An exact solution of the Navier-Stokes equations for swirl flow models through porous pipes

N. Vlachakis[1], A. Fatsis[1], A. Panoutsopoulou[2], E. Kioussis[1], M. Kouskouti[1] & V. Vlachakis[3]

[1]Technological University of Chalkis, Department of Mechanical Engineering, Greece
[2]Hellenic Defence Systems, Greece
[3]Virginia Polytechnic Institute and State University, Department of Mechanical Engineering, Blacksburg, USA

Abstract

An exact solution of the Navier-Stokes equations for laminar flow inside porous pipes simulating variable suction and injection of blood flows is proposed in the present article. To solve these equations analytically, it is assumed that the effect of the body force by mass transfer phenomena is the 'porosity' of the porous pipe in which the fluid moves. The resultant of the forces in the pores can be expressed as filtration resistance. The developed solutions are of general application and can be applied to any swirling flow in porous pipes.

The effect of porous boundaries on steady laminar flow as well as on species concentration profiles has been considered for several different shapes and systems. In certain physical and physiological processes filtration and mass transfer occurs as a fluid flows through a permeable tube. The velocity and pressure fields in these situations differ from simple Poiseuille flow in an impermeable tube since the fluid in contact with the wall has a normal velocity component. In the new flow model, a variation of the solutions with Bessel functions based on Terrill's theoretical flow model is adopted.

Keywords: exact solution, Navier-Stokes equations, pipe flow, laminar flow, porous media, blood flow characteristics.

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1 Introduction

Swirl particulate flows can be found in nature and have significant industrial applications including infiltration, blood flow and particle separation. The present study was inspired by the need to model swirl flows in such systems with the goal of developing tools for study, design and improvement of the porous and filtration process in mass fraction systems. Computation of such fields is very challenging being further complicated by each porous character and the possibility of laminar regimes.

One of the approaches to model these porous flows is based on solution of the full Navier-Stokes equations. The effect of porous boundaries on steady laminar flow as well as on species concentration profiles has been considered for several different shapes and systems [1–5]. In certain physical and physiological processes filtration and mass transfer occurs as a fluid flows through a permeable tube [6, 7]. The velocity and pressure fields in these situations differ from simple Poiseuille flow in an impermeable tube since the fluid in contact with the wall has a normal velocity component. Therefore, in processes where a combined free and porous flow occurs under the aforementioned conditions, the flow regime can be naturally modelled by coupling Darcy's law and the Navier-Stokes equations. Moreover, many factors such as the Reynolds number and transport properties of the porous media directly affect the dynamics of the flow. The diversity of underlying phenomena and the complexity of interactions between free and porous flow systems have prevented development of a general theoretical analysis of coupled flow systems. In most cases the Navier-Stokes equations are reduced to ordinary non-linear differential equations of third order for which approximate solutions are obtained by a mixture of analytical and numerical methods [8–10].

In this study, an exact solution of the Navier-Stokes equations is proposed describing the flow in a porous pipe allowing the suction or injection at the wall to vary with axial distance. In the current research work, a new exact solution of Terill proposed phenomenology [11] is presented similar to the model of blood floe through a porous pipe with variable injection and suction at the walls. In the new flows model a variation of the solutions with Bessel functions based on Terrill's theoretical flow models is adopted. This study uses biomechanical procedures to find exact solutions of the Navier-Stokes equations, governing steady porous pipe flows of a viscous incompressible fluid in a three-dimensional case including body force term.

2 Mathematical and physical modelling

The mathematical model simulates the capillary between an arteriole and a venule as a horizontal tube of constant cross-section and inner radius R with a permeable wall of thickness δ .

Assuming the flow of a Newtonian fluid through the pipe, the basic equations that describe the mechanics of blood flow in cardiovascular circulation vessels are the mass conservation equation and the equations of motion (Navier-Stokes), in a cylindrical system of coordinates $\left(r,\theta,z\right)$ where the z -axis lies along the centre of the pipe, r is the radial distance and θ is the peripheral angle. A schematic diagram of the model and coordinate system is given in figure 1.

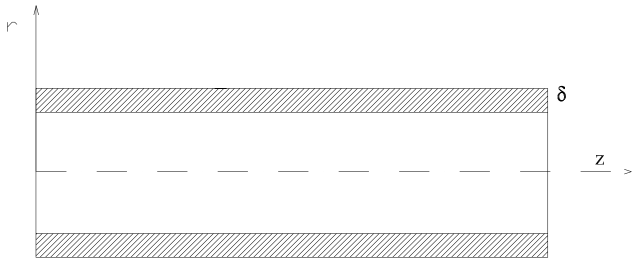

Figure 1: Representation of flow in a tubular membrane with a cylindrical coordinate system.

2.1 The Navier-Stokes equations

Starting from the solutions form suggested by Terrill [11] and taking into account body force phenomena, the following solution is proposed. It is considered that in the porous space of the pipe, mass transfer phenomena appear the body force of that is equivalent to the radial pressure gradient. Moreover, when porous spaces exist a new term is added to the radial pressure gradient which is involved in the first of the wavier-Stokes equations while the following simplified assumptions are made:

a) axial symmetry

b) the fluid is homogeneous and behaves as a Newtonian fluid

c) the pipe is considered of finite length and before the fluid enters the porous pipe its profile has already been developed

d) the permeable membrane is treated as a `fluid medium'.

The continuity equation in c cylindrical system of coordinates is:

$$\frac{1}{r^*}\cdot\frac{\partial}{\partial r}\left(r^* u_r^*\right)+\frac{1}{r^*}\cdot\frac{\partial u_\theta^*}{\partial\theta^*}+\frac{\partial u_z^*}{\partial z^*}=0 \qquad (1)$$

The Navier-Stokes equations for the case of the steady axi-symmetric motion of an incompressible fluid in a porous horizontal pipe are:

The r-direction of the momentum equation:

$$\rho\left(u_r^*\frac{\partial u_r^*}{\partial r^*}+\frac{u_\theta^*}{r^*}\cdot\frac{\partial u_r^*}{\partial\theta^*}-\frac{u_\theta^{*2}}{r^*}+u_z^*\frac{\partial u_r^*}{\partial z^*}\right)=$$

$$f_r\rho+\mu\left[\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}-\frac{\partial}{\partial r^*}\left(r^* u_r^*\right)\right)+\frac{1}{r^{*2}}\frac{\partial^2 u_r^*}{\partial\theta^{*2}}-\frac{2}{r^{*2}}\frac{\partial u_\theta^*}{\partial\theta^*}+\frac{\partial^2 u_r^*}{\partial z^{*2}}\right] \qquad (2)$$

The θ-direction of the momentum equation:

$$\rho\left(u_r^*\frac{\partial u_\theta^*}{\partial r^*}+\frac{u_\theta^*}{r^*}\cdot\frac{\partial u_\theta^*}{\partial\theta^*}+\frac{u_r^*u_\theta^*}{r^*}+u_z^*\frac{\partial u_\theta^*}{\partial z^*}\right)=-\frac{1}{r^*}\cdot\frac{\partial p^*}{\partial\theta^*}$$

$$+\mu\left[\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial\left(r^*u_\theta^*\right)}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2 u_\theta^*}{\partial\theta^{*2}}+\frac{2}{r^{*2}}\cdot\frac{\partial u_r^*}{\partial\theta^*}+\frac{\partial^2 u_\theta^*}{\partial z^{*2}}\right] \quad (3)$$

The z-direction of the momentum equation:

$$\rho\left(u_r^*\frac{\partial u_z^*}{\partial r^*}+\frac{u_\theta^*}{r^*}\cdot\frac{\partial u_z^*}{\partial\theta^*}+u_z^*\frac{\partial u_z^*}{\partial z^*}\right)=$$

$$-\frac{\partial p^*}{\partial z^*}+\mu\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}+\left(r^*\frac{\partial u_z^*}{\partial r^*}\right)+\frac{1}{r^{*2}}\cdot\frac{\partial^2 u_z^*}{\partial\theta^{*2}}+\frac{\partial^2 u_z^*}{\partial z^{*2}}\right] \quad (4)$$

2.2 The porous wall equations

Introducing the dimensionless porosity parameter ξ as follows:

$$\xi=\frac{\dot{V}_\delta\cdot k\cdot\rho}{A_\delta\cdot\delta\cdot\mu} \quad (5)$$

where A_δ is the membrane area, k is the permeability coefficient, $\dot{V}_\delta$ is the volumetric flow rate through the porous space and δ is the thickness of the interstitium.

The porous wall is supposed to be homogenous and isotropic in which the main characteristic is intrinsic permeability k. The flow through the porous wall can be simply taken into account as a boundary condition of the flow through the tube at the permeable wall.

At the permeable wall, the wall suction velocity is given by Darcy's law as a 'fluid-tissue' system :

$$u_r=-\frac{k}{\mu}\left(\frac{\partial P}{\partial r}\right) \quad (6)$$

2.3 Dimensionless form of the equations

The above equations are non-dimensionalised by the following transformation:

$$
\begin{aligned}
r^* &= r \cdot R \\
z^* &= z \cdot R \\
u_r^* &= u_r \cdot U \\
u_z^* &= u_z \cdot U \\
u_\theta^* &= u_\theta \cdot U \\
P^* &= P \cdot \rho \cdot U^2
\end{aligned}
\tag{7}
$$

Taking into account the above assumptions, the continuity equation is written using the dimensionless quantities as:

$$\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \tag{8}$$

Defining the Reynolds number as:

$$\mathrm{Re} = \frac{\rho \cdot U \cdot R}{\mu} \tag{9}$$

the system of the Navier-Stokes equations takes the non-dimensional form:

$$\left(u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} \right) = -(\xi + 1)\frac{\partial p}{\partial r} + \frac{1}{\mathrm{Re}} \left[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2} \right] \tag{10}$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_r \cdot u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} = \frac{1}{\mathrm{Re}} \left[\frac{\partial u_\theta^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} \right] \tag{11}$$

$$\left(u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}} \left[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} \right] \tag{12}$$

3 Solution strategy

Extending the procedure of Terrill [11], the axial velocity u_z, the radial velocity u_r and the tangential velocity u_θ, are expressed in terms of two functions:

$$
\begin{aligned}
u_z &= J_0(rb) e^{-bz} \\
u_r &= J_1(rb) e^{-bz} \\
u_\theta &= \xi \cdot J_1(rb) e^{-bz}
\end{aligned}
\tag{13}
$$

where $J_0\left(rb\right)$ and $J_1\left(rb\right)$ are the Bessel functions of the First kind and b is the zero of

$$J_0\left(J_0\left(b\right)\right)=0 \tag{14}$$

$$J_0\left(rb\right)=-b\cdot J_1\left(rb\right) \tag{15}$$

and

$$J_1^{'}\left(rb\right)=b\cdot J_0\left(rb\right)-\frac{J_1\left(rb\right)}{r} \tag{16}$$

The functions $J_0\left(rb\right)$ and $J_1\left(rb\right)$ are shown in figure 2 in terms of $r\cdot b$.

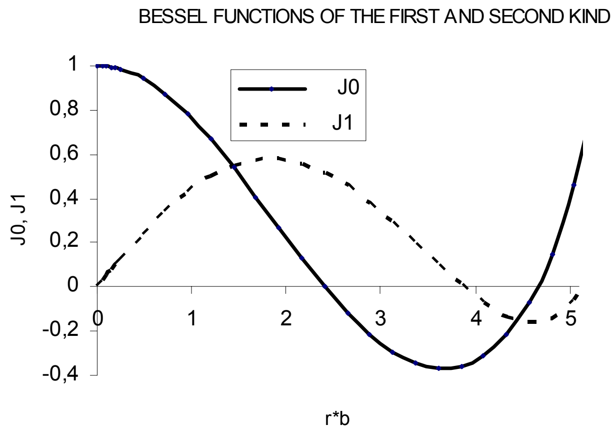

Figure 2: Bessel functions of the first kind.

The following boundary conditions are satisfied:

a. The no-slip condition at the tube wall:

$$u_z=0 \quad \text{at} \quad r=1 \tag{17}$$

b. The suction (b>0) or injection (b<0) condition at the pipe axis:

$$u_r=0 \quad \text{at} \quad r=0 \tag{18}$$

It is assumed that the speed of suction or injection has a finite value at the walls.

c. The swirl condition at the pipe axis:

$$u_\theta=0 \quad \text{at} \quad r=0 \tag{19}$$

Introducing the three velocity components – as expressed in terms of the Bessel functions – in the Navier-Stokes equations, it gives:

$$\frac{\left[J_1\left(rb\right)\right]^2 e^{-2bz}}{r} = \frac{\partial P}{\partial r} \tag{20}$$

$$b\left\{\left[J_1\left(rb\right)\right]^2 + \left[J_0\left(rb\right)\right]^2\right\} e^{-2bz} = \frac{\partial P}{\partial z} \tag{21}$$

Integration of the last equation with respect to z gives:

$$0,5\left\{\left[J_1\left(rb\right)\right]^2 + \left[J_0\left(rb\right)\right]^2\right\} e^{-2bz} = -P\left(r,z\right) + \zeta\left(r\right) \tag{22}$$

Differentiating the above equation with respect to r and combining the equations (20) and (21) it is finally found:

$$P\left(r,z\right) = -0,5\left\{\left[J_1\left(rb\right)\right]^2 + \left[J_0\left(rb\right)\right]^2\right\} e^{-2bz} \tag{23}$$

Thus the required solution of the present model is:

$$\begin{aligned} u_z &= J_0\left(rb\right)e^{-bz} \\ u_r &= J_1\left(rb\right)e^{-bz} \\ u_\theta &= \xi \cdot J_1\left(rb\right)e^{-bz} \end{aligned} \tag{24}$$

$$P\left(r,z\right) = -0,5\left\{\left[J_1\left(rb\right)\right]^2 + \left[J_0\left(rb\right)\right]^2\right\} e^{-2bz}$$

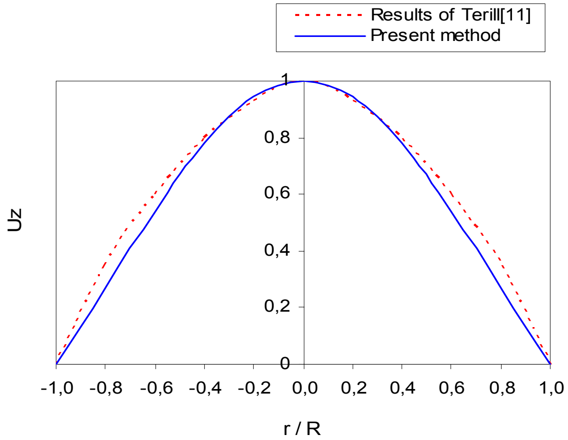

Figure 3: Axial velocity through the pipe at a given axial distance z as a function of the radius of the pipe.

In figure 3 the comparison between the axial velocity predicted by the present model (continuous line) and the couple stress theory of Terill (dashed line) can be seen. A very good comparison overall can be observed.

Figure 4 shows the predicted tangential velocity profiles at a given axial distance with respect to the radius of the pipe. It can be seen that at the center of the pipe the swirl is zero – as imposed by the boundary conditions – and as approaching the wall boundaries it is increasing.

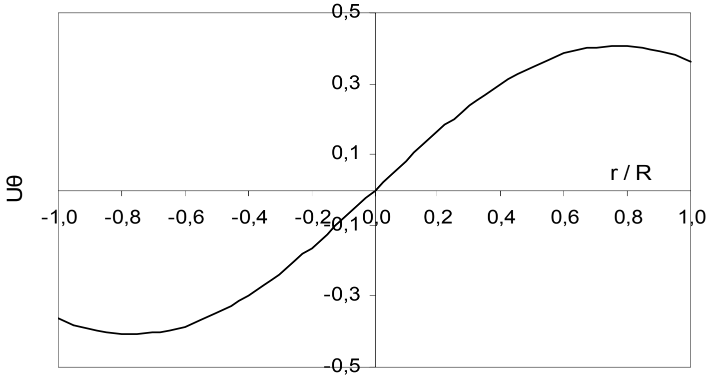

Figure 4: Tangential velocity through the pipe at a given axial distance z as a function of the radius of the pipe.

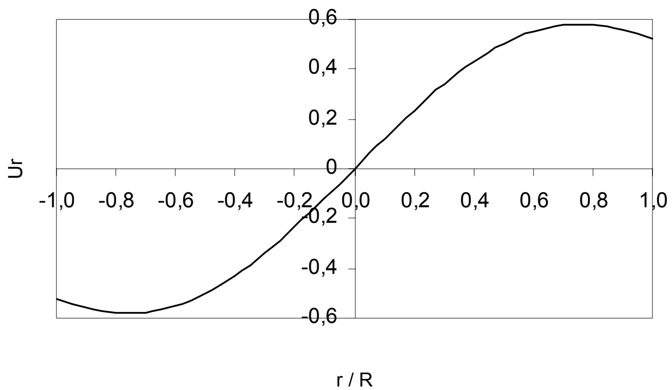

Figure 5: Tangential velocity through the pipe at a given axial distance z as a function of the radius of the pipe.

Figure 5 shows the predicted radial velocity distribution with respect of the pipe radius. At the pipe centre the radial velocity is zero and it increases rapidly up to $r/R = 0.8$. Then the wall porosity causes a deceleration due to the resistance of the fluid through the wall boundary.

4 Conclusions

In this work, a new exact solution of the Navier-Stokes equations is proposed, describing the characteristics of three-dimensional axi-symmetric pipe flows with variable suction and injection at the porous pipe walls. with application to blood flow. In figure 3 the axial velocity distribution across the pipe has been plotted concerning both the theory of Pal et al. [3] and the presented concept of the exact solution blood flow model with porous wall. The pressure and the pressure gradient are dependent on the radial coordinate r in the porous tube. Body force mechanisms in biological membranes are included because of their importance for mass transport. The body force mechanisms which represent here the volume flow rate in the porous space are strongly connected with the angular velocity (twist of the internal particles). The developed solutions are of general application [6, 7] and can be applied to any swirling flow in porous pipes.

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