(p)$ the traction vector and S the boundary of the domain.

A procedure to overcome the near-singularity problem was proposed by Richardson and Cruse for isotropic variant. The developed an outline to get the continuous form of the Somigliana´s displacement identity by using a solution corresponding to a rigid body motion to the identity. Like this, it holds

$$u_j(p)-u_j(P)=\int_S t_i(Q)U_{ji}(p,Q)dS(Q)-\int_S [u_i(Q)-u_i(P)]\,T_{ji}(p,Q)dS(Q) \tag{3}$$

A weakly singular form of the stress identity is obtained by a simple technique which is equivalent to subtracting and adding back a simple solution corresponding to a state of constant stress in the body that equals the boundary stress at point P close to the interior point p, on condition that the stress at the boundary point P is continuous. The self-regular stress identity can be written down

$$\begin{aligned}\sigma_{ij}(p)=\sigma_{ij}(P)+&\int_S\left[u_k(Q)-u_k^L(P,Q)\right]S_{kij}(p,Q)dS(Q)\\ &-\int_S\left[t_k(Q)-t_k^L(P,Q)\right]D_{kij}(p,Q)dS(Q)\end{aligned} \tag{4}$$

In Eq. (4), t_k^L and u_k^L are the linear state tractions and displacements; they are related to a constant stress state in the body corresponding to the stress at the boundary point P, as follows

$$u_k^L(P,Q)=u_k(P)+u_{km}(P)\{x_m(Q)-x_m(P)\} \tag{5}$$

and

$$t_k^L(P,Q)=\sigma_{mk}(P)\,n_m \tag{6}$$

As the linear state traction density (t_k - t_k^L) and the displacement density (u_k - u_k^L) are $O(r)$ and $O(r^2)$, respectively, the integrals in Eq. (6) work out regular or weakly singular.

3 T-stress assessment

The path-independent mutual *M integral* to determine T stress was generalized in Shah et al. [1] to the anisotropic eventuality in two dimensions. In so doing, in the analytical adaptation of the anisotropic case, Lekhnitskii´s guidelines are applied. In fig. 1 a cracked body Ω limited by the boundary Γ_o is indicated. The path-independent *J integral* may be expressed in the form

$$J=\int_{\Gamma_0}\left(W_{n_1}-t_i u_{j,1}\right)d\Gamma \tag{7}$$

where W is the strain energy density:

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{8}$$

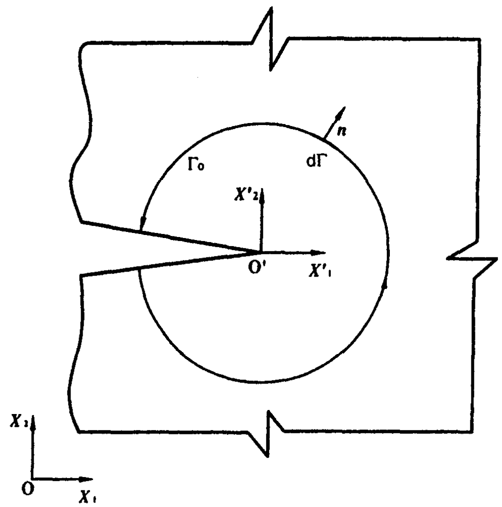

Figure 1: Contour Γ_o round the crack tip.

Contemplate two independent equilibrium states ($\sigma_{ij}^A, \varepsilon_{ij}^A, u_i^A$) and ($\sigma_{ij}^B, \varepsilon_{ij}^B$, u_i^B). The first state (*A*) befits to the boundary value problem that is analyzed with the unknown T stress. The second state (*B*) corresponds to the solution of a semi-infinite crack loaded by a point (line) force *f* acting on the crack tip in the direction parallel to the crack plane. The first state *A* is relevant to the stress and displacement fields near a crack tip.

After getting the required field expressions for states *A* and *B*, the contour *M-integral* expression for determining the T stress can be derived. For the corresponding *J integrals* with regard to the local coordinates, x_i, for states *A* and *B* it holds

$$J^{(A)} = \int_{\Gamma_0} \left[\frac{1}{2}\sigma_{ij}^A \varepsilon_{ij}^A n_1 - \sigma_{ij}^A n_j u_{i,1}^A \right] d\Gamma \tag{9}$$

and

$$J^{(B)} = \int_{\Gamma_0} \left[\frac{1}{2}\sigma_{ij}^B \varepsilon_{ij}^B n_1 - \sigma_{ij}^B n_j u_{i,1}^B \right] d\Gamma \tag{10}$$

and further, consider $J^{(A+B)}$, which denotes the value of *J integral* when both *A* and *B* fields are superimposed. Consequently,

$$J^{(A+B)} = \int_{\Gamma_0} \left[\frac{1}{2} \left(\sigma_{ij}^A + \sigma_{ij}^B \right) \left(\varepsilon_{ij}^A + \varepsilon_{ij}^B \right) n_1 - \left(\sigma_{ij}^A + \sigma_{ij}^B \right) n_j \left(u_{i,1}^A + u_{i,1}^B \right) \right] d\Gamma \tag{11}$$

In conformity with Sládek et al. [3], the *M integral* in local coordinates is expressed by the relation

$$M = J^{(A+B)} - J^{(A)} - J^{(B)} = \int_{\Gamma_0} \left[\frac{1}{2} \left(\sigma_{ij}^A \varepsilon_{ij}^B + \sigma_{ij}^B \varepsilon_{ij}^A \right) n_1 - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A \right] d\Gamma \tag{12}$$

Since the loading states are applied to the same elastic body,

$$\sigma_{ij}^A \varepsilon_{ij}^B = \sigma_{ij}^B \varepsilon_{ij}^A \tag{13}$$

That is why

$$M = \int_{\Gamma_0} \left[\left(\sigma_{ij}^A \varepsilon_{ij}^B n_1 - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A \right) \right] d\Gamma \tag{14}$$

As the *M integral* is expressed by virtue of the path independent *J integral*, it is also path independent. Thus the integration contour can be arbitrarily chosen, say, a circle with radius ε, which is then shrunk to zero.

Next, the *M integral* reads

$$M = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \left[\left(\sigma_{ij}^A \varepsilon_{ij}^B n_1 - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A \right) \right] d\Gamma \tag{15}$$

Since the *J integrals* are bounded, so is the *M integral*; it is then possible to infer that there are no contributions to the *M integral* from the singular stress terms of the asymptotic expansion. The asymptotic displacements and stresses may be separated into singular and non-singular parts in the following form:

$$\sigma_{ij}^A = \sigma_{ij}^s + \sigma_{ij}^T \tag{16.a}$$

and

$$u_i^A = u_i^s + u_i^T \tag{16.b}$$

In Eq. (16), the superscript *s* denotes the terms of the asymptotic expansion containing the stress intensity factor, and the terms with superscript *T* are proportional to the non-singular T stress. The circular contour integral from θ = -π to + π of the angular functions of the singular terms of the auxiliary field in Eq. (15) cancel out preserving only the non-vanishing contribution from the T stress. From Eq (15) it results

$$M = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \left[\left(\sigma_{ij}^T \varepsilon_{ij}^B \delta_{1k} - \sigma_{ij}^T u_{i,1}^B - \sigma_{ik}^B u_{i,1}^T \right) \right] n_k d\Gamma \tag{17}$$

where

$$\sigma_{ij}^T = T \delta_{i1} \delta_{j1} \tag{18}$$

The elastic strains corresponding to the uniform T stress σ_{ij}^{T} under plane stress conditions are given as

$$\varepsilon_{11}^{T} = a'_{11} \tag{19}$$

and

$$u_{i,1}^{T} = u_{1,1}^{T}\delta_{i1} = \varepsilon_{11}^{T}\delta_{i1} = a'_{11}T\delta_{i1} \tag{20}$$

Thus, the *M integral* is reduced and may be re-arranged into the form

$$T = M\frac{1}{a'_{11}f} \tag{21}$$

Eq. (21) yields the relationship between the *M integral*, which may be evaluated using Eq. (14) and the T stress. It can also be used to evaluate T stress in plane strain conditions provided that a_{11} is replaced by b_{11}.

It should be reminded that the terms in Eq.(14) are given in the local coordinate system about the crack tip. In the numerical implementation, the *M integral*, given by Eq. (14) is obtained in global coordinates and then transformed into the local coordinates. The transformation for the *J integral* was determined by Kishimoto et al., when similarly applied to *M integral*, it provides in local coordinates the form:

$$M_{k(GLOBAL)} = M_{1(GLOBAL)}\cos\omega + M_{2(GLOBAL)}\sin\omega \tag{22}$$

where

$$M_{k(GLOBAL)} = \int_{\Gamma_0}\left[\left(\sigma_{ij}^{A}\varepsilon_{ij}^{B}n_k - \sigma_{ij}^{A}n_j u_{i,k}^{B} - \sigma_{ij}^{B}n_j u_{i,k}^{A}\right)\right]d\Gamma\,. \tag{23}$$

4 Discussion

The implementation of the statements for the *M integral* and self-regularized Somigliana´s identities in two-dimensional anisotropic elasticity is based on the quadratic isoparametric elements. Usual and traction-singular quarter point elements were applied contiguous to the crack tip for fracture mechanics analysis. To check up path independence of the T-stress solutions gained using the *M integral*, at least two circular contours round the crack tip were selected for the assessment of the integral.

The radii of these circular contours were characteristically 0.4 – 0.6 times the simulated crack length. Each of these contours was divided into smaller circular arcs and the *M integral* was evaluated by Gaussian quadrature over each of them and summed. In general, the deviations between the numerical solutions obtained for the T stress from the different contours were less than 2 per cent.

An example considered embraces an orthotropic plate with a single edge crack that is inclined (SECP) – fig. 2. To examine the effect of the degree of orthotropy on the T-stress solutions for various relative crack lengths, the parametric study was performed. The analysis was carried out under plane stress conditions and the results of the T-stress are normalized regarding the applied stress σ_0.

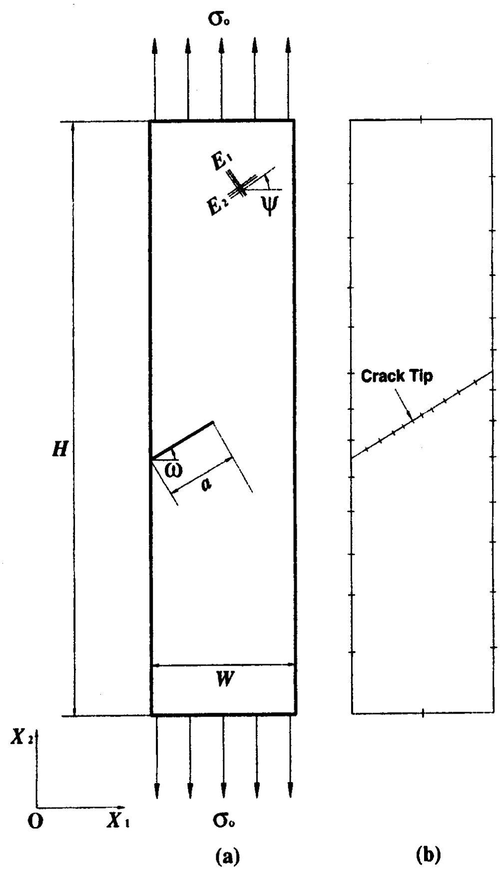

Figure 2: (a) A single-edge cracked plate subject to remote load; (b) BEM mesh taken.

For that purpose, the orthotropic material properties are defined by virtue of purely imaginary roots of the characteristic equation, denoted by $i\eta_1$ and $i\eta_2$, like this

$$\eta_1\eta_2 = \left(\frac{E_1}{E_2}\right)^{1/2} \tag{24.a}$$

$$\eta_1 + \eta_2 = \sqrt{2}\left[\left(\frac{E_1}{E_2}\right)^{1/2} + \frac{E_1}{2\mu_{12}} - \nu_{12}\right]^{1/2} \tag{24.b}$$

The quantities of these parameters investigated are: η_1 =1.5, 3, 4.5 and 6 and η_2 = 0.5 and 0.75; the Poisson´s ratio was taken ν_{12} = 0.3. Altogether, eight cases with the different combinations of values of η_1 and η_2 were studied, as listed in

Table 1. These values of η_1 and η_2 cover a relatively wide range of commonly used orthotropic materials in engineering structural applications.
Specific values of the parameters investigated are presented for the geometric eventuality of $\omega = 0°$.

Table 1: Orthotropic events analyzed for the different combinations of η_1 and η_2; $v_{12} = 0.3$.

Cases	η_1	η_2
1	1.5	0.5
2	3	0.5
3	4.5	0.5
4	6	0.5
5	1.5	0.75
6	3	0.75
7	4.5	0.75
8	6	0.75

Table 2: Characteristic properties for the orthotropic material Kevlar and the values of η_1 and η_2

	E_{11} [GPa]	E_{22} [GPa]	G_{12} [GPa]	v_{12}	η_1	η_2
Kevlar 49/epoxy	86	5.5	2.1	0.34	6.32	0.63

For the sake of brevity and primarily for the purpose of illustration here, only the results for a few specific values of the parameters investigated are presented; they are all for the geometric case of $\omega = 0°$. To be representative of a 'long' plate even for the biggest crack size analyzed, $H/W = 4$ was considered for SECP. In the case when $\psi = 0°$, variations of the T stress with relative crack size, a/W, ranging from $a/W = 0.1$ to 0.5, in SECP specimen are shown in fig. 3. The T-stress results for all the crack sizes treated reveal significant increase in the level of constraint at the crack tip with increasing values E_1/E_2, as supported by the decreasing magnitudes of their negative values.

5 Conclusion

Moreover the stress intensity factor, the T stress is extensionally acclaimed being a significant second parameter for fracture evaluation of cracked bodies. To stipulate T stress numerically, the M integral is an effective way, notably, when performing with the BEM. The paper presents the M integral for two-dimensional anisotropic elasticity and it is implemented being related to the BEM. An example demonstrates the rightness of the formulations and their suitability. It became apparent that T stress for a given cracked pattern may really be greatly influenced by anisotropy.

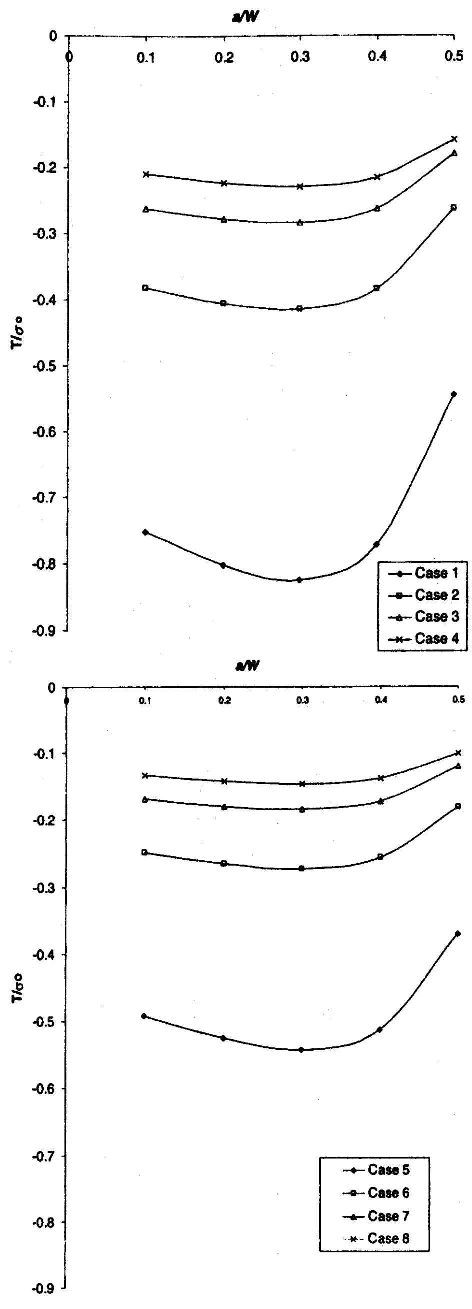

Figure 3: (Taken from [1]): (a) alteration of normalized T stress, T/σ_0, with relative crack length, a/W, ω = 0° and ψ = 0°; (b) alteration of normalized T stress, T/σ_0, with relative crack length, a/W, ω = 0° and ψ = 0°.

To determine the T-stress for radial edge cracks in thick-walled cylinders, the boundary element studies were developed. The configurations incorporated both the wide extension of radius ratios and relative crack depths. The loads contemplated are crack-face pressures embracing polynomial stress distributions. Later, the T-stress weight functions were derived from two reference T-stress solutions for crack pressures of uniform and linear distributions.

The inferred weight functions were verified for miscellaneous loading events. A satisfactory correspondence between the weight functions predictions and solutions obtained directly from the boundary element analysis was acquired. The methodology of weight functions is amenable for T-stress calculations even in conditions of more complicated loading.

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