

Stratified flow over topography: wave generation and boundary layer separation

B. R. Sutherland & D. A. Aguilar

*Department Mathematical and Statistical Sciences, University of Alberta,
Edmonton, Canada*

Abstract

We have performed laboratory experiments to study wave generation over and in the lee of model topography. We have chosen to use periodic, finite-amplitude hills which are representative of the Earth's major mountain ranges as well as the repetitious topographic features of the ocean floor. The topographic shapes are selected to encompass varying degrees of roughness, from smoothly-varying sinusoidal hills to sharper triangular and rectangular hills.

Contrary to linear theory predictions, the (vertical displacement) amplitude of the internal waves directly over the hills is generally much smaller than the hill height. This is because fluid is trapped in the valleys between the hills effectively reducing the amplitude of the hills. Thus the experiments serve to emphasize the importance of boundary layer separation upon internal waves generated by flow over rough topography.

1 Introduction

Internal waves propagate through fluids whose density, effectively, decreases continuously with height. Like surface waves, fluid parcels associated with internal waves move up and down due to buoyancy. However, the motion is not confined to an interface and so it is possible for internal waves to move vertically as well as horizontally through the fluid.

The strongest source of internal waves in the atmosphere results when stratified air flows over mountain ranges. For hills that are periodic and sufficiently small amplitude, linear theory predicts that waves are generated in the overlying air with the same period as the time for flow from one hill crest to the next. The waves propagate vertically away from the hills provided this period is longer than the



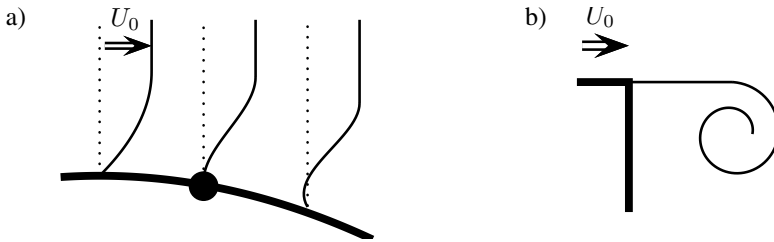


Figure 1: a) Flow profiles surrounding a separation point (solid circle) in the lee of a hill crest. b) Boundary layer separation from a sharp corner.

buoyancy period, which is the natural period of vertical oscillations of the stratified fluid. The buoyancy period is shorter of the stratification is stronger.

Linear theory also predicts that the vertical-displacement amplitude of the waves is half the hill-to-valley distance. Again this assumes the hills are such small amplitude that the air can flow over the surface with negligible change to its speed. If the hills are sufficiently large amplitude, however, we expect the air to move significantly more slowly in the valleys than over the hills.

This occurs due to two reasons. In uniform-density fluid, it is well known that an adverse pressure gradient can develop in an expanding flow leading to deceleration and ultimately reversal of the flow at the boundary, as depicted in fig. 1a. The point where the shear at the boundary is zero, and hence where there is no surface stress, is called a separation point. When such boundary-layer separation occurs the downstream flow typically is unstable. Streamlines wrap into vortices (fig. 1b) and eventually break down turbulently,

In a stratified fluid, boundary layer separation can also occur because the fluid in a valley is significantly more dense than the fluid at the surrounding hill tops. The kinetic energy of the flow relative to the available potential energy of the fluid in the valley in part dictates the location of the separation point.

But the situation is more complicated than this. When unstratified or sufficiently weakly stratified air flows over a set of hills, a perturbation pressure field is established with relatively low pressure overlying hill crests and high pressure at the base of the valleys, fig. 2a. Thus an adverse pressure gradient is established in the lee of the crest and this can lead to boundary layer separation.

However, if the fluid is more strongly stratified so that the forcing period is long compared to the buoyancy period, then internal waves are generated and the pressure variation on the surface changes. Now the centre of low pressure lies at the midpoint between the hill and valley on the lee of the crest (fig. 2b). Meanwhile, the high now lies at the midpoint on the facing side of the hill. This means that the adverse pressure gradient is strongest at the valley floor and so the separation point will shift downslope from its position in the absence of waves. That is, internal waves retard or at least forestall the formation of separated boundary layers.

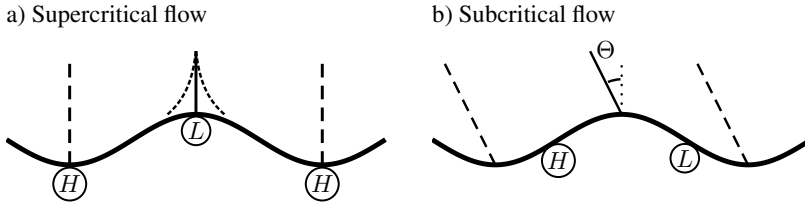


Figure 2: a) Vertical phase lines and surface pressure associated with evanescent disturbances for which $Fr > 1$. Such disturbances have exponentially decreasing amplitude with height as illustrated by the short dashed lines above the hill crest. b) Tilted phase lines associated with vertically propagating internal waves for which $Fr < 1$.

Although the literature on boundary layer separation in uniform density fluids is vast, remarkably few studies have examined the influence of stratification. Baines and Hoinka [1] examined boundary layer separation and resulting boundary-trapped lee waves behind an isolated hill and Baines [2] examined separation due to flow over an isolated valley. Both studies ignored the influence of internal waves upon separation and, conversely, ignored the impact of boundary layer separation upon internal wave generation. In experiments studying stratified flow over a step [3], vertically propagating waves and boundary-trapped lee waves were found to couple resonantly. The period of both was an approximately constant fraction of the buoyancy period.

By way of numerical simulations, Welch *et al.* [4] were the first to study internal waves generated above large amplitude sinusoidal hills. They showed that the wave amplitude increases with the hill height until the hills were so tall that separation occurs. At still larger hill heights they showed that wave amplitudes remained fixed and the depth of the fluid trapped in the valleys increased. These results were corroborated in laboratory experiments [5] examining flow over moderate and large-amplitude sinusoidal hills.

Although these studies have focused upon flow over smooth topography, recent ocean observations [6] suggest that substantial mixing and associated internal wave activity occurs as a result of tidal flow over abyssal canyons associated with the Mid-Atlantic Ridge.

This raises the issue of the influence of topographic shape upon boundary layer separation and internal wave generation, and is the point of the work presented here.

2 Experimental set-up

Details of the experimental setup are given by Aguilar *et al.* [5]. All experiments were performed in a 197 cm long and 17.5 cm wide glass tank. The tank was filled with uniformly salt-stratified fluid to a depth of 27 cm. The resulting density gra-

dient is proportional to the squared buoyancy frequency, which in the Boussinesq approximation is given by

$$N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}. \quad (1)$$

Here g is the acceleration of gravity and ρ_0 is a characteristic density. The buoyancy period is $2\pi/N$.

One of four model hills was then suspended on the surface of this stratified fluid. Each topographic shape consisted of four hills with crest-to-crest distance $\lambda = 13.7$ cm. Three of the hills had half crest-to-trough amplitude $h_0 = 1.30$ cm having either sinusoidal, triangular or rectangular shape. The fourth hill was sinusoidal but had amplitude $h_0 = 0.65$ cm ($h_0/\lambda \sim 0.047$). The qualitative discussion here will focus on the three large-amplitude hills, although amplitude measurements will also include data from the moderate-amplitude hill experiments.

From an initial state of rest, the hills were towed at constant horizontal speed, U , until they reached the end of the tank. Whether or not internal waves are generated by periodic topography is determined by the value of the Froude number

$$\text{Fr} \equiv \frac{Uk}{N}, \quad (2)$$

in which $k = 2\pi/\lambda$ is the wavenumber based on the crest-to-crest distance, λ , of the hills.

Nonlinear effects associated with the half crest-to-valley hill height, h_0 , can be represented nondimensionally in a variety of ways. A measure of the influence of stratification upon boundary layer separation is given by a quantity sometimes called an inverse vertical Froude number. For convenience as well as conceptual accuracy [2], here we call it the ‘Long number’, after Robert Long [7]. Explicitly, we define the Long number by

$$\text{Lo} \equiv \frac{Nh_0}{U}. \quad (3)$$

We expect stratification to play an important role if $\text{Lo} \gg 1$. In the limit $\text{Lo} \ll 1$, linear theory should be valid. But, with N and h_0 fixed, this limit corresponds to increasing flow speed. So boundary layer separation may still occur due to adverse pressure gradients in the lee of a hill crest.

A stratification-independent non-dimensional parameter that represents the importance of boundary-layer separation is $h_0/\lambda = \text{LoFr}/2\pi$, the aspect ratio of the hills. For the three large-amplitude model hills, $h_0/\lambda \sim 0.094$. Because topographic shape is relevant to this study, we define a third parameter $S_{\max}$, which is the maximum slope of the hills. For large amplitude sinusoidal hills, $S_{\max} = \text{LoFr} \simeq 0.60$, for triangular hills, $S_{\max} \simeq 0.38$, and for rectangular hills $S_{\max} \rightarrow \infty$.

Internal waves and boundary layer separation were observed in experiments using synthetic schlieren [8], by which a digital camera records how an image of horizontal black and white lines becomes distorted when the stratified fluid between them is disturbed. In this sense the fluid acts like a time-varying lens.



If fluid separates from a boundary, the associated density gradients are large and greatly distort the image both at the separation point and along the separated streamline. Where the flow is turbulent but still stratified, the image is distorted to such a degree that it blurs and the black and white lines can no longer be distinguished.

The distortion associated with propagating internal waves is not so large and the disturbance field in the tank can be treated as quasi two-dimensional. Knowing how light travels through such a disturbance, the degree of image distortion can straightforwardly be related back to the amplitude of the waves.

For convenience, the results are presented here in terms of the time rate of change of the buoyancy frequency, N^2_t ,

$$N^2_t = \frac{\partial}{\partial t} \left(-\frac{g}{\rho_0} \frac{\partial \rho}{\partial z} \right). \quad (4)$$

in which $\rho = -\bar{\rho}'\xi$ is the perturbation density field. For periodic waves, the N^2_t field is proportional to ξ . Using power spectra to measure peak frequencies and horizontal wavenumbers of the observed waves, the vertical-displacement amplitude, A_ξ , is thereby determined [5].

Although the experiments are conducted so that waves move downward from topography towed along the surface, for conceptual convenience, all the images shown here will be flipped upside-down so that it will appear as if disturbances and waves occur above the hills. Within the Boussinesq approximation, there is no dynamical distinction between upward and downward propagating disturbances.

3 Experimental results

3.1 Qualitative results

First we examine boundary layer separation in experiments with three different topographic shapes towed at relatively slow speeds so that in all three cases $Fr \simeq 0.4$ and $Lo \simeq 1.5$. The distortion pattern associated with streamlines separating in the lee of hill crests is shown in fig. 3. These time series illustrate increased density gradients in the tank where the image of black and white lines behind the tank is magnified.

Separation behind the sinusoidal and triangular hills occurs midway between the hill crests and troughs and the detached streamline reconnects with the following hill near its crest. Apparently the presence of the cusp at the triangular hill crests does not influence boundary layer separation. However, separation occurs right at the lee-side corner of the rectangular-shaped hills and a substantially larger proportion of fluid is trapped between hills. Thus we expect the effective hill height to be relatively smaller for flow over rectangular hills.

In fig. 4 we examine the effect of increasing U (and, hence, increasing Fr and decreasing Lo) upon boundary layer separation. As Fr increases, the separation point occurs further below the hill crest. In part, this is because faster flow has



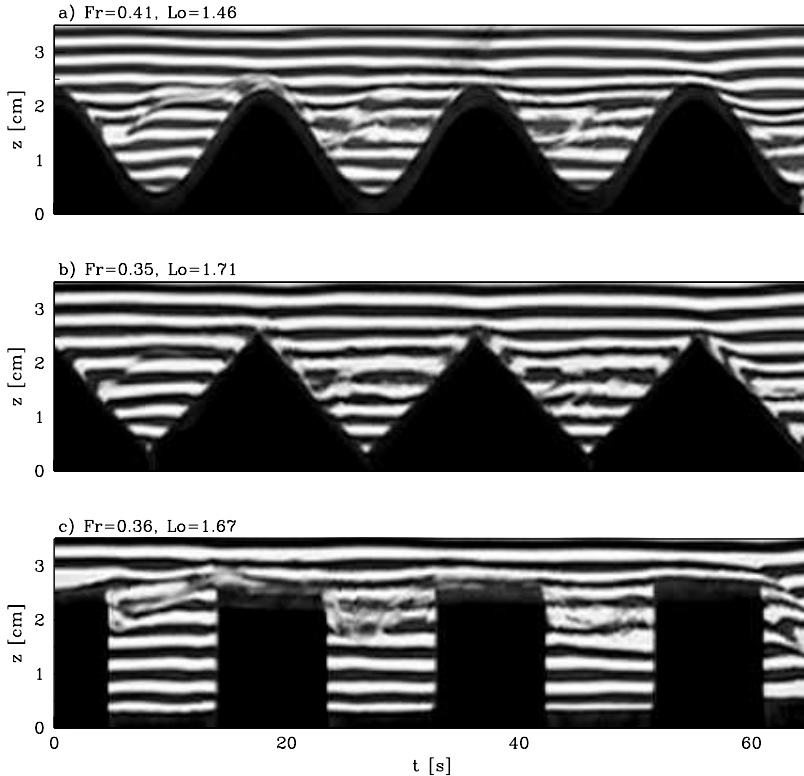


Figure 3: Time series showing distorted image due to flow over a) sinusoidal, b) triangular and c) rectangular topography. In all three cases the flow is subcritical with $Fr \approx 0.4$.

more kinetic energy with which to sweep dense fluid out of the valleys. But is also because the pressure in the lee of the hills decreases as the flow speed increases and so fluid is sucked into the valleys.

Stratification plays less of a role than adverse pressure gradients in establishing a separation point as Fr increases. Although Lo correspondingly decreases to values much smaller than unity, the experiments show that the flow becomes increasingly turbulent as evident from the blurring of the image in figs. 4e and f.

The results are shown for rectangular topography experiments, but the conclusions are qualitatively similar for all three topographic shapes. This serves as a reminder that Lo alone does not establish whether the flow regime is linear or not.

Next we examine the effect of boundary layer separation upon wave excitation. Fig. 5 shows the near hill flow structure up to 5 cm above the hill valleys and shows the wave field between 5 and 20 cm. In the three subcritical cases (figs. 5a,b,c), waves are generated above the hills with phase lines tilting upstream at the slope

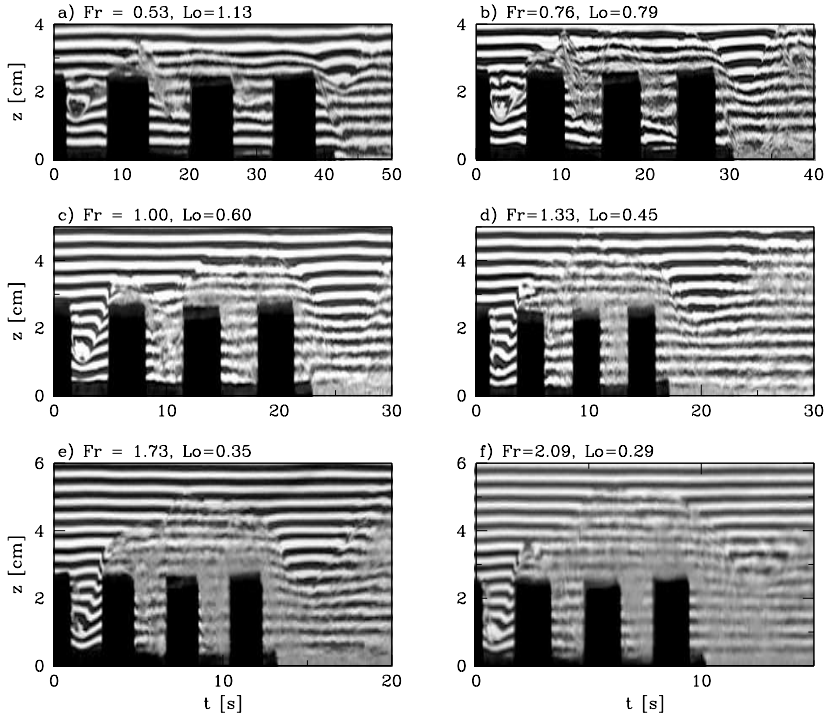


Figure 4: As in fig. 4 but for flow over rectangular topography with successively larger flow rates as indicated by the Froude number.

predicted by linear theory. As expected from our boundary layer separation observations, the wave amplitudes are largest above the sinusoidal and triangular-shaped hills.

In the three supercritical cases (figs. 5d,e,f) shown the waves directly over the hills are evanescent. However, vertically propagating waves are excited in the lee of the four hills in part as a consequence of the low pressure behind the trailing hill. Because there is no fluid trapping behind this hill the streamlines descend much closer to the surface. The flow then rebounds dynamically in response to buoyancy forces, as is evident from measurements showing that the rebound frequency is a nearly constant fraction of N [5].

3.2 Quantitative results

From measurements of the amplitude of the N^2_t field and the corresponding wave frequencies and horizontal wavelengths, we compute the vertical displacement amplitude of the waves. These are plotted in fig. 6.

Fig. 6a shows the measured amplitude of the waves observed directly over the

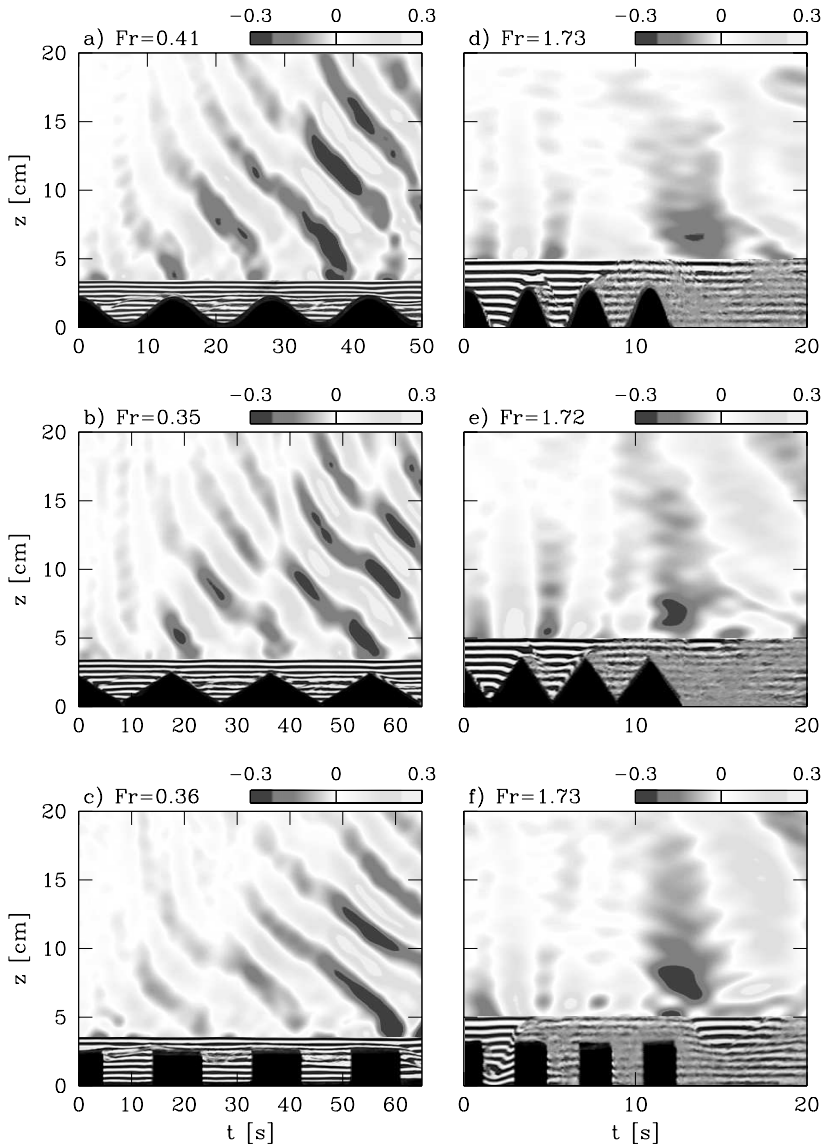


Figure 5: Raw image of near-hill flow and processed images showing N^2_t fields associated with internal waves generated by subcritical flow over a) sinusoidal, b) triangular and c) rectangular topography and by supercritical flow over d) sinusoidal, e) triangular and f) rectangular topography. Colour contours correspond to values of the N^2_t field measured in units of s^{-3} .

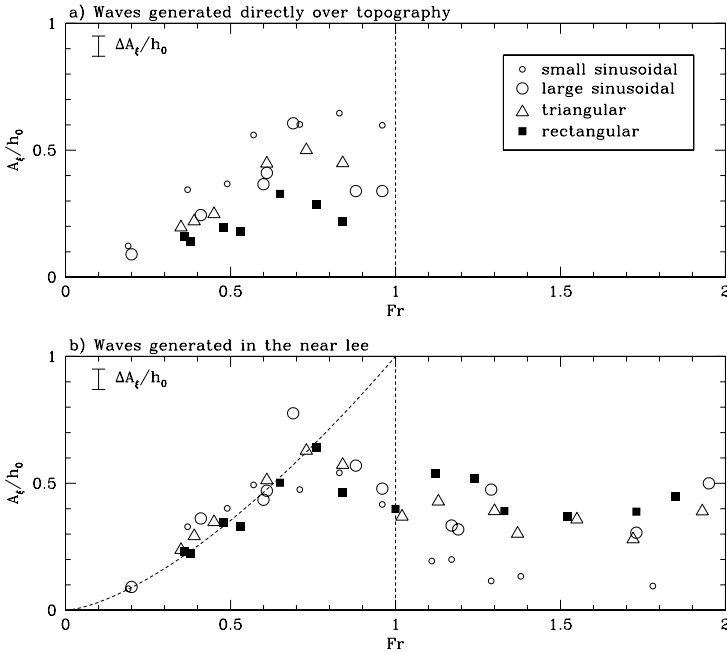


Figure 6: Propagating internal wave amplitudes measured a) over and b) in the lee of 4 sets of hills with shapes indicated in the legend. The dashed line in b) corresponds to the curve $A_{\xi}/h_0 = Fr^{3/2}$.

hills which, consistent with linear theory, were non-evanescent only for subcritical flow with $Fr < 1$. Whereas linear theory predicts the amplitude should equal the topographic height, the experiments show that the waves are consistently smaller. The discrepancy is greater for smaller Fr because Lo correspondingly increases, meaning that stratification more strongly influences blocking. Consistent with this hypothesis, we see that the relative amplitudes of waves above small-amplitude sinusoidal hills is larger than those above large-amplitude hills. Presumably for even smaller amplitude hills, and corresponding smaller Lo , the relative amplitude should approach unity.

In the lee of subcritical topography at fixed Fr (fig. 6b), the relative amplitude of the lee waves is approximately the same for all experiments independent of shape and hill amplitude. Crudely, we find $A_{\xi} \simeq (Fr)^{3/2}h_0$ for $Fr < 0.8$. In supercritical experiments with $Fr > 1$, the relative amplitude of waves in the lee of small sinusoidal hills is smaller than that of waves in the lee of large-amplitude hills. For a range of supercritical Froude numbers, the amplitude behind the large hills is approximately $0.38h_0$.

4 Conclusion

The laboratory experiments reported upon here show is little qualitative or quantitative distinction between the results of the triangular and sinusoidal hill experiments. At fixed Fr and Lo , boundary layers separate at similar locations and internal waves have comparable amplitudes. This gives some hope that the roughness associated with mountain ranges may not play so significant a role in influencing wave excitation.

On the other hand, we find that internal waves have substantially smaller amplitude when generated directly over rectangular topography. This occurs because enhanced boundary layer separation traps fluid in the valleys between the hills and so reduces the effective hill height. This result is important when taken in connection with internal wave generation above the rough, canyon-shaped topography associated with the Mid-Atlantic Ridge.

Acknowledgements

The discussion regarding the influence of internal waves upon boundary layer separation came out of discussions with Richard Rotunno. We are grateful to Caspar Williams for his assistance in preparing some of the figures. This research was supported by the Canadian Foundation for Climate and Atmospheric Science (CFCAS).

References

- [1] Baines, P.G. & Hoinka, K.P., Stratified flow over two-dimensional topography in fluid of infinite depth: a laboratory simulation. *J Atmos Sci*, **42**, pp. 1614–1630, 1985.
- [2] Baines, P.G., *Topographic Effects in Stratified Flows*. Cambridge University Press: Cambridge, England, p. 482, 1995.
- [3] Sutherland, B.R., Large-amplitude internal wave generation in the lee of step-shaped topography. *Geophys Res Lett*, **29(16)**, p. art. no 1769, 2002.
- [4] Welch, W.T., Smolarkiewicz, P., Rotunno, R. & Boville, B.A., The deepening of a mixed layer in a stratified fluid. *J Atmos Sci*, **58(12)**, pp. 1477–1492, 2001.
- [5] Aguilar, D., Sutherland, B.R. & Muraki, D.J., Generation of internal waves over sinusoidal topography. *Deep-Sea Res II*, p. in press, 2006.
- [6] Ledwell, J.R., Montgomery, E., Polzin, K., St.Laurent, L.C., Schmitt, R. & Toole, J., Evidence for enhanced mixing over rough topography in the abyssal ocean. *Nature*, **403**, pp. 179–182, 2000.
- [7] Long, R.R., Some aspects of the flow of stratified fluids. III Continuous density gradients. *Tellus*, **7**, pp. 341–357, 1955.
- [8] Sutherland, B.R., Dalziel, S.B., Hughes, G.O. & Linden, P.F., Visualisation and measurement of internal waves by “synthetic schlieren”. Part 1: Vertically oscillating cylinder. *J Fluid Mech*, **390**, pp. 93–126, 1999.

