A meshless solution for potential equations using a continuous-valued circular line source

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Abstract

We find solutions to $\nabla^2 U = 0$ in a simply-connected 2-D domain D, using a continuous line source associated with a concentration function comprising n undetermined parameters. This choice reduces ill-conditioning effects by reducing the number of parameters involved. The choice of a continuous circular line source C around D follows from previous results indicating that, when solving the same problem with discrete point sources, the result is independent of precise placement of sources. The circle is associated with a concentration function that is constrained to satisfy the problem's boundary conditions. Accuracy is achievable using a number of parameters which, had discrete sources been used, would be insufficient to represent the geometry of D, thus giving inaccurate results. Empirical investigations with various forms of concentration function show that with some domains, the error in calculated values of U can be less than 0.1%: an order of magnitude improvement over discrete methods. More complex domains yield less accuracy, and, after testing on a range of domains, we formulate an empirical rule for an appropriate form for the concentration function for a generic domain. Code requiring high-precision arithmetic was developed in *Mathematica*, which also simplifies routine tasks of solving linear systems and integrations.

1 Introduction

Previous research has shown that when using meshless discrete sources in the MFS, the configuration of sources relative to the domain D is extremely flexible. In [1] we showed that, within certain limits, the source distribution can be random, and in [2] we showed that sources "at infinity" (i.e. a large distance

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from D compared to the size of D) can produce very accurate results for the simple domain discussed in this paper. Fam's study [4] confirmed our findings on source distributions near D, using extensions of the MFS. Studies [1, 2, 4] show that previous attempts to analyse meshless source distributions are incorrect. For example, Alves [3], in an analysis of Poisson equations, considers a 'natural' radius for a circular distribution of discrete sources of "5~10 times the diameter of D" (without any precise definition for the term 'diameter' for a non-circular domain). We claim, using [1, 2], that a 'natural' radius is infinite, and reiterate that view from the results of this paper. Alves says that ill-conditioning problems preclude the use of very large radii, but we use Mathematica to analyse potential problems with sources at arbitrary distances from D without significant ill-conditioning.

2 The continuous line source method

For discrete point sources S_j exterior to a domain D, the potential $U(m)$ at a point m in D or on the boundary of D is given by

$$U(m) = \sum_j U^*(S_j)\, c_j \tag{1}$$

where U^* is the fundamental solution at S_j with respect to m and the c_j are undetermined coefficients. Figure 1 shows such a domain D (with boundary ∂D and interior point $m(x,y)$) in which the point sources have been replaced by a continuous circular line source with constant radius R, of which AB is an arc. This circular source has an associated concentration function $c(\theta,\mathbf{c})$ (abbreviated to $c(\theta)$), where the parameter θ is the angle shown in Figure 1 and $\mathbf{c}$ is a vector of n undetermined coefficients.

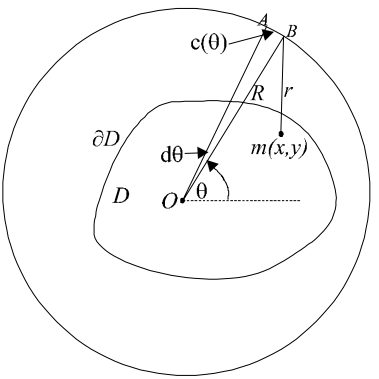

Figure 1: Circular source and domain.

The continuous equivalent of eqn. (1) is derived by considering the potential dU due to the infinitesimal line source S_θ on the arc AB, on which it is assumed that $c(\theta)$ is approximately constant. Assuming throughout that the Origin O is in the interior of D.and that that m is not at O,

$$dU(m,\theta) = U^*(S_\theta)c(\theta)d\theta \tag{2}$$

Hence the total potential at m due to the whole circular source is

$$U(m) = \int_0^{2\pi} U^*(S_\theta)\, c(\theta)d\theta = \int_0^{2\pi} \frac{\ln(r)}{2\pi}\, c(\theta)d\theta\,, \tag{3}$$

where
$$r^2 = \left(R\cos(\theta) - x\right)^2 + \left(R\sin(\theta) - y\right)^2 \tag{4}$$

Similarly the flux $Q(m)$ at m due the entire circular source is given by

$$Q(m) = \int_0^{2\pi} Q^*(S_\theta)\, c(\theta)d\theta = \int_0^{2\pi} \frac{1}{2\pi r}\, \frac{n_x r_x + n_y r_y}{r} c(\theta)d\theta \tag{5}$$

where $r_x = R\cos(\theta) - x$, $r_y = R\sin(\theta) - y$ and (n_x , n_y) are direction cosines at m.

In the case where m is at O, eqn (3) simplifies to

$$U(m) = \frac{\ln(R)}{2\pi} \int_0^{2\pi} c(\theta)\, d\theta \tag{6}$$

and eqn (5) simplifies to

$$Q(m) = \frac{1}{2\pi R} \int_0^{2\pi} \left(n_x \cos(\theta) + n_y \sin(\theta)\right) c(\theta)d\theta\,. \tag{7}$$

Our method then proceeds by discretising ∂D and setting up a set of linear equations based on known potentials on ∂D. Let $\mathbf{m} = (m_0, m_1, \cdots, m_{N-1})$ be a vector of the midpoints of N boundary elements on ∂D with corresponding known boundary values (either potential or flux) $\mathbf{b} = (b_0, b_1, \cdots, b_{N-1})$. For each element of **m** in turn we use one of eqns (3, 5) (depending on whether the relevant boundary condition is a known potential or flux) to obtain a system of linear equations in the undetermined parameters **c** for $c(\theta)$. This gives a matrix equation

$$\mathbf{Mc = b} \tag{8}$$

where the coefficients of **M** come from one of eqns (3,5). *Mathematica* provides a convenient inversion for **M** using its pseudo-inverse, and this accounts for over-determined systems and can be done to arbitrary precision (within the limits of computer memory). Thus we can obtain a relatively accurate approximation $\bar{\mathbf{c}}$ for **c**:

$$\bar{\mathbf{c}} = \text{PseudoInverse}[\mathbf{M}]\ \mathbf{b}. \tag{9}$$

The potential and flux at an interior or boundary point **p** can then be calculated from any one of eqns (3,5,6,7), depending on whether a flux or potential is required and whether **p** is at the Origin or not, using the known elements of $\overline{\mathbf{c}}$ in the integrals involving $c(\theta)$.

2.1 Choice of concentration function

Choosing a concentration function involves finding suitable linearly independent basis functions, and a subjective judgement about the number of undetermined parameters involved. On balance we have found that there is rarely much to be gained by choosing over-complicated basis functions, since simple polynomials suffice in all the cases we have tried. The numbers of undetermined parameters is more of a problem. Choosing too few cannot reflect the domain geometry accurately, and often gives completely wrong solutions. Choosing too many often has little effect but produces a progressive loss of accuracy due to ill-conditioning. In some cases this loss of accuracy is significant.

The only other constraint on the concentration function is a continuity condition $c(0)=c(2\pi)$. Collocation polynomials satisfying this continuity condition and evaluating to the undetermined parameter c_i of **c** at ordinates i initially produced promising results in simple rectangular domains, even with high values of n (>30). But for more complex domains, either ill-conditioning or a rapidly oscillating function (or both) gave less accurate results. Hence we concentrated on piecewise linear functions only. The n-parameter continuous piecewise linear function *pwlin*, is defined in eqn (10).

$$pwlin(\theta,n)=\begin{cases}\alpha_r\theta+\beta_r & :\theta\in\left[\dfrac{2\pi r}{n},\dfrac{2\pi(r+1)}{n}\right), \quad r=0..n-1,\\ 0 & :\theta\notin\left[\dfrac{2\pi r}{n},\dfrac{2\pi(r+1)}{n}\right), \quad r=0..n-1, \quad \theta\in[0,2\pi]\end{cases} \tag{10}$$

$$\alpha_r=\frac{n}{2\pi}(c_{r+1}-c_r), \quad \beta_r=(r+1)c_r-rc_{r+1}$$

This set of concentration functions is useful because the integrals in matrix **M** in eqn (8) are simple for all values of n (although it's technically harder to define their domains correctly). No assumption are made about placement of the c_r. This fact, and the results using piecewise linear concentration functions, effectively counter Poullikkis's assertion [7], that point sources must be placed uniformly at a fixed distance from the boundary.

2.2 Conjectures on parameters of the concentration function

In the examples that follow, it is apparent that in some cases the choice of parameters of the concentration function matters very little, whereas in other cases it matters a great deal. We aim to determine whether or not it is possible to

find general results for the number and position of the parameters of the concentration function which are independent of the domain. If this can be done, our method can become a generic method to solve any potential problem $\nabla^2 U = 0$ in any D subject to boundary conditions on ∂D. Trials from this study and results from [1] and [9] suggest that, given the definition for Nominal Radius of the domain, $NR[D]$, in section 2.2.1, we should consider placing the enclosing circle either 'at infinity' or between 10 and 20 times $NR[D]$. Furthermore, the number of concentration function parameters should be a simple function of the number of boundary elements on the convex and concave parts of the discretised boundary. .

2.2.1 Definition: *Nominal Radius*

For a simply-connected domain D, the *Nominal Radius* of D, denoted by $NR[D]$ is half of the maximum (straight line) distance between pairs of points on ∂D.

$$NR[D] = \frac{1}{2} \sup_{x,y \in \partial D} \left(|x - y| \right) \tag{11}$$

This definition is intended to be no more than a general guideline in choosing the radius of the enclosing circle. It is used to relate the circle radius to a single spatial characteristic of D. $NR[D]$ is "loosely" the radius of the smallest circle that can be drawn around D without intersecting with the boundary of D.

3 Convex domain example: torsion of an elliptical bar

This is an example of torsion of a bar with an elliptical cross-section, and is taken from [10]. Figure 2 shows the configuration of a quarter of the domain. Dirichlet boundary conditions U=0 apply on the straight sides and a Neumann boundary condition $Q1(x,y) = \dfrac{75xy}{\sqrt{25x^2 + 10000y^2}}$ applies on the curved side.

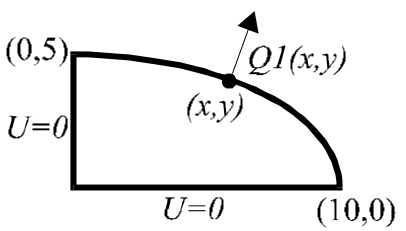

Figure 2: Torsion of an elliptical bar.

Brebbia and Dominguez [10] use reference points (2,2) and (4,3.5) for their calculations. At these points their quadratic BEM gives U as –2.431 and –8.472 respectively, with analytical results of –2.400 and –8.400 respectively. Table 1 shows our results at these reference points. In this case the nominal radius for the

domain is $NR(D)=\frac{1}{2}\sqrt{5^2+10^2}\sim 5.6$. Rounded up to 6, this is the minimal radius that yields reasonable results The discretisation used 12 boundary elements, so we consider among others, 12 concentration function parameters (CFP). With high numbers of parameters such as these, using the collocation concentration function is not practical, as failure to converge is frequent, and the results in Table 1 are for the piecewise linear concentration function.

Table 1: Results for torsion of an elliptical bar.

CFP	10	10	12	12	15	15	20	20
Radius	U(2,2)	U(4,3.5)	U(2,2)	U(4,3.5)	U(2,2)	U(4,3.5)	U(2,2)	U(4,3.5)
6	-2.0862	-6.6759	-2.0862	-6.6759	-2.2935	-8.258	-2.4444	-8.61
50	-2.4116	-8.4406	-2.4116	-8.4406	-2.4116	-8.44053	-2.4116	-8.4406
100	-2.4116	-8.4406	-2.4116	-8.4406	-2.4116	-8.44064	-2.4116	-8.4406
200	-2.4116	-8.4407	-2.4116	-8.4407	-2.4117	-8.44068	-2.4116	-8.4406
400	-2.4115	-8.4406	-2.4115	-8.4406	-2.4116	-8.44066	-2.4116	-8.4407
1000	-2.4116	-8.4406	-2.4116	-8.4406	-2.4116	-8.44063	-2.4116	-8.4406
5000	-2.4116	-8.4407	-2.4116	-8.4407	-2.4121	-8.44109	-2.4116	-8.4406

The results in Table 1 show an improvement on the quadratic BEM calculations and those obtained in [1] and [9], but there appears to be a limit on the maximum accuracy achievable (about 0.5% in this case). In all cases, the results are largely independent of the circle radius, except at a radius which is marginally larger than $NR(D)$). There is also very little dependence of the results on the number of parameters in the concentration function. Other purely convex domains yielded results of similar accuracy, with maximum errors in U and Q of ~1%.

4 Convex domain example: flow past a circular cylinder

Zhang [6] provides an example of a concave domain, for which our meshless method needs more careful investigation. In Figure 3, fluid flows past the circular arc AB. U is the stream function and $Q=\nabla U$. There are 'natural' boundary conditions on the boundary segments OA, AB and BC, and the boundary conditions on CD and DO are calculated from the analytical solution, eqn (13) with y=2 and x=0 respectively.

$$U(x,y)=1-\frac{1}{y^2+(x-4)^2} \tag{13}$$

In contrast to the number of nodes used by Zhang [6] to discretise the boundary (ranging from 26 to 104), we obtained similar accuracy using a crude discretisation with at most 20 parameters. We used 18 boundary elements: 3 on OA, 8 on AB, 1 on BC, 4 on CD and 2 on DO. For the following results we

calculated the potential at a selection of the same 19 interior points as Zhang: $\left\{\left(\frac{n}{5},\frac{n}{10}\right), n=1\ldots19\right\}$ (equally spaced on diagonal *OC* in Figure 3). Table 2 gives summary results for these 19 reference points. The Nominal Radius is $\sqrt{5}$: half the diagonal *OC* in Figure 3, and the column marked 'R/*NR*' shows the radius of the enclosing circle *R* as a multiple of the Nominal Radius. CFP is the number of Concentration function parameters and "% error" is the mean absolute percentage error for all 19 reference point.

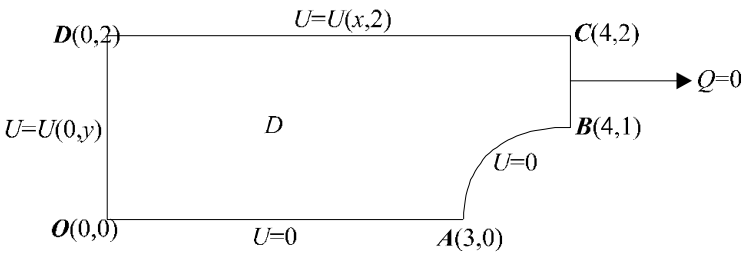

Figure 3: Fluid flow.

Table 2: Fluid flow.

CFP		10	13	15	20	25
R	R/*NR*	% error	% error	% error	% error	% error
2.5	1.1	1.2	1.2	22.2	50.3	254.3
10	4.5	9.5	3.6	56.6	84.0	72.3
20	8.9	10.0	3.2	27.2	33.1	33.8
30	13.4	10.1	3.1	4.8	31.1	28.8
40	17.9	10.1	3.0	3.5	5.2	4.7
50	22.4	10.1	2.9	3.4	5.2	3.5
100	44.7	10.1	2.8	5.2	2.7	2.6
200	89.4	10.1	6.6	6.6	6.6	6.6
500	223.6	9.9	3.5	9.9	9.9	9.9
1000	447.2	9.9	9.9	7.4	9.9	9.9

There is clearly a much wider variation in results than there was for the convex domain example. With 13 concentration function parameters (the number of boundary elements on the convex sides), there is some consistency of results for a wide range of radius. For cases where the radius is between 10 and 20 times the nominal radius (actual radius is roughly between 20 and 50), the mean relative % error is about 3%. For an "infinite" radius ($R > 100$), the mean relative % error is more than 5%, which is unacceptable. Using a "minimal" radius only gives good accuracy with a careful choice for the number of concentration function parameters. Solving other problems with concave boundaries leads us to

believe that this meshless method results in ill-conditioned with respect to the number of concentration function parameters

The main source of error in the overall error measure in Table 2 is nearly always due to the reference point (0.2, 0.1), which is the reference point nearest *O* in Figure 3. This point typically accounts for between 35% and 65% of the total error. The next reference point (0.4, 0.2) usually accounts for a further 10% of the total error.

Table 3 shows the results of calculations of *U* for five reference points using the optimal 13 concentration function parameters. Most results are within 2% of the exact values. In this case the major source of error is the point (3.8, 1.9).

Table 3: Fluid flow - variation of *U* with radius at five reference points.

Radius	U(0.2, 0.1)	U(1.0, 0.5)	U(2.0, 1.0)	U(3, 1.5)	U(3.8, 1.9)
2.5	0.100555	0.439841	0.802767	1.03351	1.42095
10	0.116463	0.431713	0.802797	1.05629	1.27069
20	0.112426	0.434135	0.801896	1.05633	1.27231
30	0.110742	0.435021	0.801614	1.0563	1.27302
40	0.109861	0.43547	0.801478	1.05627	1.27344
50	0.109323	0.43574	0.801396	1.05625	1.27372
100	0.108171	0.436266	0.80118	1.05615	1.2742
200	0.136966	0.415435	0.819446	1.05093	1.24058
500	0.108966	0.444447	0.821415	1.03478	1.22172
1000	0.0081242	0.469391	0.822426	1.00444	1.31153
exact	*0.0930796*	*0.445946*	*0.8*	*1.03846*	*1.37945*

Since the domain is theoretically infinite, we tried larger values for the distances *CD* and *OD*, and used the 'natural' approximations $U_{CD} \sim 1$ and $U_{OD} \sim 1$. We achieved slightly better accuracy than that reported in Table 3 with the values $CD = OD = 10$, which this is probably a more realistic scenario than the boundary conditions stated in Figure 3.

5 Boundary element requirements

In general we have tried to use sufficient boundary elements to reflect the geometry of the domain and the boundary conditions, and no more. Using more did not improve accuracy materially in the case of convex domains. In some concave domains, accuracy deteriorated as the number of boundary elements increased. We attribute this effect to a forced high concentration density on some portions of the enclosing circle.

Hence we never needed the large number of nodes required in some other studies. For example, [5], needed a minimum of 72 boundary elements to achieve acceptable accuracy. Liu does confirm our view that too few parameters

cannot model D effectively, and that too many increases ill-conditioning unacceptably. Zhang [6] reports similar results. Similarly, Fam [8] solves a rectangular domain problem using dipole sources. The domain boundary is discretised into 12 or 24 boundary points. Our solution achieved the same accuracy with only 8 boundary points.

6 Conclusion

Our distributed source method can produce accurate results for very simple domains and can achieve a significant reduction in the number of parameters (of the concentration function) required, and the necessary number of boundary elements. Of the two classes of concentration function considered (collocation and piecewise linear), the piecewise linear class allows for more concentration function parameters if required. It generally gives less accurate but more reliable results. We have found similar results for other simple convex domains, although achievable accuracy diminishes with increasing complexity of the domain. Within very broad limits, for convex domains, the radius of the circular source and the number of collocation function parameters have little impact on the accuracy of the calculations.

When the domain contains at least one concave element, the configuration of the enclosing circle is more stringent. In addition to the fluid flow example discussed here, we have considered other cases and found, with all of them, that optimal accuracy can only be achieved by using particular circle configurations. The only combination of parameters that works tolerably well in every (concave) case is the combination:

number of concentration function parameters =
number of boundary elements on the convex faces of the discretised domain;

and

circular source radius = n NR(D) where $10 \le n \le 20$.

Within this parameter set, $n = 10$ giving marginally better results. Using this combination necessarily involves some trading of accuracy for generality. We stress that this conclusion derives from not only the examples presented here, but also from additional investigation of other domains, with particular attention paid to concave domains. Furthermore, applying the rule suggested above may not give optimal accuracy in any given case. The number of boundary elements should be minimised: there should be sufficient to reflect the geometry of the domain and the boundary conditions, but no more. In particular, the combination "one concentration function parameter per boundary element with infinite radius" would have been appealing.

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